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On locally nilpotent endomorphisms of Gorenstein injective modules

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Abstract

In this paper, we generalize the idea of injective envelopes to the Ω-envelopes. We find more general conditions for local nilpotent homorphisms with the Ω-envelopes. Also we define the Gorenstein injective envelopes and prove similar theorems about the envelopes and find relations among the injective envelopes, Gorenstein injective envelopes and Ω-envelopes. Finally we can prove that if the base ring R is an n-Gorenstein and $\{(G_i), \psi_j^i\}$ is a directed system of Gorenstein injective R-modules, then $\varinjlim G_i$ is Gorenstein injective.

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1. Ω-Envelopes

Let Ω be a collection of R-modules of a left Noetherian ring R.

Definition 1. *By an Ω-envelope of a left R-module N we mean a linear map $\phi : N \longrightarrow E(N)$ where $E(N)$ is in Ω such that*

1. *if $f : N \longrightarrow F$ is a linear map where $F \in \Omega$, then there is a linear map $g : E(N) \longrightarrow F$ such that $f = g \circ \phi$, i.e., the diagram is commutative.*

$$\begin{tikzcd} N \arrow[r, "\phi"] \arrow[dr, "f"] & E(N) \arrow[d, dotted, "g"] \\ & F \end{tikzcd}$$

2. *if $F = E(N)$ and $f = \phi$ then g is necessarily an automorphism.*

We note that an Ω-envelope of M, when it exists, is unique up to isomorphism.

Theorem 1. *If every left R-module has an Ω-envelope and if Ω is closed under direct sums of a countable number of modules in Ω, then any direct sum $\oplus_{i=1}^{\infty} M_i \longrightarrow \oplus_{i=1}^{\infty} E(M_i)$ of Ω-envelopes $M_i \longrightarrow E_i$ is an Ω-envelope.*

Proof. The proof of corollary 1 [3] gives that if

$$\begin{tikzcd} \oplus_{i\in I} M_i \arrow[r] \arrow[dr] & \oplus_{i\in I} E_i \arrow[d, dotted] \\ & \oplus_{i\in I} E_i \end{tikzcd}$$

is commutative, then $\oplus_{i\in I} E_i \longrightarrow \oplus_{i\in I} E_i$ is an injection. But by hypothesis $\oplus_{i\in I} M_i$ has an Ω-envelope $\oplus_{i\in I} M_i \longrightarrow E$. But then

$$\begin{tikzcd} \oplus_{i\in I} M_i \arrow[r] \arrow[dr] & \oplus_{i\in I} E_i \arrow[d, dotted] \\ & E \end{tikzcd}$$

can be completed to a commutative diagram since each $M_i \longrightarrow E_i$ is an Ω-envelope and $\oplus_{i\in I} E_i \in \Omega$. Since $\oplus_{i\in I} M_i \longrightarrow E$ is an envelope, the diagram

$$\begin{tikzcd}
\oplus_{i\in I} M_i \arrow[r] \arrow[dr] & E \arrow[d, dotted] \\
 & \oplus_{i\in I} E_i
\end{tikzcd}$$

can be completed to a commutative diagram. Then consider the following diagram

$$\begin{tikzcd}
 & E \arrow[d] \\
\oplus_{i\in I} M_i \arrow[ur] \arrow[r] \arrow[dr] & \oplus_{i\in I} E_i \arrow[d] \\
 & E
\end{tikzcd}$$

shows $E \longrightarrow \oplus_{i\in I}E_i \longrightarrow E$ is an automorphism of E. So it makes E isomorphically to a summand of $\oplus_{i\in I}E_i$. If $E \longrightarrow \oplus_{i\in I}E_i$ were not an isomorphism, then $ker(\oplus_{i\in I}E_i \longrightarrow E) \neq 0$. But then $ker(\oplus_{i\in I}E_i \longrightarrow \oplus_{i\in I}E_i) \neq 0$ which is impossible by the corollary 1 [3]. $\square$

Let Ω be some class of left R-modules such that every left R-module has an Ω-envelope and Ω is closed under a countable direct sum of left R-modules. Then we can extend the idea about locally nilpotent endomorphisms of the class of injective modules to an arbitrary class Ω of modules defined in definition 1.

Lemma 1. *If $N \xrightarrow{\phi} E(N)$ is an Ω-envelope of N and $N \xrightarrow{g} N$ is an automorphism of N, then $g(N) \xrightarrow{\phi\cdot g^{-1}} E(N)$ is an Ω-envelope of $g(N)$.*

Proof. Consider the following diagram :

$$\begin{tikzcd}
N \arrow[r, "\phi"] \arrow[d, "g"'] & E(N) \\
g(N) \arrow[ur, dotted, "\phi\circ g^{-1}"'] \arrow[r, "\phi\circ g^{-1}"] & E(N)
\end{tikzcd}$$

The map $\phi \circ g^{-1}$ satisfies all the conditions of Ω-envelope. $\square$

Lemma 2. *Let $N \in \Omega$ and $h : N \longrightarrow E(N)$ be a Ω-envelope of N. If $\phi : E(N) \longrightarrow E(N)$ is a linear map such that $\phi|_N$ is an automorphism of N, then ϕ is an automorphism of $E(N)$.*

Proof. Consider the map

$$\begin{tikzcd}
N \arrow[r, "h"] \arrow[d, "\phi|_N"'] \arrow[dr, dotted, "h"'] & E(N) \arrow[d, "\phi"] \\
N \arrow[r, "h \circ (\phi|_N)^{-1}"] & E(N)
\end{tikzcd}$$

Since $\phi|_N$ is an automorphism, $(\phi|_N)^{-1}$ is defined. So the left triangle is commutative. By the definition of Ω-envelope, ϕ is an automorphism of $E(N)$. $\square$

Theorem 2. *If $f \in End(E)$ where $E \in \Omega$ such that $ker(f) \subset E$ is an Ω-envelope, then f is locally nilpotent on E.*

Proof. Let K be the kernel of f, then $K \subset E$ is an Ω-envelope. Then by the hypothesis of Ω and by Thoerem 1, $K \oplus K \oplus \cdots \subset E \oplus E \oplus \cdots$ is an Ω-envelope. So define a map

$$\begin{array}{rcl} \phi : E \oplus E \oplus \cdots & \longrightarrow & E \oplus E \oplus \cdots \\ (x_1, x_2, \cdots) & \mapsto & (x_1, x_2 - f(x_1), x_3 - f(x_2), \cdots) \end{array}$$

Then ϕ is a homomorphism and $\phi|_{K \oplus K \oplus \cdots} = id_{K \oplus K \oplus \cdots}$. So by lemma 2, ϕ is an automorphism of $E \oplus E \oplus \cdots$ and in particular ϕ is onto. Let $x \in E$ and consider $(x, 0, 0, \cdots)$. Then $\phi(x_1, x_2, x_3, \cdots) = (x, 0, 0, \cdots)$ for some $(x_1, x_2, x_3, \cdots) \in E \oplus E \oplus \cdots$. Then $x_1 = x, x_2 - f(x_1) = 0, x_3 - f(x_2) = 0$ and so on. So $x_n = f^{n-1}(x)$ for all $n \geq 2$. But since $x_{n+1} = 0$ for some n, $f^n(x) = 0$ i.e., f is locally nilpotent on E. $\square$

¿From the fact that $ker(f) \subset ker(f^2) \subset \cdots \subset ker(f^n) \subset \cdots$ we can generalize the hypothesis of theorem 2.

Theorem 3. *If R is left Noetherian ring, E is in Ω, and $f \in End(E)$ is such that $\bigcup_{n=1}^{\infty} ker(f^n) \subset E$ is an Ω-envelope, then f is locally nilpotent on E .*

Proof. Let $K = \cup_{n=1}^{\infty} ker(f^n) \subset E$. Define

$$\begin{array}{rcl} \psi : E \oplus E \oplus \cdots & \longrightarrow & E \oplus E \oplus \cdots \\ (x_1, x_2, \cdots) & \longmapsto & (x_1, x_2 - f(x_1), x_3 - f(x_2), \cdots) \end{array}$$

Then easily ψ is an endomorphism. If $(x_1, x_2, \cdots)$ in $K \oplus K \oplus \cdots$, then there exist $a_i, b_i \geq 1$ such that $f^{a_i}(x_i) = 0$ and $f^{b_i}(x_{i-1}) = 0$. So $f^{m_i}(x_i - f(x_{i-1})) = 0$ for some $m_i \geq max\{a_i, b_i\}$ for each i. So ψ maps $K \oplus K \oplus \cdots$ into $K \oplus K \oplus \cdots$. And

$\psi|_{K\oplus K\oplus\cdots}$ is injective since its kernel is 0. Now let $(y_1, y_2, \cdots) \in K\oplus K\oplus\cdots$. Since each $y_i \in K = \bigcup_{n=1}^{\infty} Ker(f^n)$ there exists $t_i \in \{1, 2, \cdots\}$ such that $f^{t_i}(y_i) = 0$. Given $n \in \{1, 2, \cdots\}$ choose m such that

$$m \geq max\{t_n, t_{n-1} - 1, t_{n-2} - 2, \cdots, t_1 - (n-1)\}.$$

Then

$$\begin{aligned} & f^m(y_n + f(y_{n-1}) + f^2(y_{n-2}) + \cdots + f^{n-1}(y_1)) \\ = \;& f^m(y_n) + f^{m+1}(y_{n-1}) + f^{m+2}(y_{n-2}) + \cdots + f^{m+n-1}(y_1) \\ = \;& 0 \end{aligned}$$

so for every $n \in \{1, 2, \cdots\}$

$$y_n + f(y_{n-1}) + f^2(y_{n-2}) + \cdots + f^{n-1}(y_1) \in K$$

Then

$$\begin{aligned} & \psi(y_1, y_2 + f(y_1), \cdots, y_n + f(y_{n-1}) + f^2(y_{n-2}) + \cdots + f^{n-1}(y_1), \cdots) \\ = \;& (y_1, y_2 + f(y_1) - f(y_1), y_3 + f(y_2) + f^2(y_1) - f(y_2) - f^2(y_1), \cdots) \\ = \;& (y_1, y_2, \cdots, y_n, \cdots). \end{aligned}$$

So $\psi|_{K\oplus K\oplus\cdots}$ is onto. Hence $\psi|_{K\oplus K\oplus\cdots}$ is an isomorphism on $K \oplus K \oplus \cdots$ and ψ is an automorphism of $E \oplus E \oplus \cdots$. Let $x \in E$ and consider $(x, 0, 0, \cdots)$. Then $\psi(x_1, x_2, \cdots) = (x, 0, 0, \cdots)$ for some $(x_1, x_2, \cdots) \in E \oplus E \oplus \cdots$. Then

$$\begin{aligned} x_1 &= x, \\ x_2 - f(x_1) &= 0 \\ x_3 - f(x_2) &= 0 \\ &\cdots \\ x_n - f(x_{n-1}) &= 0 \\ &\cdots \end{aligned}$$

So $x_n = f(x_{n-1}) = f^2(x_{n-2}) = \cdots = f^{n-1}(x_1) = f^{n-1}(x)$. But for some n, $x_{n+1} = f^n(x) = 0$. Therefore f is a locally nilpotent on E. □

Lemma 3. *If R is left Noetherian ring and $f : E \longrightarrow E$ is an endomorphism of a left R-module in Ω such that $\cup_{n=1}^{\infty} ker(f^n) \subset E$ is an Ω-envelope, then $\cup_{n=1}^{\infty} ker(f^n) = E$.*

Proof. Obvious. □

Lemma 4. *Let R be a left Noetherian. Let E be a left R-module in Ω and let $f_0, f_1, f_2, \cdots$ be a sequence of elements in $End(E)$. Consider those $x \in E$ such that $f_n \circ f_{n-1} \circ \cdots f_1 \circ f_0(x) = 0$ for some n(depending on x). If K is the set of such x, then K is a submodule of E.*

Proof. Obvious. □

Theorem 4. *If K is closed for all $f_0, f_1, \cdots$ i.e., $f_i(x) \in K$ for all $x \in K$ and $K \subset E$ is an Ω-envelope, then $K = E$.*

Proof. Consider the direct sum $K \oplus K \oplus \cdots$ of countable number of Ks. Since $K \subset E$ is an Ω-envelope, $K \oplus K \oplus \cdots \subset E \oplus E \oplus \cdots$ is an Ω-envelope. So define a map

$$\begin{array}{rcl} \phi : E \oplus E \oplus \cdots & \longrightarrow & E \oplus E \oplus \cdots \\ (x_1, x_2, x_3, \cdots) & \mapsto & (x_1, x_2 - f_0(x_1), x_3 - f_1(x_2), \cdots) \end{array}$$

Then ϕ is a homomorphism, and $\phi|_{K \oplus K \oplus \cdots}$ is an injection since $\ker\phi|_{K \oplus K \oplus \cdots} = 0$. And if $(y_1, y_2, y_3, \cdots) \in K \oplus K \oplus \cdots$, then let

$$\begin{array}{rcl} x_1 & = & y_1 \\ x_2 & = & y_2 + f_0(x_1) \\ x_3 & = & y_3 + f_1(x_2) \\ & & \cdots \end{array}$$

Then

$$\begin{array}{rcl} \phi(x_1, x_2, \cdots) & = & (x_1, x_2 - f_0(x_1), x_3 - f_1(x_2), \cdots) \\ & = & (y_1, y_2 + f_0(x_1) - f_0(x_1), y_3 + f_1(x_2) - f_1(x_2), \cdots) \\ & = & (y_1, y_2, y_3, \cdots) \end{array}$$

So ϕ is onto on $K \oplus K \oplus \cdots$. Hence $\phi|_{K \oplus K \oplus \cdots}$ is an isomorphism of $K \oplus K \oplus \cdots$. So by lemma 2, ϕ is an isomorphism of $E \oplus E \oplus \cdots$ and in particular ϕ is onto. Let $x \in E$ and consider $(x, 0, 0, \cdots)$. Then $\phi(x_1, x_2, x_3, \cdots) = (x, 0, 0, \cdots)$ for some $(x_1, x_2, x_3, \cdots) \in E \oplus E \oplus \cdots$. Then $x_1 = x, x_2 - f_0(x_1) = 0, x_3 - f_1(x_2) = 0$ and so on. So $x_n = f_{n-2}(x_{n-1})$ for all $n \geq 2$. But for some n, $x_{n+2} = 0$. Choose any $x \in E$ then

$$\begin{array}{rcl} 0 & = & x_{n+2} \\ & = & f_n(x_{n+1}) \\ & = & f_n(f_{n-1}(x_n)) \\ & = & f_n(f_{n-1}(f_{n-2}(x_{n-1}))) \\ & = & \cdots \\ & = & (f_n \circ f_{n-1} \circ \cdots \circ f_0)(x) \end{array}$$

So $x \in K$ for all $x \in E$. This ends the proof. $\square$

For examples, let Ω be the collection of injective modules then Ω is closed under direct sums of a countable number of modules in Ω. The definition of Ω-envelopes is equivalent to the definition of the injective envelopes of modules.

Corollary 1. *If R is left Noetherian, E is an injective left R-module, and $f \in End(E)$ is such that E is an essential extension of $ker(f)$ denoted $ker(f) \subset' E$, then f is locally nilpotent on E.*

Corollary 2. *If R is left Noetherian, E is an injective left R-module, and $f \in End(E)$ is such that $\bigcup_{n=1}^{\infty} ker(f^n) \subset' E$, then f is locally nilpotent on E .*

Corollary 3. *Let R be a left Noetherian ring, E be an injective left R-module and $f_0, f_1, f_2, \cdots$ be a sequence of elements in $End(E)$. Let $K = \{x \in E | f_n \circ f_{n-1} \circ \cdots \circ f_0(x) = 0$ for some $n\}$. If $f_i(K) \subset K$ for all $i \geq 0$ and $K \subset' E$ then $K = E$.*

2. Gorenstein injective modules over Gorenstein rings

Definition 2. *A ring R is said to be Iwanaga-Gorenstein (or simply Gorenstein) if R is left and right Noetherian and if R has finite self-injective dimension on both sides.*

Zaks [8] has shown that when R is Gorenstein the two self-injective dimensions are the same.

Definition 3. *A left R-module G is said to be Gorenstein injective if there is an exact sequence*

$$\cdots \to E^{-2} \to E^{-1} \to E^{0} \to E^{1} \to E^{2} \to \cdots$$

of injective left R-modules such that $G = ker(E^0 \to E^1)$ and such that the functor $Hom(E, -)$ leaves the sequence exact whenever E is an injective left R-module.

All injective modules are trivially Gorenstein injective. Over a Gorenstein ring R the injective modules are the only Gorenstein injective modules precisely when R has finite global dimension. The following result gives a manageable criterion in order that a module G be Gorenstein injective over a Gorenstein ring.

Proposition 1. *If R is a Gorenstein ring and $n = inj.dim._R R$ (also $= inj.dim.R_R$) then a left R-module G is Gorenstein injective if and only if there is an exact sequence*

$$0 \to M \to E^0 \to \cdots \to E^{n-1} \to G \to 0$$

with $E^0, \cdots, E^{n-1}$ injective left R-modules, i.e. if and only if G is an n-th cosyzygy.

Proof. [9], Proposition 4.4 □

Let Ω_G be the collection of Gorenstein injective left R-modules of a Gorenstein ring R. Then the Ω_G-envelope is also called the Gorenstein injective envelope. In [9] all modules over a Gorenstein ring were proved to have such envelopes.

Lemma 5. *The collection Ω_G of Gorenstein injective modules over a Gorenstein ring is closed under direct sums. And if $M_i \longrightarrow G_i$ are Ω_G-envelopes over a Gorenstein ring for all $i \in I$, then $\oplus M_i \longrightarrow \oplus G_i$ is an Ω_G-envelope.*

Proof. Suppose $n = inj.dim_R R$. Let $(G_i)_{i \in I}$ be a family of Gorenstein injective modules. By Proposition 1 above there exist exact sequences

$$0 \longrightarrow M_i \longrightarrow E_i^0 \longrightarrow E_i^1 \longrightarrow \cdots \longrightarrow E_i^{n-1} \longrightarrow G_i \longrightarrow 0$$

such that $E_i^0, E_i^1, \cdots, E_i^{n-1}$ are all injective. Then

$$0 \longrightarrow \oplus M_i \longrightarrow \oplus E_i^0 \longrightarrow \oplus E_i^1 \longrightarrow \cdots \longrightarrow \oplus E_i^{n-1} \longrightarrow \oplus G_i \longrightarrow 0$$

is an exact sequence. Since R is left Noetherian, each $\oplus E_i^j$ is an injective module. So again by Proposition 1, $\oplus G_i$ is Gorenstein injective. Hence Ω_G is closed under the direct sums. □

Corollary 4. *If R is a Gorenstein ring, and $f \in End(G)$ is such that $ker(f) \subset G$ is an Ω_G-envelope, then f is locally nilpotent on E .*

Proof. Obvious from Theorem 2. □

Corollary 5. *If R is a Gorenstein ring, and $f \in End(G)$ is such that $\bigcup_{n=1}^{\infty} ker(f^n) \subset G$ is an Ω_G-envelope, then f is locally nilpotent on E .*

Proof. Obvious from Theorem 3. □

Corollary 6. *Let R be a Gorenstein ring and let $K \subset G$ be an Ω_G-envelope. Let $f_0, f_1, f_2, \cdots$ be elements of $End(G)$ such that $f_n(K) \subset K$ for all $n \geq 0$. If for each $x \in K$, there is an $n \geq 0$ such that $(f_n \circ f_{n-1} \circ \cdots \circ f_0)(x) = 0$, then the same is true for all $x \in G$.*

Proof. Obvious from Theorem 4. □

Remark: If in the corollary 4 it were true that $ker(f)$ is essential in E (as is the case of an injective envelope) then the result would follow from the proof of the corresponding result for injective envelopes. However there are Gorenstein injective envelopes which are not essential extensions (see [10]).

Theorem 5. *Let R be an n-Gorenstein ring and $\{(G_i), \psi_j^i\}$ be a directed system of Gorenstein injective R-modules. Then $\varinjlim G_i$ is Gorenstein injective.*

Proof. Let $\mathcal{X}$ be a representative set of indecomposable injective left R-modules. For left R-modules M, we form the injective modules $E^* = \oplus_{Hom(E,M)} E$, and $E' = \oplus_{E \in \mathcal{X}} E^*$ and let $E' \to M$ be defined by the evaluation maps. Then it is easy to see that $E' \to M$ is an injective precover and that this procedure is functorial. We now appeal to Proposition 4.3 [9]. That result says that if G is Gorenstein injective and if $E \to G$ is an injective cover then $E \to G$ is surjective and $ker(E \to G)$ is Gorenstein injective. But then if $E' \to G$ is an injective precover, it is easy to see that $E' = E_1 \oplus E_2$ where $E_1 \to G$ is an injective cover and where $E_2 \subset ker(E' \to G)$. So the result holds if we only assume $E \to G$ an injective precover. From this it follows that if G is Gorenstein injective, we can functorially construct an exact sequence

$$0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^{n-1} \longrightarrow G \longrightarrow 0$$

with $E^0, E^1, \cdots, E^{n-1}$ injective. Now let $\{(G_i), \psi_j^i\}$ be a directed system of Gorenstein injective R-modules. Then we functorially construct such exact sequences

$$0 \longrightarrow N_i \longrightarrow E_i^0 \longrightarrow E_i^1 \longrightarrow \cdots \longrightarrow E_i^{n-1} \longrightarrow G_i \longrightarrow 0$$

with $E_i^0, E_i^1, \cdots, E_i^{n-1}$ injective for each i. Then

$$0 \longrightarrow \varinjlim N_i \longrightarrow \varinjlim E_i^0 \longrightarrow \cdots \longrightarrow \varinjlim E_i^{n-1} \longrightarrow \varinjlim G_i \longrightarrow 0$$

is also exact. Since the ring R is Gorenstein, each $\varinjlim E_i^j$ is injective module. Hence $\varinjlim G_i$ is an n-th cosyzygy of $\varinjlim N_i$ and so $\varinjlim G_i$ is Gorenstein injective. □

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