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ON FAMILIES OF CAUCHY TRANSFORMS

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Abstract

In this paper we prove a number of results on Cauchy transforms of generalized type given by Borel measures supported on the class of analytic functions mapping the unit disc into the unit disk.

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1. Introduction

Let $\Delta = \{z \in C : |z| < 1\}$, $\Gamma = \partial\Delta$ and let B (equipped with the topology of uniform convergence on compact subsets) denote the set of functions ϕ that are analytic in Δ such that $|\phi(z)| < 1$ and $\phi(0) = 0$. Let $\mathcal{M}, \mathcal{N}$ denote the sets of complex–valued Borel measures on Γ and B respectively. Here, $\mathcal{M}$ is equivalent to the subset of $\mathcal{N}$ consisting of all those measures supported on the set $\{xz : |x| = 1\}$.

For $z \in \Delta$ and $\alpha \geq 0$, let A_α denote the family of functions f for which there exists a measure $\mu \in \mathcal{N}$ such that

$$f(z) = \begin{cases} \displaystyle\int_B \frac{1}{(1-\phi(z))^\alpha}\, d\mu(\phi) & \text{for } \alpha > 0 \\ \displaystyle\int_B \log\frac{1}{1-\phi(z)}\, d\mu(\phi) + f(0) & \text{for } \alpha = 0 \end{cases} \tag{1.1}$$

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The classes F_α consisting of those functions g for which there exists a measure $\mu \in \mathcal{M}$ such that,

$$g(z) = \begin{cases} \displaystyle\int_B \frac{1}{(1-\overline{x}z)^\alpha}\, d\mu(x) & \text{for } \alpha > 0 \\ \displaystyle\int_B \log\frac{1}{1-\overline{x}z}\, d\mu(x) + f(0) & \text{for } \alpha = 0 \end{cases} \tag{1.2}$$

have been well studied [3], [4], [9] and [10]. The classes F_α are subsets of A_α when the measures μ in (1.1) are in $\mathcal{M}$.

The class A_α is a Banach space with respect to the norm

$$\|f\|_{A_\alpha} = \begin{cases} \inf \|\mu\|, & \text{for } \alpha > 0 \\ \inf \|\mu\| + |f(0)|, & \text{for } \alpha = 0 \end{cases} \tag{1.3}$$

where μ varies over all measures in $\mathcal{N}$ for which (1.1) holds and where $\|\mu\|$ denotes the total variation norm of μ .

Clearly, for $f \in F_\alpha$, $\|f\|_{F_\alpha} \geq \|f\|_{A_\alpha}$. It is also known from [1] that for $\alpha \geq 1$, $F_\alpha = A_\alpha$.

We will show in this paper that for $0 \leq \alpha < \beta$,

$$A_\alpha \subset A_\beta \quad \text{and} \quad \|f\|_{A_\beta} \leq \|f\|_{A_\alpha} \tag{1.4}$$

This generalizes similar results for F_α in [6] and [7].

We will also show that $A_0 = BMOA$ and that the norm $\|.\|_{A_0}$ is equivalent to well known BMO norms. Furthermore we will show that, for all $n \geq 1$ and $\alpha \geq 0$

$$\|z^n\|_{A_\alpha} \leq k$$

where the constant k is independent of n or α.

2. The Classes A_α

In this section we will establish for $0 \leq \alpha < \beta$ the relationship between A_α and A_β as well as their respective norms.

Theorem 1. *If $0 \leq \alpha < \beta$, then $A_\alpha \subset A_\beta$ and $\|f\|_{A_\beta} \leq \|f\|_{A_\alpha}$*

Proof. Note that since $A_\alpha = F_\alpha$ for $\alpha \geq 1$ [1], and for $0 < \alpha < \beta$,$F_\alpha \subset F_\beta$ and $\|f\|_{F_\beta} \leq \|f\|_{F_\alpha}$ [7], then all we have to prove is the case $0 \leq \alpha < \beta < 1$.

(i) Let $f \in A_\alpha$ where $0 < \alpha < \beta$, then we can write

$$f(z) = \int_B \frac{1}{(1-\psi(z))^\alpha} d\mu(\psi), \tag{2.1}$$

Since $\dfrac{1}{(1-z)^\alpha} \in F_\alpha \subset F_\beta$, we can write

$$\frac{1}{(1-z)^\alpha} = \int_\Gamma \frac{1}{(1-\overline{x}z)^\beta}\, d\nu(x) \tag{2.2}$$

and

$$\left\|\frac{1}{(1-z)^\alpha}\right\|_{F_\beta} \leq \left\|\frac{1}{(1-z)^\alpha}\right\|_{F_\alpha} = 1 \tag{2.3}$$

Now by replacing z in (2.2) by $\psi(z)$ and putting the result in (2.1) we get

$$f(z) = \int_B \int_\Gamma \frac{1}{(1-\overline{x}\psi(z))^\beta} d\nu(x) d\mu(\psi). \tag{2.4}$$

Suppose, without loss of generality that ν is a positive measure and let

$$g_n(z) = \sum_{k=1}^{n} \frac{\nu_k}{(1-\overline{x}z)^\beta}.$$

Then by (2.2),

$$\int_B g_n(\psi)\ d\mu(\psi)$$

converges locally uniformly to

$$f(z) = \int_B \int_\Gamma \frac{1}{(1-\overline{x}\psi(z))^\beta} d\nu(x) d\mu(\psi).$$

Let $\eta_n(\psi) = \sum_{k=1}^{n} \nu_k \mu(\psi)$ then,

$$\int_B g_n(\psi)\ d\mu(\psi) = \int_B \frac{1}{(1-\psi)^\beta} d\eta_n(\psi),$$

where $\|\eta_n\| \leq \|\nu\| \|\mu\|$ for all n. Hence by compactness, there exists a measure σ, such that,

$$f(z) = \int_B \frac{1}{(1-\psi(z))^\beta} d\sigma(\psi) \tag{2.5}$$

which shows that $f \in A_\beta$. Furthermore, $\|\sigma\| \leq \|\nu\| \|\mu\|$. Consequently,

$$\|f\|_{A_\beta} \leq \|\sigma\| \leq \|\mu\| \|\nu\| \quad \text{for all } \mu$$

however since μ and ν are arbitrary measures that give (2.1) and (2.2) then,

$$\|f\|_{A_\beta} \leq \inf\{\|\mu\|\} \inf\{\|\nu\|\}$$

Hence by (2.3)

$$\|f\|_{A_\beta} \leq \|f\|_{A_\alpha}$$

(ii) Now let $f \in A_0$. We want to show that $f \in A_\alpha$ for any $\alpha > 0$. By definition,

$$f(z) = \int_B \log \frac{1}{1-\phi(z)}\, d\mu(\phi) + f(0) \tag{2.6}$$

Since $\log \dfrac{1}{1-z} \in F_0 \subset F_\alpha$ see [5], then

$$\log \frac{1}{1-z} = \int_\Gamma \frac{1}{(1-\overline{x}z)^\alpha}\, d\nu(x)$$

where ν depends only on α. Hence (2.6) becomes

$$f(z) = \int_B \int_\Gamma \frac{1}{(1-\overline{x}\phi(z))^\alpha} d\nu(x)\ d\mu(\phi) + f(0) \tag{2.7}$$

where the integral in (2.7) looks exactly like the one in (2.3) with α replacing β and hence using an argument similar to the one in (i) we will get that

$$f(z) = \int_B \int_\Gamma \frac{1}{(1-\psi(z))^\alpha}\, d\sigma(\psi) + f(0) \tag{2.8}$$

which shows that $f \in A_\alpha$. Furthermore $\|\sigma\| \leq \|\nu\|\|\mu\|$, and hence

$$\|f\|_{A_\alpha} \leq \inf\{\|\mu\|\} + |f(0)| = \|f\|_{A_0} \tag{2.9}$$

□

3. CHARACTERIZATION OF A_0

It is known from ([2], p248) that a function $\phi \in$ BMO if and only if there exists functions ϕ_1 and ϕ_2 in L^∞ such that

$$\phi = \phi_1 + \widetilde{\phi_2} + \alpha$$

where both $\|\phi_1\|_\infty$ and $\|\phi_2\|_\infty$ are less than $C\|\phi\|_\star$, C is a constant and $\|\ \|_\star$ is the classical BMO norm ([2], p248).

Consequently $f \in$ BMOA if and only if there are analytic functions f_1 and f_2 such that

$$f = f_1 + f_2 + \alpha \tag{3.1}$$

where $\|\mathrm{Re} f_1\|_\infty \leq C$ and $\|\mathrm{Im}\ f_2\|_\infty \leq C$.

If we define on BMOA the norm

$$\|f\|^\star = \inf\{\ \|\mathrm{Re} f_1\|_\infty + \|\mathrm{Im} f_2\|_\infty : f = f_1 + f_2 + \alpha\ \} \tag{3.2}$$

then by ([2], p248), the norms $\|f\|^\star$ and $\|f\|_\star$ are equivalent.

Now we have the following proposition which establishes a set equality between A_0 and BMOA.

Theorem 2. $A_0 = BMOA$

Proof. Suppose that $f \in A_0$, then according to (1.1) and (1.2) there exists a measure $\mu \in \mathcal{N}$ such that,

$$f(z) = \int_B \log \frac{1}{1-\phi(z)}\, d\mu(\phi) + f(0) \tag{3.3}$$

Assume without loss of generality, that μ is a probability measure. Then f is subordinate to $\log \dfrac{1}{1-z} + f(0)$ and consequently by (3.1), $f \in$ BMOA. The proof of the other inclusion follows from (3.1) and subordination.

Theorem 3. *The norms $\|.\|_\star$ and $\|.\|_{A_0}$ are equivalent, namely there exists positive constants c_1 and c_2 such that*

$$c_1\|f\|_\star \leq \|f\|_{A_0} \leq c_2\|f\|_\star \tag{3.4}$$

Proof. Suppose $f \in \mathrm{BMOA}$, then f can be decomposed as in (3.1). Let d_1 denote $\|\mathrm{Re} f_1\|_\infty$ and d_2 denote $\|\mathrm{Im}\ f_2\|_\infty$. Then

$$\left|\frac{\pi}{2d_1}\mathrm{Im} i f_1(z)\right| \leq \frac{\pi}{2} \tag{3.5}$$

and

$$\left|\frac{\pi}{2d_2}\mathrm{Im} f_2(z)\right| \leq \frac{\pi}{2} \tag{3.6}$$

for all $z \in \Delta$. Consequently, by subordination

$$i f_1(z) = \frac{2d_1}{\pi}\left(\log\frac{1}{1-\phi(z)} - \log\frac{1}{1+\phi(z)}\right) + i f_1(0) \tag{3.7}$$

$$f_2(z) = \frac{2d_2}{\pi}\left(\log\frac{1}{1-\psi(z)} - \log\frac{1}{1+\psi(z)}\right) + f_2(0) \tag{3.8}$$

for all $z \in \Delta$ and where $\phi, \psi \in B$. Therefore

$$\|f\|_{A_0} \leq \frac{4}{\pi}(d_1 + d_2), \tag{3.9}$$

and hence

$$\|f\|_{A_0} \leq \frac{4}{\pi}\|f\|^\star \leq c_2\|f\|_\star \tag{3.10}$$

which gives the right inequality in (3.3).

Next, we show the left inequality. Let us write f as in (1.2) and assume without loss of generality that μ is a positive measure. Then,

$$|\mathrm{Im} f(z)| \leq c\|\mu\| \tag{3.11}$$

where $c > 1$. Thus

$$|\mathrm{Im} f(z)| \leq c\|f\|_{A_0} \tag{3.12}$$

and since

$$\|\mathrm{Im} f\|_\infty \geq \|\mathrm{Im} f\|_\star \tag{3.13}$$

we have

$$\|f\|_\star \leq k_1\|\mathrm{Im} f\|_\star \leq k_2\|f\|_{A_0} = \frac{1}{c_1}\|f\|_{A_0}. \tag{3.14}$$

where $c_1 = \dfrac{1}{k_2}$ and the left inequality in (3.13) follows by ([2], p235) and this concludes the proof. □

Theorem 4. *$\|z^n\|_{A_\alpha} \leq k$ for $n \geq 1$*

Proof. It is enough to show that $\|z^n\|_{A_0} \leq k$ for $n \geq 1$. Since we showed that $\| \,.\, \|_\star$ and $\| \,.\, \|_{A_0}$ are equivalent, let us approximate $\|z^n\|_\star$. It is known from ([2], p240)that,

$$\|g\|_\star^2 \approx \sup_\psi \iint_\Delta |\bigtriangledown g|^2(1-|z|^2)|\psi'(z)|dA, \tag{3.15}$$

where $\psi(z) = \dfrac{z+z_0}{1+\overline{z_0}z}$ is a mobius transformation.
Replace g in (3.15) by z^n and $|z|$ by r to get,

$$\begin{aligned}
I &= \iint_\Delta |\bigtriangledown(z^n)|^2 \,(1-r^2)|\psi'(z)|\, dA \\
&\leq \iint_\Delta n^2r^{2n-2}(1-r^2)|\psi'(z)|dA \\
&\leq \int_0^{2\pi}\int_0^1 n^2r^{2n-2}(1-r^2)|\psi'(z)|drd\theta \\
&\leq 2\pi n^2 \int_0^1 r^{2n-2}(1-r)\, dr, \text{ because} \int_0^{2\pi} |\psi'(z)|d\theta \leq 2\pi \\
&\leq 2\pi n^2 \int_0^1 (r^{2n-2}-r^{2n})\, dr = \frac{4\pi n^2}{(4n^2-1)} \leq \frac{4\pi}{3}
\end{aligned} \tag{3.16}$$

which gives us the desired result and completes the proof. ☐

The following theorem is a direct consequence of Proposition 4.

Theorem 5. *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is analytic and if* $\sum_{n=1}^{\infty} |a_n| < \infty$ *then* $f \in A_\alpha$ *for all* $\alpha \geq 0$.

Proof. It is sufficient to prove that $f \in A_0$ since $A_0 \subset A_\alpha$. To show that $f \in A_0$ all we have to show is that the norm $\|f(z)\|_{A_0}$ is bounded.

$$\|f(z)\|_{A_0} = \left\|\sum_{n=0}^{\infty} a_n z^n\right\|_{A_0} \leq \sum_{n=0}^{\infty} |a_n|\, \|z^n\|_{A_0} \leq \frac{4\pi}{3}\, \sum_{n=0}^{\infty} |a_n| < \infty. \quad \square$$

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