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A UNIFYING SEMILOCAL CONVERGENCE THEOREM FOR NEWTON-LIKE METHODS IN BANACH SPACE

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Abstract. We provide a unifying semilocal convergence theorem for Newton-like methods, in order to approximate a solution of a nonlinear equation in Banach space. We use a certain positive functional as an upper bound on the distances involved. This way we can obtain sharper error bounds than before. Earlier results are special cases of our theorem. Several choices of the majorizing functional are also provided.

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1. INTRODUCTION.

In this study we are concerned with approximating a solution x^* of an equation

$$F(x) = 0, \tag{1}$$

where F is a continuous nonlinear operator defined on an open convex subset D of a Banach space E_1 with values in a Banach space E_2.

Newton-like methods of the form

$$x_{n+1} = x_n - A(x_n)^{-1} F(x_n) \quad (n \geq 0) \tag{2}$$

have been used by many authors to approximate x^*. For a survey of such results we refer the reader to [2], [3], [4]. Here, $A(x) \in L(E_1, E_2)$ $(x \in D)$ which denotes the space of all bounded linear operators from E_1 into E_2. For $A(x) = F'(x)$ $(x \in D)$ we obtain the famous Newton–Kantorovich method [3], [4], [5]. Several other choices for operator A are also possible [1], [2], [4].

Here, we introduce a majorizing functional and use it to provide a unifying semilocal convergence theorem for Newton-like method (2). Results already in

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the literature are considered special cases of our theorem if we simply choose the functional appropriately. Moreover, we show that in case this functional exists, then our error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ are sharper than earlier ones.

2. CONVERGENCE ANALYSIS.

We state and prove the following semilocal convergence theorem for Newton-like method (2).

Theorem. *Let $F\colon D \subseteq E_1 \to E_2$ be a continuous operator, defined on a convex subset D of a Banach space E_1 with values in a Banach space E_2.*

Assume

(a) *for some $x_0 \in D$ there exists $R \geq 0$ such that*

$$\bar{U}(x_0, R) = \{x \in E_1 \mid \|x - x_0\| \leq R\} \subseteq D;$$

(b) *$A(x)^{-1} \in L(E_2, E_1)$ for all $x \in U(x_0, R)$;*

(c) *there exists a functional $\alpha\colon \bar{U}(x_0, R) \to [0, +\infty)$ such that*

$$\|A(x)^{-1}F(x)\| \leq \alpha(x) \quad \text{for all } x \in \bar{U}(x_0, R); \tag{3}$$

(d) *iteration $\{t_n\}$ $(n \geq 0)$ given by*

$$\begin{aligned} t_{n+2} &= t_{n+1} + \alpha_{n+1}, \quad (n \geq 0),\ t_0 = 0, \\ t_1 &\geq \|A(x_0)^{-1}F(x_0)\|,\ \alpha_n = \alpha(t_n) \quad (n \geq 1) \end{aligned} \tag{4}$$

is monotonically increasing and bounded above by R.

Then

(i) *Iteration $\{t_n\}$ $(n \geq 0)$ generated by (4) converges to some limit $r_0 \in [0, R]$.*

(ii) *Newton-like iteration $\{x_n\}$ $(n \geq 0)$ generated by (2) is well defined, remains in $\bar{U}(x_0, r_0)$ for all $n \geq 0$ and converges to a solution x^* of equation $F(x) = 0$. Moreover, the following error bounds hold for all $n \geq 0$:*

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \tag{5}$$

and

$$\|x^* - x_n\| \leq r_0 - t_n. \tag{6}$$

Proof. (i) Scalar iteration $\{t_n\}$ $(n \geq 0)$ is monotonically increasing and bounded above by R (hypothesis (d)), and as such it converges to some limit $r_0 \in [0, R]$.

(ii) By hypothesis (b) Newton-like method (2) is well defined for all $n \geq 0$. We will show estimate (5) for all $n \geq 0$. For $n = 0$, (2), (4) give $\|x_1 - x_0\| \leq t_1 = t_1 - t_0$, which shows (5) for $n = 0$, and $x_1 \in \bar{U}(x_0, r_0)$. Let us assume that (5) holds and $x_{n+1} \in \bar{U}(x_0, r_0)$ for $n = 0, 1, 2, \ldots, k$. Using (2), (3) and (4) we get

$$\begin{aligned} \|x_{k+2} - x_0\| &\leq \|x_{k+2} - x_{k+1}\| + \|x_{k+1} - x_0\| \\ &\leq \cdots \leq (t_{k+2} - t_{k+1}) + (t_{k+1} - t_k) + \cdots + (t_1 - t_0) \\ &\leq t_{k+2} - t_0 = t_{k+2} \leq r_0, \end{aligned}$$

and

$$\|x_{k+2} - x_{k+1}\| = \| - A(x_{k+1})^{-1} F(x_{k+1})\| \le \alpha(x_{k+1}) = \alpha_{k+1} = t_{k+2} - t_{k+1},$$

which completes the induction.

It follows from estimate (5) that Newton-like method $\{x_n\}$ $(n \ge 0)$ is Cauchy in a Banach space E_1, and as such it converges to some limit $x^* \in \bar{U}(x_0, r_0)$ (since $\bar{U}(x_0, r_0)$ is a closed set). By letting $n \to \infty$ in (2) and using the continuity of $A(x)^{-1}$ and F we get $F(x^*) = 0$. That is x^* is a solution of equation (1).

Finally estimate (6) follows from (5) by using standard majorization techniques [3], [4], [5].

That completes the proof of the Theorem. ♠

Remark. The selection of the functional α is not easy in general. Here are some choices:

(1) An "optimum" choice is given by

$$\alpha(x) = \sup_{\substack{x \in \bar{U}(x_0, r) \\ r \in [0, R]}} \|A(x)^{-1} F(x)\|. \tag{7}$$

If such a functional can be found then since all upper bounds (see (3)) on the expression $\|A(x)^{-1}F(x)\|$ $(x \in D)$ already in the literature are "larger" than α, it follows that our error bounds on the distances are smaller than all earlier ones (see [1]–[5] and the references there).

(2) Let $\gamma_0 \colon \bar{U}(x_0, R) \to [0, +\infty)$ be a functional; γ be a nonnegative nondecreasing, continuous function defined on $[0, R]$ such that:

$$\|A(x_0)^{-1}(A(x) - A(x_0))\| \le \gamma(x) \le \gamma_0(\|x - x_0\|) \quad \text{for all } x \in \bar{U}(x_0, r) \subseteq \bar{U}(x_0, R) \tag{8}$$

and

$$\gamma_0(R) < 1; \tag{9}$$

β be a functional such that $\beta \colon \bar{U}(x_0, R) \to [0, +\infty)$; β_0 be a nonnegative, nondecreasing continuous function of three variables defined on $[0, R]^3$ such that:

$$\begin{aligned} \left\| \int_0^1 A(x_0)^{-1}[F'(x + t(y - x) - A(x)](y - x)dt \right\| &\le \beta(x) \\ &\le \beta_0(\|x - x_0\|, \|y - x_0\|, \|x - y\|) \end{aligned} \tag{10}$$

for all $x, y \in \bar{U}(x_0, r) \subseteq \bar{U}(x_0, R)$ and

$$y = x - A(x)^{-1} F(x). \tag{11}$$

It then follows from the estimate

$$\|A(x)^{-1} F(x)\| \le \|A(x)^{-1} A(x_0)\| \cdot \|A(x_0)^{-1} F(x)\|, \tag{12}$$

the Banach Lemma on invertible operators [4], [5], (3), (8)–(11) that functional α can be chosen by

$$\alpha(x) = \delta(\|x - x_0\|)\beta_0(\|x - x_0\|, \|x - x_0 - A(x)^{-1}F(x)\|, \|A(x)^{-1}F(x)\|), \tag{13}$$

where

$$\delta(t) = \frac{1}{1 - \gamma_0(t)}, \quad t \in [0, R], \tag{14}$$

and that $A(x)^{-1}$ is invertible for all $x \in \bar{U}(x_0, R)$. Note that functionals β, γ can be given by

$$\beta(x) = \sup_{\substack{x \in \bar{U}(x_0, r),\ r \in [0,R] \\ y = x - A(x)^{-1}F(x)}} \left\| \int_0^1 A(x_0)^{-1}[F'(x) + t(y-x)) - A(x)](y-x)dt \right\| \tag{15}$$

and

$$\gamma(x) = \sup_{\substack{x \in \bar{U}(x_0, r) \\ r \in [0,R]}} \|A(x_0)^{-1}(A(x) - A(x_0))\|. \tag{16}$$

It follows from (12), (13), and (14) that if α is chosen as in (7) then

(17)

$$\begin{aligned} \alpha(x) &\leq \beta(x)\frac{1}{1 - \gamma(x)} \\ &\leq \beta_0(\|x - x_0\|, \|x - x_0 - A(x)^{-1}F(x)\|, \|A(x)^{-1}F(x)\|)\delta(\|x - x_0\|), \end{aligned}$$

for all $x \in \bar{U}(x_0, r)$, $r \in [0, R]$. That is, error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ $(n \geq 0)$ with α chosen as in (7) are smaller or equal than the corresponding ones when α is chosen as in (13) or by

$$\alpha(x) = \beta(x)\frac{1}{1 - \gamma(x)},\ x \in \bar{U}(x_0, r),\ r \in [0, R] \tag{18}$$

and β, γ given by (15) and (16) respectively.

Our theorem can be reduced to any theorem on Newton methods already in the literature for special choices of the functional α. We will not show that for any such theorem in existence. However we will only show it under Newton–Kantorovich hypotheses [4], [5] for Newton's method. That is we set $A(x) = F'(x)$ $(x \in D)$ in (2). Indeed, let ℓ, b be such that

$$\|F'(x) - F'(y)\| \leq \ell\|x - y\| \quad \text{for all } x, y \in D \tag{19}$$

$$\|F'(x_0)^{-1}\| \leq b. \tag{20}$$

Then it can easily be seen that we can choose α as in (18) for

$$\gamma(x) = \ell b\|x - x_0\| \ \text{ and } \ \beta(x) = \frac{1}{2}\ell b\|F'(x)^{-1}F(x)\|^2 \ \ (x \in D). \tag{21}$$

Finally, for a list of choices for the functional α other than (7) and relevant ideas we refer the reader to [1], [2], [4].

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