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DERIVATIVE OPERATOR ON A HYPERGEOMETRIC FUNCTION OF THREE COMPLEX VARIABLES

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Abstract. In recent years stydies have been don on the effect of some differential operators on the entire function of several complex variables. In this paper a study of the effect of one of these operators (for example $D^{(N)}$-diferential operator) on the hypergeometric function of three complex variables. The effect of this operator has shown some contiguous function relations. A hypergeometric differential equation with its general solution these type of hypergeometric function hane been obtained.

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1.Introduction

The hypergeometric function $H(a,b,b^{'};c;z_1,z_2,z_3)$ of three complex variables z_1, z_2 and z_3, is defined and studied in ([2], [4], [7]) in the form

$$H(a,b,b^{'};c;z_1,z_2,z_3) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+p}(b)_{m+n}(b^{'})_{n+p}}{(c)_{m+n+p}} \frac{z_1^m}{m!}\frac{z_2^n}{n!}\frac{z_3^p}{p!} \tag{1.1}$$

$$= \sum_{m,n,p}^{\infty} \alpha_{m,n,p} z_1^m z_2^n z_3^p.$$

The following form gives the radius of regularity of this function

$$\rho = \limsup_{m+n+p\to\infty} \left\{ \frac{|\alpha_{m,n,p}|}{\sigma_{m,n,p}} \right\}^{\frac{1}{m+n+p}} = 1 \tag{1.2}$$

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where

$$\sigma_{m,n,p} = \begin{cases} (\frac{m+n+p}{m})^{\frac{m}{2}}(\frac{m+n+p}{n})^{\frac{n}{2}}(\frac{m+n+p}{p})^{\frac{p}{2}} & ;m,n,p \neq 0 \\ 1 & ;m,n,p = 0 \end{cases} \tag{1.3}$$

for $a, b, b^{'}$ and c are neither zero or nor negative integers. It is easy to see that the hypergeometric function $H(a, b, b^{'}; c; z_1, z_2, z_3)$ is regular in the closed hypersphere $\bar{S}_r; r = 1$ where is given by $\bar{S}_r = |z_1|^2 + |z_2|^2 + |z_3|^2 = 1$.

The operator D, as given in [3] in the case of two complex variables, takes the form

$$D = \begin{cases} d_1 + d_2 + d_3, & m, n, p \geq 1 \\ 1, & otherwise \end{cases} \tag{1.4}$$

where $d_1 = z_1 \frac{\partial}{\partial z_1}$, $d_2 = z_2 \frac{\partial}{\partial z_2}$ and $d_3 = z_3 \frac{\partial}{\partial z_3}$.

Operating with D on the hypergeometric function $H(a, b, b^{'}; c; z_1, z_2, z_3)$ of three complex variables yields

$$\begin{aligned} DH(a, b, b^{'}; c; z_1, z_2, z_3) &= [z_1(b^{'} + a - c + d_1) + z_2(b' + b - c + d_2) \\ &+ z_3(a + b^{'} - c + d_3)]H(a, b, b^{'}; c; z_1, z_2, z_3) \\ &+ \frac{1}{c}\{z_1[(a - c) - d_2][(b - c) + d_3] \\ &+ z_2[(b^{'} - c) - d_1][(b - c) + d_3] \\ &+ z_3[(b^{'} - c) - d_1][(a - c) + d_2]\}H(a, b, b^{'}; c + 1; z_1, z_2, z_3), \end{aligned} \tag{1.5}$$

alternatively, equation (1.5) can be written in the form

$$\begin{aligned} DH(a, b, b^{'}; c; z_1, z_2, z_3) &= z_1\frac{ab}{c}H(a + 1, b + 1, b^{'}; c + 1; z_1, z_2, z_3) \\ &+ z_2\frac{bb^{'}}{c}H(a, b + 1, b^{'} + 1; c + 1; z_1, z_2, z_3) \\ &+ z_3\frac{ab^{'}}{c}H(a + 1, b, b^{'} + 1; c + 1; z_1, z_2, z_3) \end{aligned} \tag{1.6}$$

since $Dz_1^m z_2^n z_3^p = (m + n + p)z_1^m z_2^n z_3^p$.

2. The Contiguous function relations

For the hypergeometric function $H(a, b, b^{'}; c; z_1, z_2, z_3)$, the following contiguous function relations can be defined

$$
\begin{aligned}
(D+a)H(a,b,b^{'};c;z_1,z_2,z_3) =& aH(a+1,b,b^{'};c;z_1,z_2,z_3) \\
&+ d_2H(a,b,b^{'};c;z_1,z_2,z_3), \qquad (2.1)\\
(D+b)H(a,b,b^{'};c;z_1,z_2,z_3) =& bH(a,b+1,b^{'};c;z_1,z_2,z_3) \\
&+ d_3H(a,b,b^{'};c;z_1,z_2,z_3), \qquad (2.2)\\
(D+b^{'})H(a,b,b^{'};c;z_1,z_2,z_3) =& b^{'}H(a,b,b^{'}+1;c;z_1,z_2,z_3) \\
&+ d_1H(a,b,b^{'};c;z_1,z_2,z_3) \qquad (2.3)
\end{aligned}
$$

and

$$
(D+c-1)H(a,b,b^{'};c;z_1,z_2,z_3) = (c-1)H(a,b,b^{'};c-1;z_1,z_2,z_3). \qquad (2.4)
$$

From (2.1), (2.2), (2.3) and (2.4), it follows at once that

$$
\begin{aligned}
(a-b)H(a,b,b^{'};c;z_1,z_2,z_3) =& aH(a+1,b,b^{'};c;z_1,z_2,z_3) \\
&- bH(a,b+1,b^{'};c;z_1,z_2,z_3) + (d_2-d_3)H(a,b,b^{'};c;z_1,z_2,z_3),
\end{aligned}
\qquad (2.5)
$$

$$
\begin{aligned}
(a-b^{'})H(a,b,b^{'};c;z_1,z_2,z_3) =& aH(a+1,b,b^{'};c;z_1,z_2,z_3) \\
&- b^{'}H(a,b,b^{'}+1;c;z_1,z_2,z_3) + (d_2-d_1)H(a,b,b^{'};c;z_1,z_2,z_3),
\end{aligned}
\qquad (2.6)
$$

$$
\begin{aligned}
(b-b^{'})H(a,b,b^{'};c;z_1,z_2,z_3) =& bH(a,b+1,b^{'};c;z_1,z_2,z_3) \\
&- b^{'}H(a,b,b^{'}+1;c;z_1,z_2,z_3) + (d_3-d_1)H(a,b,b^{'};c;z_1,z_2,z_3)
\end{aligned}
\qquad (2.7)
$$

and

$$
\begin{aligned}
(a-c+1)H(a,b,b^{'};c;z_1,z_2,z_3) =& aH(a+1,b,b^{'};c;z_1,z_2,z_3) \\
&- (c-1)H(a,b,b^{'};c-1;z_1,z_2,z_3) + d_2H(a,b,b^{'};c;z_1,z_2,z_3). \qquad (2.8)
\end{aligned}
$$

From these relations the other contiguous function relations can be deduced.

3. The hypergeometric Differential Equation

The operator D, that is used in the derivation of the contiguous function relations, is also in deriving a differential equation satisfied by (1.1) From (1.1) we obtain

$$\begin{aligned}
&D(D+c-1)H(a,b,b^{'};c;z_1,z_2,z_3)\\
&\qquad=\sum_{m,n,p}\frac{(m+n+p)(m+n+p-1)(a)_{m+p}(b)_{m+n}(b^{'})_{n+p}}{(c)_{m+n+p}}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\frac{z_3^p}{p!}\\
&\qquad=\sum_{m,n,p}\frac{(m+n+p)(a)_{m+p}(b)_{m+n}(b^{'})_{n+p}}{(c)_{m+n+p-1}}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\frac{z_3^p}{p!}\\
&\qquad=z_1\sum_{m,n,p}\frac{(a)_{m+1+p}(b)_{m+1+n}(b^{'})_{n+p}}{(c)_{m+n+p}}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\frac{z_3^p}{p!}\\
&\qquad\quad+z_2\sum_{m,n,p}\frac{(a)_{m+p}(b)_{m+n+1}(b^{'})_{n+1+p}}{(c)_{m+n+p}}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\frac{z_3^p}{p!}\\
&\qquad\quad+z_3\sum_{m,n,p}\frac{(a)_{m+p+1}(b)_{m+n}(b^{'})_{n+p+1}}{(c)_{m+n+p}}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\frac{z_3^p}{p!}\\
&\qquad=\{z_1(a+D-d_2)(b+D-d_3)+z_2(b+D-d_3)(b^{'}+D-d_1)\\
&\qquad\quad+z_3(a+D-d_2)(b^{'}+D-d_1)\}H(a,b,b^{'};c;z_1,z_2,z_3).
\end{aligned}$$

It is easy to see that the hypergeometric function $H(a,b,b^{'};c;z_1,z_2,z_3)$ should be a solution of a partial differential equation given by

$$\begin{aligned}
\{D(D+c-1)&-[(a+D)-d_2][(b+D)-d_3]z_1\\
&-[(b+D)-d_3][(b^{'}+D)-d_1]z_2\\
&-[(a+D)-d_2][(b^{'}+D)-d_1]z_3\}H(a,b,b^{'};c;z_1,z_2,z_3)=0
\end{aligned}\tag{3.1}$$

4. $D^{(N)}$-operator of hypergeometric function of three complex variables

The $D^{(N)}$-operator which is defined and studied in [1,3] in the case of two complex variables and refined here in the case of three complex variables in the form

$$D^{(N)}=\begin{cases}(d_1+d_2+d_3)^N, & m,n,p\geq 1\\ 1, & otherwise\end{cases}\tag{4.1}$$

where N is a finite positive integer. In terms of the $D^{(N)}$-operator, equation (1.5) can be written in the form

$$DH(a,b,b';c;z_1,z_2,z_3) = (c-1)[H(a,b,b';c-1;z_1,z_2,z_3) - H(a,b,b';c;z_1,z_2,z_3)]. \tag{4.2}$$

Now, we employ the differential operator

$$D^2 = z_1^2\frac{\partial^2}{\partial z_1^2} + z_2^2\frac{\partial^2}{\partial z_2^2} + z_3^2\frac{\partial^2}{\partial z_3^2} + 2z_1z_2\frac{\partial^2}{\partial z_1\partial z_2} + 2z_1z_3\frac{\partial^2}{\partial z_1\partial z_3} + 2z_2z_3\frac{\partial^2}{\partial z_2\partial z_3} \tag{4.3}$$

on the hypergeometric function $H(a,b,b';c;z_1,z_2,z_3)$ to get

$$D^2H(a,b,b';c;z_1,z_2,z_3) = (c-1)(c-2)H(a,b,b';c-2;z_1,z_2,z_3) - 2(c-1)^2H(a,b,b';c-1;z_1,z_2,z_3) + c(c-1)H(a,b,b';c;z_1,z_2,z_3). \tag{4.4}$$

Then, the following relation can be derived by induction

$$D^NH(a,b,b';c;z_1,z_2,z_3) = (c-1)\sum_{k=0}^{N}(-1)^k\binom{N}{k}\prod_{j=0}^{N-2}(c-N+j+k)\, H(a,b,b';c-N+k;z_1,z_2,z_3) \tag{4.5}$$

References

[1] M.S.Metwally; *Representation of Analytic Function and Basic Sets of Polynomials in Deferent Regions*, Thesis from M.Sc. Degree, Assiut Univ. (1989).

[2] S. Saran, Hypergeometric Function of Three Variables, Ganita 5, 77-91 (1954).

[3] K.A.M.Sayyed; *Basic Sets of Polynomials of Two Complex Variables and Their Convergence Properties*, Thesis for Ph.D. Degree, Assiut Univ. (1975)

[4] H.M.Srivastava; *Hypergeometric Function of Three Variables*, Ganita 15, 97-108 (1964).

[5] H.M.Srivastava; *On the Reducibility of Certain Hypergeometric Functions* , Revista Mat. Fis. Teorica, 16, 7-14(1966).

[6] H.M.Srivastava; *Relation Between Functions Contiguous to Certain Hypergeometric Function of Three Variables*, Proc. Nat. Acad. Sci. India, Sec. A, 36, 377-385 (1966).

[7] H.M.Srivastava; *Some Integrals Representing Triple Hypergeometric Functions*, Rend. Circ. Mat. Palermo(2) 16(1-3) (1967) 99-115