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CAUSAL STRUCTURES IN LINEAR SPACES

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Abstract. Linear kinematic spaces (i.e linear topological spaces with partial ordering) are studied. They are defined by a set of 8 axioms implying that topology, linear structure and ordering are compatible with each other. The weakest topology of a linear kinematics is shown to be determined by the open cone of future of any its points. This topology allows decomposition of a linear kinematics in a Cartesian product of Euclidean, antidiscrete and discrete spaces. The structure of linear kinematics is proven to be preserved by factorization along the antidiscrete component of this decomposition. Linear topological spaces with pseudometrics which satisfy a time inequality instead of the triangle inequality are studied (3 axioms). A pseudometric (which is determined by a pseudonorm) is shown to define a topology on a linear space, it being a continuous mapping in this topology. For a space with a pseudometric to be a linear kinematics it is necessary and sufficient that the mapping of multiplication by -1 be continuous. A pseudometric topology of a linear kinematics is identical with the weakest topology of a linear kinematics with a given cone of future. The Minkovskii space of special relativity theory is proved to be a pseudometric linear kinematics.

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1.INTRODUCTION.

1.1. There are no doubts that the idea of causality is one of the main principles of physics. However the appropriate mathematical relation defined on some space typical for the relativity theory (e.g., Minkovskii linear space or a pseudo-Riemannian manifold) was studied only in the 1960s by H. Busemann [3] and R.I. Pimenov [5]. After this time only a few scientists paid attention to this mainly geometrical subject. However both Busemann and Pimenov missed important features of possible systems of axioms developed for description of the causal relationship. They do not discuss non-Euclidean topologies and they assumed that a pseudometric (which does not satisfy the triangle inequality) does not determine a topology on a given space. In this paper we begin our analysis of causal structures defined on spaces relevant for relativity theory. We discuss spaces which simultaneously possess three structures: the structure of linear space L over the field $\mathbb{R}$, topological structure (the system Δ of open sets) and partial ordering. In physics the latter is interpreted as causal structure. We are interested in properties which can be derived from the following system of axioms.

Definition. *Linear space L with topology Δ and partial ordering $<$ is called a linear kinematics if it satisfies the following three groups of axioms.*

1.2. First group of axioms.

$\mathbf{H}_1$. The mapping of addition $+ : L \times L \to L$ is continuous.

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$\mathbf{H}_2$. For any $\alpha\in\mathbb{R}$ the mapping of multiplication by α is continuous.

It can be easily deduced that for any $a \in L$ and for any open set U its translation $a+U$ is open, the translation of a closed set is closed, homothety $a\alpha(\alpha a)$ with non-zero coefficient α transfers an open set onto an open set and a closed set onto a closed set. The axiom H_2 is much weaker than the usual requirement that the mapping of multiplication as a whole (i.e.$\mathbb{R} \times L \to L$)be continuous. We shall not require this.

1.3. Second group of axioms.

$\hat{\mathbf{I}}_1$. Partial ordering $<$ is transitive: from $a<b$ and $b<c$ follows that $a<c$.

$\hat{\mathbf{I}}_2$. Partial ordering $<$ is strict: $a<a$ is impossible.

1.4. Third group of axioms.

$\mathbf{L\hat{E}}_1$. $\forall a,b \in L \qquad a<b \qquad \Rightarrow \qquad \forall u \in L \qquad a+u<b+u.$

$\mathbf{L\hat{E}}_2$. $\forall a,b \in L \qquad a<b \qquad \Rightarrow \qquad \forall \alpha \in \mathbb{R}_+ \qquad \alpha a<\alpha b$, where $\mathbb{R}_+ := (0,+\infty)$.

$\mathbf{L\hat{E}}_3$. $\forall a \in L \qquad \forall U \in \Delta \qquad a \in U \qquad \Rightarrow \qquad \exists b \in U \qquad a<b$, i.e. in any neighbourhood U of any point a there is at least one point which follows a (next point).

$\mathbf{L\hat{E}}_4$. The past of the zero point $Q_0^- := \{u \in L | u<0\}$ is open.

Axioms $H_1, H_2, L\hat{E}_1, L\hat{E}_2$, imply that the topology Δ, linear structure and ordering ($<$) are compatible with each other. Axiom $L\hat{E}_3$ implies that the relation of ordering is topologically non-empty. Axiom $L\hat{E}_4$ implies that all points which are close enough to some point in the past of the zero point also belong to the past of the zero point.

We do not imply that L has the Euclidean topology as it was assumed by Pimenov in [5]. It creates to problems. 1. To find which topologies are compatible with the given set of axioms. 2. To determine which structures introduced in [5] are preserved by this set of axioms.

2. COROLLARIES.

2.1. $L\hat{E}_3$ implies that the topology in L is not discrete. Otherwise there would be no next point in a one point neighborhood of a given point. $L\hat{E}_4$ implies that the topology in L is not antidiscrete.

2.2. $\forall a,b \in L \qquad a<b \qquad \Rightarrow \qquad \forall \alpha \in \mathbb{R}_- \qquad \alpha a>\alpha b, \qquad where \qquad \mathbb{R}_- = (-\infty,0).$
(We skip proofs in this part due to their simplicity).

2.3. $\forall a \in L \qquad \forall U \in \Delta \qquad a \in U \qquad \Rightarrow \qquad \exists c \in U \qquad c<a$, $i.\dot{a}.$ in any neighborhood U of any point a there is a least one preceding point.

2.4. For any point $a \in L$ its past $Q_a^- := \{u \in L | u<a\}$ and its future $Q_a^+ := \{u \in L | u>a\}$ are open sets.

2.5. $\forall a,b \in L \qquad a<b \qquad \Rightarrow \qquad Q_a^+ \supset Q_b^+, \qquad Q_a^- \subset Q_b^-$. It follows from the transitivity of order.

2.6. For any $a \in L$ the sets Q_a^+, Q_a^- are convex.

2.7. For any $a \in L$ the sets Q_a^+, Q_a^- are cones with apex a. It follows from the easily verified fact that Q_0^+ is a cone with apex 0. $\hat{I}_2$ implies that apexes do not belong to cones.

3. POSSIBLE TOPOLOGIES.

3.1. In this part we shall answer the question which topologies are compatible with the given system of axioms for finite-dimensional linear kinematics. Since convex cones Q_0^+ and $Q_0^- = -Q_0^+$ are open and the property of a set to be open is preserved by translations, one can easily find the weakest topology in the finite-dimensional linear space L, which is compatible with these two facts. It is obvious that the weakest topology in L is closely related with the type of the cone Q_0^+. Let us begin with the following note.

3.2. If aff $Q_0^+ \neq L$, then the space L can be decomposed to a Cartesian product of the subspaces $L^{'}=$ aff Q_0^+ and $M=\frac{L}{L^{'}}$. The ordering and the weakest topology in $L^{'}$ are defined by the cone Q_0^+, $L^{'}$ and all its translates are open, hence the topology in M is discrete. Therefore it is sufficient to find an answer to the question formulated in 3.1 for the case where aff $Q_0^+ = L$.

Let us first consider the spaces of small dimensions.

3.3. Let $L = \Re^1$. Simulate $\Re^1$ by the points of real axis $\mathbb{R}^1$ and let Q_0^+ be an open ray in the standard (Euclidean) topology of $\mathbb{R}^1$. Then the ray Q_0^- is also open. The intersections of their translates are also open, i.e. all segments $a < t < b$ are open. Therefore the weakest topology is L is Euclidean. It is compatible with the axioms of part 1.

3.4. Let $L = \Re^2$. Simulate $\Re^2$ by the points of a Cartesian plane $\mathbb{R}2$ (preserving linear structure). We shall consider 5 cases:

1. Q_0^+ is an open in $\mathbb{R}^2$ sharp (i.e. it does not contain straight lines) convex sector with angle $< \pi$.
2. Q_0^+ is an open in $\mathbb{R}^2$ convex sector containing straight lines, i.e. an open half-plane.
3. Q_0^+ is a sector like (2) with addition of one of its edge rays.
4. Q_0^+ is a sector like (1) with addition of one of its edge rays.
5. Q_0^+ is a sector like (1) with addition of both of its edge rays. In the case (2) the addition of both rays is impossible due to axiom $\hat{I}_2$.

3.4.1. In the case (1) the weakest topology is Euclidean. If we introduce coordinates (x, t) on the Cartesian plane $\mathbb{R}^2$ directing the t axis along the bisectrix of the cone Q_0^+, we shall see that it is a model of two dimensional space-time of the special relativity theory. All axioms of part 1 are fulfilled.

3.4.2. In the case (2) let us introduce Cartesian coordinates (x, t) on the plane $\mathbb{R}^2$ directing the x axis along the edge of the half-plane Q_0^+. The weakest topology can be defined if we choose all strips $a < t < b$ as the base of a topology. The topology is different from the Euclidiean, $L = \mathbb{R}^1 \times \mathbb{R}_a^1$, where $\mathbb{R}_a^1$ is the real axis with *antidiscrete* topology. It is a model of two-dimensional Halilean space-time. All axioms of part 1 are fulfilled.

3.4.3. In the case (3) let us introduce coordinates (x_1, x_2) on the plane $\mathbb{R}^2$ directing the x axis along the edge of the half-plane Q_0^+, its direction being the same with the direction of the ray which is included in $Q_0^+ = Q_{(0,0)}^+$, Fig.1. The ordering which is introduced by this cone corresponds to a strict lexicographical ordering: $(x_1, x_2) < (y_1, y_2) \Leftrightarrow (x_1 < y_1 \, or \, (x_1 = y_1 \, and \, x_2 < y_2))$. Since cones $Q_{(0,a)}^+$ and $Q_{(0,b)}^-$ are open, their intersection, i.e. segment $a < x_2 < b$ on the x_2 axis, is also open. Therefore the x_2 axis itself and all its translates (lines $x_1 = const$) are open sets. Hence the weakest topology or $\Re^2$ is the topology of a Cartesian product of the x_2 axis (with Euclidean topology) and the x_1 axis with *discrete* topology, $\Re^2 = \mathbb{R}^1 \times \mathbb{R}_d^1$. All axioms of part 1 are fulfilled.

3.4.4. In the case (4), the ordering is not lexicographic since it is just partial ordering. But for the same reason as in the case (3), all lines $x_1 = const$ are open and $\Re^2$ has a topology of a Cartesian product of the x_2 axis (with Euclidean topology) and the x_1 axis with the *discrete* topology. All axioms of part 1 are fulfilled.

3.4.5. In the case (5) let us introduce oblique coordinates on the plane $\mathbb{R}^2$ directing the x, y axes along two edge rays of the cone Q_0^+. For the same reason as in 3.4.3, each line parallel to any of the axes should be an open set. Therefore their intersection (point) is also open, and the topology in $\Re^2$ is discrete. It is not compatible with $I\hat{E}_3$. Hence the case (5) for a linear kinematics is impossible.

3.5. For any finite dimension n let us simulate $L = \Re^n$ by the points of the Euclidean pace $\mathbb{R}^n$ (preserving linear structure) and consider a case when the cone Q_0^+ is open not only in $\Re^n$ but also in $\mathbb{R}^n$. An open convex cone $Q_0^+ \subset \mathbb{R}^n$ which does not contain its apex and therefore is not equal to the whole $\mathbb{R}^n$ can be represented as $Q_0^+ = \tilde{N} \times \mathbb{R}^k$, where C is a sharp cone (i.e., it does not contain straight lines) and $O \leq k < n$. Therefore the entire space can be represented as a Cartesian product of linear spaces $\Re^n = \Re^{n-k} \times \Re^k$. A closed sharp cone C is open in the Euclidean space $\mathbb{R}^{n-k}$ and hence, the weakest topology introduced by C in $\Re^{n-k}$ is Euclidean. Therefore all $(n - k) - dimensional$ open balls B can be taken as a base of the topological space $\Re^{n-k}$. In this case cylindrical tube-like neighborhoods $B \times \Re^k$ of the subspace $\Re^k$ and all their translates will be the base of topology of the space $\Re^n$. Topological space $\Re^n$ will be represented as $\Re^n = \mathbb{R}^{n-k} \times \mathbb{R}_a^k$, where $\mathbb{R}_a^k$ is a k-dimensional linear space with *anti-discrete* topology. All axioms of part 1 are fulfilled. Yet it should be noted that in this part we discuss only the weakest topology compatible with this system of axioms.

3.6. Let the frontier ∂Q_0^+ (after simulation of $\Re^n$ by $\mathbb{R}^n$) contain at least one ray. There is a supporting plane P for ∂Q_0^+ which contains this ray. Translating n-dimensional cones Q_0^+ and Q_0^-, which are open in $\Re^n$, one can see that both closed (in Euclidean topology) half-spaces separated by P are open. Therefore the hyperplane P itself and all its translations are open. Hence $\Re^n$ is represented as a Cartesian product of a line $\mathbb{R}_d$ with the discrete topology and a hyperplane P, the topology in P being determined by the cone $Q_0^+ \cap P$. Repeating with $P = \Re^{n-1}$ the same process and lowering dimension we shall arrive at the case 3.5, because a completely discrete topology for a linear kinematics is impossible.

Hence the following theorem is proved.

3.7 Theorem 1. *Let in a finite-dimensional linear kinematics $(L, \Delta, <)$ the dimension of the cone Q_0^+ be equal to dim L, and let Δ be the weakest topology of a linear kinematics. Then L is a Cartesian product of three linear topological spaces, $L = L' \times A \times D, L'$ being a space with Euclidean topology, A with anti-discrete and D with discrete topology. If "the cone of future" Q_0^+ contains straight lines or a ray passing within its frontier (frontier in a sense of the Euclidean topology in L), then this decomposition is a non-trivial one, in the first case being $dim A > 0$ and in the second case $dim D > 0$.*

3.8. The discrete component in this representation is a good illustration of the concept of "parallel worlds" [4]. Let for example $dim\, D > 0$. Then there is no path (continuous mapping, segment) connecting points from different fibers (x_1, d_1) and (x_2, d_2), $d_1 \neq d_2$. At the same conditions the causal relation is common for all fibers, i.e. causal link between "worlds "(fibers of space-time) is preserved.

3.9. If dim $Q_0^+ < dim\, L$ then in the representation $L = L' \times A \times D$ there will be another factor D' (as well as D with discrete topology), which did not preserve causal link between its fibers.

3.10. R.I. Pimenov did not consider (finite dimensional) linear spaces with non-Euclidean topology, but he needed properties which in our system of axioms belong to spaces with Euclidean topology and to spaces with anti-discrete layer of maximal possible dimension (co-dimension 1). Appropriate causal relations are interpreted as Einstein causality (i.e. relativity theory) and Newtonian causality (preselected time axis), correspondingly. We can see that there are also other types of causality which are not equivalent to these two because they determine different topologies. Since Pimenov considered only Euclidean topology, to describe these two types of causality he had to introduce very extensive characteristics which cannot be generalized for intermediate cases.

4. PREORDERING AND SPACE OF FIBERS OF ABSOLUTE SIMULTANEITY.

4.1 Definition. *We introduce preordering $\leq$ defining it by the condition*

$$\forall a, b \in L \qquad a \leq b \qquad \Leftrightarrow \qquad a \in C1\, Q_b^-$$

Relation $\leq$ is a relation of preordering, i.e. it is reflexive and transitive. Let us point out first the properties of the cones Q_a^+, Q_a^-.

4.1.1. $\forall a \in L \qquad Q_a^+ = a + Q_0^+, \qquad Q_a^- = a + Q_0^-.$

4.1.2. $\forall \alpha \in \mathbb{R}_+ \qquad \alpha Q_0^+ = Q_0^+, \qquad \alpha Q_0^- = Q_0-, \qquad \alpha C1 Q_0^+ = C1 Q_0^+ \qquad \alpha C1 Q_0^- = C1 Q_0^-$ (self-similarity).

Therefore the relation $\leq$ is stable to addition and multiplication on scalars from $\mathbb{R}_+$.

4.1.3. $\forall a, \qquad b \in L \qquad a \leq b \qquad \Rightarrow \qquad \forall u \in L \qquad a + u \leq b + u.$

4.1.4. $\forall a, \qquad b \in L \qquad a \leq b \qquad \Rightarrow \qquad \forall \alpha \in \mathbb{R}_+ \qquad \alpha a \leq \alpha b.$

Reflexivity of $\leq$ follows from 2.3

4.1.5. $\forall a \in L \qquad a \leq a.$

In advance of the proof of transitivity let us point out two useful properties.

4.1.6. $\forall a, b, c \in L \qquad a < b \qquad \& \qquad b \leq c \qquad \Rightarrow \qquad a < c;$
$\forall a, b, c \in L \qquad a \leq b \qquad \& \qquad b < c \qquad \Rightarrow \qquad a < c.$

4.1.7. $\forall a, b, c \in L \qquad a \leq b \qquad \& \qquad b \leq c \qquad \Rightarrow \qquad a \leq c$ (transitivity).

Proof. Let $a \in C1Q_b^-, b \in C1Q_c^-, U$ be any neighbourhood of a point a. Then there is $u \in U \cap Q_b^- \neq \emptyset$, and due to (6) $u < b \leq c \Rightarrow u < cu \in Q_c^- \Rightarrow U \cap Q_b^- \subset Q_c^- \Rightarrow U \cap Q_c^- \neq \emptyset$, i.e. any neighbourhood of a point a intersects with Q_c^-. Therefore $a \in C1Q_c^-$, i.e. $a \leq c$.

4.1.8.

For any $a \in L$ the sets $C1Q_a^+, C1Q_a^-$ are convex cones with apex a (apx belongs to the cone due to (5)). (Proof is quite analogous to the proofs of 2.6 and 2.7 with substitution of $<$ with $\leq$).

4.1.9.

$\forall a \in L \qquad IntC1Q_a^+ = Q_a^+, \qquad IntC1Q_a^- = Q_a^-.$

Proof. Let us check as an example the first statement. It is obvious that $IntC1Q_a^+ \supset Q_a^+$. Prove that no point $u \in C1Q_a^+$, $u \notin Q_a^+$ can belong to $IntC1Q_a^+$. Any point u with these properties should belong to ∂Q_a^+. Due to $L\hat{E}_3$ in each neighbourhood U of point u there is $v < u$. If $v \in C1Q_a^+$, then $a \leq v < u$, and that makes a contradiction with $u \notin Q_a^+$. Therefore in any neighbourhood of the point u there are points which do not belong to $C1Q_a^+$, i.e. $u \notin IntC1Q_a^+$.

4.2 Definition. *The preordering $\leq$ defines the relation of equivalence õn L:*

$$\forall a, b \in L \qquad a \sim b \qquad \Leftrightarrow \qquad a \leq b \qquad \& \qquad b \leq a.$$

In physics this relation is called *absolute simultaneity*. It is symmetric by definition, and reflexive and transitive due to 4.1.5 and 4.1.7.

4.3 Definition. *Class of points equivalent to a given point c is designated as $\tilde{c} := \{b \in L | c \sim b\}$. This set can be also called fiber of simultaneity of the point c.*

We would like to point out connection of this term with the results of part 3. Let L be a Cartesian product of the three linear topological spaces, $L = L' \times A \times D$ with Euclidean, antidescrete and discrete topology, respectively. Then the class of points equivalent to $a = (a_{L'}, a_A, a_D)$ is $\tilde{a} = a_{L'} \times A \times a_D$, i.e. if the cone Q_0^+ contains straight lines then the fiber of simultaneity of the point a consists exactly of the sum of all translates of these lines which pass through the point a. Indeed, since in definition 4.1 there is a closure operation, the entire set A (or more exactly, its translate) is contained in $\tilde{a}$. On the other hand, the fiber of simultaneity cannot be bigger because the cone $Q_0^+ \cap L'$ is sharp.

4.3.1. For any $a \in L$ the sets $Q_a^+, Q_a^-, C1Q_a^+, C1Q_a^-$ consists of complete classes of equivalence.

4.3.2. Class $\tilde{0}$ is a linear subspace in L. (Under conditions of Theorem 1 $\tilde{0}$ is identical with A).

4.3.3. For any $a \in L\, \tilde{a} = a + \tilde{0}$.

4.3.4. For any $a \in L$ the set $\tilde{a}$ is closed ($\tilde{a} = C1\, Q_a^+ \cap C1\, Q_a^-$).

4.3.5. For any $a \in L\, Int\, \tilde{a} = \emptyset$. (It follows from 4.3.4 and 4.1.9). (Under conditions of Theorem 1 it means that $\dim L' \geq 1$).

4.4 Space of fibers of absolute simultaneity.

Definition. *The set of classes of equivalence $\sim$ we designate as $\tilde{L} := L/$.*

Since all classes of equivalence are the translates of the subspace $\tilde{0}$ (4.3.2, 4.3.3), the set $\tilde{L}$ inherits from L its linear structure, L being a Cartesian product $L \approx \tilde{L} \times \tilde{0}$.

$\tilde{L}$ as factor-space inherits from L topological structure (Δ). Let us designate $\pi : L \to \tilde{L}$ -factorization mapping (projection). Then $A \subset \tilde{L}$ is open in $\tilde{L}$ iff $\pi^{-1}(A)$ is open in L.

Due to Theorem 1 of part 3.7 the following statement is obvious: *$\tilde{L}$ is a Hausdorff space.* Indeed factor-space is Hausdorff even in the weakest topology of a finite dimensional linear kinematics. However this statement is true also for infinite dimensional linear kinematics. The proof is relatively simple and we shall not discuss it here.

4.4.1 Lemma. *$\pi : L \to \tilde{L}$ is an open mapping, i.e. if $A \subset L$ is open, then $\pi(A) \subset \tilde{L}$ is also open.*

Proof. We must add to A all classes of equivalence for all element from A and prove that thus extended set $B := \pi^{-1}(\pi(A))$ is open.

Let $b \in B$, then $\exists a \in A$ so that $a \sim b$ and there is an open set U so that $a \in U \subset A$. Consider an open (due to axiom $\acute{I}_1$) set $V := (b-a) + U$, $b \in V$. We should prove that $V \subset B$. Indeed for any $v \in V\, \exists u \in U\, v = b - a + u$. Since $a \sim b \Rightarrow a - b \sim 0 \Rightarrow v \sim u$ and $u \in A$, then $v \in B$. Hence, any point $b \in B$ belongs to B together with its neighbourhood, i.e. B is open.

4.4.2.. We had already noted that equivalence $\sim$ is stable to addition of vectors and multiplication by scalars. Then ordering $<$ is also stable to this equivalence. It immediately follows from 4.1.6.

1. $\forall a, b, c \in L \qquad a < b \,\&\, b \sim c \Rightarrow a < c.$

2.. $\forall a, b, c \in L \qquad a \sim b \,\&\, b < c \Rightarrow a < c.$ Therefore factor-space $\tilde{L}$ inherits ordering from $L(<)$.

Definition. Let $\tilde{a}, \tilde{b} \in \tilde{L}$, then $\tilde{a} \tilde{<} \tilde{b} \Leftrightarrow a < b$.

4.4.3. Theorem 2. *Linear space $\tilde{L}$ with ordering $\tilde{<}$ and factor-topology $\tilde{\Delta}$ is a linear kinematics.*

Proof. We must verify that all axioms of a linear kinematics are fulfilled.

The continuity of addition follows from the commutativity of the diagram

$$\begin{array}{ccc} L \times L & \xrightarrow{+} & L \\ {\scriptstyle (\pi,\pi)}\downarrow & & {\scriptstyle \pi}\downarrow \\ \tilde{L} \times \tilde{L} & \xrightarrow{+} & \tilde{L} \end{array} \qquad \forall a, b \in L \pi(a) + \pi(b) = \pi(a+b)$$

Let $A \subset \tilde{L}$ be open. Then $\pi^{-1}(A) \subset L$ is also open. Addition in L is continuous, therefore the pre-image B of a set $\pi^{-1}(A)$ at the mapping $+ : L \times L \to L$ is open in $L \times L$. $\pi : L \to \tilde{L}$ is an open mapping, therefore $(\pi, \pi) : L \times L \to \tilde{L} \times \tilde{L}$ is also an open mapping. Then image $C := (\pi, \pi)(B)$ of an open set B is open in $\tilde{L} \times \tilde{L}$. Since the above diagram is commutative, the set C is identical with the pre-image of A at mapping $+ : \tilde{L} \times \tilde{L} \to \tilde{L}$. Hence this pre-image is open and addition in factor-space $\tilde{L}$ is continuous.

Let us check the axiom $\acute{I}_2$: for any $\alpha \in \mathbb{R}$ the mapping $\alpha. : \tilde{L} \to \tilde{L} : \tilde{a}\alpha\, \alpha\tilde{a}$ is continuous. If $\alpha = 0$, then this is a constant mapping and therefore continuous. Let $\alpha \neq 0$. Corresponding diagram is commutative, since $\pi o \alpha. = \alpha o \pi$. The same reasons as above prove our statement.

$\grave{\mathbf{I}}_2$. $\forall a, b \in F\, a > 0 \,\&\, b > 0 \Rightarrow f(a+b) \geq f(a) + f(b)$ *(Einstein inequality, or time inequality.)*

$\grave{\mathbf{I}}_3$. $\forall a \in F\, \forall \alpha \in \mathbb{R}\, f(\alpha a) = \alpha f(a)$ *(uniformness).*

Since Q_0^+ is a convex cone, $\grave{I}_2$ is correct ($a \in Q_0^+ \& b \in Q_0^+ \Rightarrow a + b \in Q_0^+$). $\grave{I}_3$ is correct because F is a cone symmetrical with respect to the zero point. Besides $f(0) = f(0x) = 0f(x) = 0$ since the cone Q_0^+ is not empty.

Each convex cone defines a relation of partial ordering on L : for any $a, b \in L$

$$a < b \Leftrightarrow b - a \in Q_0^+.$$

The relation $<$ is transitive ($\forall a, b, c \in L\, a < b \,\&\, b < c \Rightarrow b - a \in Q_0^+ \,\&\, c - b \in Q_0^+ \Rightarrow c - a \in Q_0^+ \Rightarrow a < c$) and anti-reflexive ($0 \notin Q_0^+ \Rightarrow \forall a \in L \neg(a < a)$). Thus the axioms $\hat{I}_1$ and $\hat{I}_2$ of part 1 are fulfilled.

Because f is strictly positive only on Q_0^+, one can give another definition of partial ordering which is equivalent to the first:

$$\forall a, b \in L \qquad a < b \qquad \Leftrightarrow \qquad b - a \in F \& f(b-a) > 0.$$

Under this definition transitivity of ordering $<$ follows from Einstein inequality, anti-reflexivity - from the fact that $f(0) = 0$. Axiom $\grave{I}_1$ can be formulated in a pure function form:

$\grave{\mathbf{I}}_1'$. $\forall a \in F\, f(a) > 0 \Rightarrow \forall b \in L\, \exists \alpha \in \mathbb{R}_+\, a + \alpha b \in F \,\&\, f(a + \alpha b) > 0.$

Definition. *Designate* $W := \{(a,b)|a,b \in L, b-a \in F\} \subset L \times L$. *We shall call function* $\rho : W \to \mathbb{R}, \rho(x,y) := f(x-y)$ *the time like metric (or pseudometric) in* W. (In physics pseudometric has a sense of oriented distance between two events separated by the time like interval).

Proofs of other axioms are trivial.

4.4.4. It should be noted that fibers of absolute simultaneity appear in linear kinematics due to very different physical reasons. Thus in galilean kinematics $\mathbb{R}^3 \times \{t\}$, where the cone of future is the half-space $t > 0$, the fiber of absolute simultaneity is $\mathbb{R}^3$. In kinematics $\mathbb{R}^4 \times \mathbb{R}$ (Minkovskii space with the fiber $\mathbb{R}$ over each point used in [5, p. 399] to simulate electromagnetism) the fiber of simultaneity is $\mathbb{R}$.

5. METRIC LINEAR KINEMATICS.

5.1. Definitions. In physics a definition of ordering using some function (for example, which is positive if its argument is 'greater' than the zero point) would be more convenient than an abstract definition used in part 1. Since ordering must be transitive, this function should satisfy time inequality (triangle inequality with opposite sign). We shall build a linear kinematics where both ordering and topology are defined by such a function. Most of its properties will prove to be determined by the domain of this function.

Definition. *Let* L *be a linear space over the field* $\mathbb{R}, Q_0^+ -$ *convex cone in* L *with apex* $0, 0 \notin Q_0^+$. *Designate* $Q_o^- := (-1)Q_0^+, F := Q_0^+ \cup Q_0^- \cup \{0\}$. *We shall consider only non-empty cones with the following property:*

$\grave{\mathbf{I}}_1$. $\forall a \in Q_0^+ \, \forall b \in L \, \exists \alpha \in \mathbb{R}_+ \; a + \alpha b \in Q_0^+$

Function $f : F \to \mathbb{R}$ *which is strictly positive on* Q_0^+ *is called a time-like norm (or a pseudonorm) if it satisfies the following conditions:*

Einstein inequality for the time like metric has the form

$\grave{\mathbf{I}}_2$. $\forall a,b,c \in L \; a > b \,\&\, b > c \Rightarrow \rho(a,c) \geq \rho(a,b) + \rho(b,c)$

Busemann ([3], pp. 5,7) wrote that a pseudometric which satisfies the Einstein inequality did not define a topology on a space L. Pimenov ([5], p.257) also assumed it. We shall prove that with the additional axioms $\grave{I}_1$ and M_3 pseudometric defines a topology on the space L, and this topology is quite different from the Euclidean one. We also shall find an additional condition which is sufficient and necessary for a linear space L with pseudonorm f to be a linear kinematics.

5.2. Corollaries.

5.2.1. $\forall a \in Q_0^+ \, \forall v \in L \, \forall \alpha \in \mathbb{R}_+ \qquad a + \alpha b \in Q_0^+ \Rightarrow (\forall \beta \in \mathbb{R}_+ \qquad \beta < \alpha \Rightarrow a + \beta b \in Q_0^+)$.

It immediately follows from the convexity of the cone Q_0^+. This corollary can be formulated also in a function form. Let $a \in F, b \in L, f(a) > 0, \alpha \in \mathbb{R}_+, a + b\alpha \in F, f(a + \alpha b) > 0$. Then $\forall \beta \in \mathbb{R}_+ \, \beta < \alpha \Rightarrow a + \beta b \in F \& f(a + \beta b) > 0$.

5.2.2. $\forall a \in F \, \forall \varepsilon \in \mathbb{R}_+ \; f(a) > \varepsilon \Rightarrow \forall b \in L \, \exists \alpha \in \mathbb{R}_+ \; a + \alpha b \in F \& f(a + \alpha b) > \varepsilon$.

Proof. Let us take $\delta > 0$ and represent a as $a = a - \delta a + \delta a$. Due to $\grave{I}_1$ there is $\beta \in \mathbb{R}_+$ so that $f(\delta a + \beta b) > 0$. Therefore, $f(a + \beta b) \geq f(a - \delta a) + f(\delta a + \beta b) > (1-\delta) f(a) > (1-\delta)\varepsilon$. The left part of this inequality does not depend on δ. If we take $sup_{\delta \in \mathbb{R}_+}$ we get $f(a + \beta b) \geq \varepsilon$. Then for any $\alpha \in \mathbb{R}_+ \alpha < \beta \Rightarrow f(a + \alpha b) = f(\frac{\alpha}{\beta}(\frac{\beta}{\alpha} a + \beta b)) = \frac{\alpha}{\beta} f(a + \beta b + (\frac{\beta}{\alpha} - 1)a) \geq \frac{\alpha}{\beta}(f(a + \beta b) + (\frac{\beta}{\alpha} - 1) f(a)) \geq \frac{\alpha}{\beta}\varepsilon + (1 - \frac{\alpha}{\beta}) f(a) > \varepsilon$, since at $\alpha = \beta$ the latter statement is an equality, and in the left part of it there is a linear function of α, which increases when α decreases.

5.2.3. Let $a \in F, b \in L, \varepsilon \in \mathbb{R}_+, f(a) > \varepsilon, \alpha \in \mathbb{R}_+, a + \alpha b \in F, f(a + \alpha b) > \varepsilon$. Then for any $\beta \in \mathbb{R}_+ \beta < \alpha \Rightarrow a + \beta b \in F \& f(a + \beta b) > \varepsilon$. (This was actually proved in the proof of the previous statement).

5.3. Definition of topology by means of pseudometric..

5.3.1 Definition. *The set* $B_\varepsilon(a) := \{x \in L | x - a \in F \& \rho(x,a) > \varepsilon\}$ *will be called extra ball with radius* ε *and center* a.

It is more natural to search a topology with the base of extra balls which are convex sets (obvious) than to consider balls which are not convex for a space with pseudometrics.

Theorem 3. *The set of extra balls* $\Gamma := \{B_3(a) | a \in L, \varepsilon \in \mathbb{R}_+\}$ *is a base of a topology on* L.

Proof. 1. Extra balls from Γ cover all L. Let us take an arbitrary point $x \in L$ and vector $u \in Q_0^-$. It is obvious that $x + u < x$ and if we designate $f(u) =: -2\varepsilon, \varepsilon \in \mathbb{R}_+$, then $x \in B_\varepsilon(x + u)$.

2. Intersection of two sets from Γ can be represented as a sum of sets from Γ. Let $c \in B_\varepsilon(a) \cap B_\delta(b)$. We must check that $\exists h \in L\, \exists \alpha \in \mathbb{R}_+$ so that $c \in B_\alpha(h) \subset B_\varepsilon(a) \cap B_delta(b)$. Let us take $u \in Q_0^-$. Due to 5.2.2 there is $r_1 \in \mathbb{R}_+$ and $r_2 \in \mathbb{R}_+$ so that $f(c + r_1 u - a) > \varepsilon$ and $f(c + r_2 u - b) > \delta$. Assigning $r := min\{r_1, r_2\}, h := c + ru, f(h - c) =: -2\alpha, \alpha \in \mathbb{R}_+$, we have that $h \in B_\varepsilon(a) \cap B_\delta(b), h < c$ and $c \in B_\alpha(h)$. If $x \in B_\alpha(h)$, then $f(x - a) \geq f(x - h) + f(h - a) > \varepsilon$ and $f(x - b) \geq f(x - h) + f(h - b) > \delta$, therefore $B_\alpha(h) \subset B_\varepsilon(a) \cap B_\delta(b)$.

Definition. *The topology on the space* L *with the base* Γ *is called the time like topology determined by the pseudonorm* f.

5.3.2. It is well known that in a finite dimensional linear space all norms determine the same topology. We shall prove (without using the condition of finite dimension) that *the time like topology determined by the pseudonorm* f *actually depends only on the domain of* f. Let us designate $Q_a^+ := a + Q_0^+, \Gamma' := \{Q_a^+ | a \in L\}$.

Lemma 1. *The set of cones* Γ' *is a base of a topology on* L.

Proof is quite analogous to the proof of Theorem 3.

Lemma 2. *In topology with the base* Γ *all cones from* Γ' *are open.*

Proof. This statement immediately follows from the equality $Q_0^+ = Y_{\varepsilon \in \mathbb{R}_+} B_\varepsilon(0)$ and the fact that the mapping of addition is continuous under this set of axioms $(M_1 - M_3)$ (see the proof of Theorem 5 in 5.4.2).

Lemma 3. *In topology with the base* Γ' *all extra balls from* Γ *are open.*

Proof. We must check that for any $x \in B_\varepsilon(a)$ there is $y \in L$ so that $x \in Q_y^+ \subset B_\varepsilon(a)$. It is sufficient to take $u \in Q_0^-$ and choose (using 5.2.2) $\alpha \in \mathbb{R}_+$ so that $x + \alpha u \in B_\varepsilon(a)$. It is obvious that $y = x + \alpha u$.

Hence the following theorem was proved.

Theorem 4. *Topologies with the base* Γ *and* Γ' *are the same.*

5.3.3. This topology is quite different from the standard topology of a metric space. For example, it cannot be Hausdorff. If $a < b$ and $a \in B_r(x)$, then $b \in B_r(x)$. $(a < b\, \&\, a \in B_r(x) \Rightarrow f(b - x) \geq f(b - a) + f(a - x) > r \Rightarrow b \in B_r(x))$. Also, multiplication by -1 in this topology is not continuous. This is related with the fact that all sets from Γ are unlimited towards the direction of "time increasing "Q_0^+ and limited on the other side.

5.4. Strengthening of topology and construction of a linear kinematics. To build a linear kinematics the increasing of the time like topology is necessary.

5.4.1. Lemma.. *The set* $\Delta := \{U \cap (-V) \,|\, U, V \in \Gamma\}$ *is a base of a topology on* L*, this topology being stronger than the time like topology.*

Proof of this lemma is a quite direct one (using the fact that Γ is a base of a topology on L).

Example. Let us take $\mathbb{R}$ as a linear space and $f = id$ as a pseudonorm. Then the time like topology on $\mathbb{R}$ is a topology with the base of rays $\{(x, +\infty) \,|\, x \in \mathbb{R}\}$, and the topology with the base Δ is Euclidian.

Definition. *The topology on* L *with the base* Δ *is called the topology determined by the pseudonorm* f.

5.4.2. Theorem 5. *Linear space* L *with the topology determined by the pseudonorm* f *and with the ordering determined by the same pseudonorm is a linear kinematics.*

Proof. Axioms $\hat{I}_1, \hat{I}_2$ have been already checked in definition of partial ordering on L (5.1). Let us check the axiom $\acute{I}_1$. We designate the sets from the base Δ as $B_{r,q}(a, b) := B_r(a) \cap -B_q(b)$, where $a, b \in L, r, q \in \mathbb{R}_+$.

Lemma. $\forall a,b,c,d \in L \forall r,q,p,t \in \mathbb{R}\, B_{r,q}(a,b) + B_{p,t}(c,d) \subset B_{r+p,q+t}(a+c,b+d)$.

Proof is quite direct and uses only the time inequality.

Continue the proof of the theorem 5. Let a set $U \subset L$ be open. We must verify that its pre-image at mapping $+ : L \times L \to L$ is open in $L \times L$, i.e. for any $(x,y) \in L \times L\, x+y \in U \Rightarrow \exists V,W \subset L-$ open sets so that $x \in V, y \in W, V+W \subset U$. One can assume that $U,V,W \in \Delta$. Let $U = B_{\alpha,\beta}(u,v), e \in Q_0^+$ and $\gamma \in \mathbb{R}_+$. Considering $\rho(x, x-\gamma e) = f(\gamma e) = \gamma f(e) > 0$ let us select $\gamma_1 \in \mathbb{R}_+$ so that $f(\gamma_1 e) < \alpha$ and let $\gamma < \gamma_1$. Then $x \in B_r(x-\gamma e)$, where $f(\gamma_1 e) =: 2r < 2\alpha$. Consider $\rho(y, u-(x-\gamma e)) = f(x+y-u-\gamma e)$, where according to conditions $f(x+y-u) > \alpha$. Due to $\grave{I}_1$ there is $\gamma_2 \in \mathbb{R}_+$ so that $f(x+y-u-\gamma_2 e) > \alpha$. Let us select $\gamma = min\{\gamma_1, \gamma_2\}$ and assign $a := x-\gamma e$. Then $x \in B_r(a)$ and $y \in B_{\alpha-_r}(c)$, where $c := u-a$.

Analogous, there are $q < \beta$ and $b \in L$ so that $x \in -B_q(b)$ and $y \in -B_{\beta-g}(d)$, where $d := v-b$. Now the inclusion $B_{r,q}(a,b) + B_{\alpha-r,\delta-q}(c,d) \subset B_{\alpha,\beta}(u,v)$ is ensured by the lemma.

Let us check the axiom $\acute{I}_2$. Since multiplication on a scalar $\alpha \in \mathbb{R}$ with $\alpha \neq 0$ is a one-to-one mapping, the continuity of multiplication is equivalent to the following proposition:*if $U \subset L$ is open, then for any $\alpha \in \mathbb{R} (\alpha \neq 0)\, \alpha U \subset L$ is also open.* It is enough to check this statement for the sets from Δ only. Let $U := B_r(a) \cap -B_q(b)$. If $\alpha \in \mathbb{R}_+$, then $\alpha B_r(a) = B_{\alpha_r}(\alpha a)$. If $\alpha \in \mathbb{R}_-$, then $\alpha B_r(a) = -B_{|\alpha|_r}(|\alpha| a)$. These sets are open.

Stability of ordering $<$ to addition $(axiomL\hat{E}_1)$ follows directly from definition. Stability of ordering to multiplication (LK_2) follows from the axiom $\grave{I}_3$ and definition of topology determined by the pseudonorm f.

Let us check the axiom $L\hat{E}_3$: in any neighbourhood of any point there is a next point. It is sufficient to check this statement for the sets from Δ. Let $a \in B_{r,q}(u,v)$ and $b \in Q_0^+$. Then for any $\alpha \in \mathbb{R}_+\; a+\alpha b > a$. Statement 5.2.2 implies that $\exists \alpha_1 \in \mathbb{R}_+\; \rho(-a-\alpha_1 b, v) > q$. Then $a+\alpha_1 b \in B_{r,q}(u,v)$. This is the "next point belonging to the same neighbourhood.

Let us check the axiom $L\hat{E}_4$. We know (see the proof of Lemma 2 in 5.3.2) that Q_0^+ is open. By definition of the base Δ, multiplication on -1 is continuous, therefore $Q_0^- = -Q_0^+$ is also open.

5.4.3. Thus we proved that for a space L with pseudonorm f to be a linear kinematics it is necessary and sufficient that the mapping of multiplication by -1 be continuous. In physics it corresponds to the requirement that the reversion of time is an allowed operation.

Corollary. *Let L be a linear space, $f : F \to \mathbb{R}$ - pseudonorm in $F, F := Q_0^+ \cup Q_0^- \cup \{0\}$. Then the topology in L determined by the pseudonorm f is identical with the weakest topology of the linear kinematics L with the cone of future Q_0^+.* (It immediately follows from 5.4.2 and 5.3.2).

5.5 The continuity of the pseudonorm.

Theorem 6. *Function f is continuous.*

Proof. It is sufficient to consider only the pre-images of rays $(\varepsilon, +\infty)$ and $(-\infty, \varepsilon)$ (where $\varepsilon \in \mathbb{R}$) and prove that they are open in F. If $\varepsilon \in \mathbb{R}_+$, then $f^{-1}((\varepsilon,+\infty)) = B_\varepsilon(0)$. If $\varepsilon \in \mathbb{R}_-$, then $f^{-1}((-\infty,\varepsilon)) = -B_{|\varepsilon|}(0)$. If the ray contains the zero point, then $f^{-1}((-\infty,\varepsilon)) \cap F = F \setminus C1B_\varepsilon(0)$ for $\varepsilon \in \mathbb{R}_+$ and $f^{-1}((\varepsilon,+\infty))$ $\cap F = F \setminus C1 - B_{|\varepsilon|}(0)$ for $\varepsilon \in \mathbb{R}_-$. Therefore these pre-images are open in F.

The space of fibers of absolute simultaneity.

Definition. *Linear kinematics $(L, \Delta, <)$ is called metric linear kinematics, if there is a pseudonorm f : $F \to \mathbb{R}$ so that the topology Δ is determined by f and ordering $<$ is compatible with f(i.e. $\{x \in L | x > 0\} = \{x \in L | f(x) > 0\}$).*

5.6.1. Theorem 7. *Let (L,f) be a metric linear kinematics. Then factorspace $\tilde{L}$ is also a metric linear kinematics, and there is a topological immersion $\tilde{L} \to L$. For any topological immersion $\phi : \tilde{L} \to L$ there is a homomorphism $L \cong \phi(\tilde{L}) \times \tilde{0}$ where $\tilde{0} \subset L$ has antidiscrete and $\phi(\tilde{L})$ has a Hausdorff topology.*

Proof. Let (L,f) be a metric linear kinematics, $\tilde{L}$ - its factor-kinematics. Prove that $\tilde{L}$ is also a metric kinematics with a pseudonorm defined by the following rule. Let us designate factorization mapping as $\pi : L \to \tilde{L}, \tilde{F} := \pi(F)$. For any $\tilde{a} \in \tilde{F}$ we define $\tilde{f}(\tilde{a}) := f(a)$.

This definition is correct: if $a \sim b$, then $f(a) = f(b)$. Let on the contrary $a \sim b$ and $f(a) \neq f(b)$. One can assume that $f(a) > f(b) > 0$ ($f(a)$ and $f(b)$ cannot have different signs because ordering $<$ is stable to equivalence $\sim$). There is an $\varepsilon \in \mathbb{R}_+$ so that $f(a) > \varepsilon > f(b)$. Therefore there is $\alpha \in \mathbb{R}_+$ so that $f(\alpha a) = \varepsilon$. Hence $f(a) > f(\alpha a) \Rightarrow (1-\alpha)f(a) > 0 \Rightarrow f(a - \alpha a) > 0$. Yet $a \sim b \Rightarrow a - \alpha a \sim b - \alpha a \Rightarrow f(b - \alpha a) > 0$. Therefore $f(b) = f(b - \alpha a + \alpha a) \geq f(b - \alpha a) + f(\alpha a > \varepsilon$ - contradiction with assumption $f(b) < \varepsilon$.

Let us continue the proof of the theorem 7. We must check that f is a pseudonorm. Designate $\tilde{Q}_0^+ := \pi(Q_0^+), \tilde{Q}_0^- := \pi(Q_0^-)$. Projection π is a linear mapping (see 4.3.2. and 4.3.3.), therefore $\tilde{Q}_0^+$ is a convex cone in L with apex $\tilde{0}, \tilde{0} \notin \tilde{Q}_0^+$.

Let us check $\grave{I}_1$. Let $\tilde{a} \in \tilde{Q}_0^+, \tilde{b} \in \tilde{L}$. Then for any representatives in these classes $\exists \alpha \in \mathbb{R}_+\ a + \alpha b \in Q_0^+$. Projecting it into $\tilde{L}$ we have $\tilde{a} + \alpha\tilde{b} \in \tilde{Q}_0^+$.

Let us check $\grave{I}_2$. If $a > 0$ and $b > 0$, then for any representatives in these classes $a > 0, b > 0$ and $f(a+b) \geq f(a) + f(b)$. Since $f = \tilde{f}0\pi$ and π is a linear mapping, we have time inequality in factor-space. Axiom $\grave{I}_3$ can be verified in the same way. Hence the function $\tilde{f}$ is a pseudonorm.

It is obvious that $\tilde{f}$ is strictly positive only on $\tilde{Q}_0^+$, therefore ordering $\tilde{<}$ is compatible with $\tilde{f}$.

Pseudonorm $\tilde{f}$ defines a topology on the factor-space $\tilde{L}$ which is identical with the factor-topology $\frac{L}{\sim}$, because bases of the two topologies match:$\tilde{B}_\varepsilon(\tilde{a}) \cap -\tilde{B}_\delta(\tilde{b})\ == \pi(B_\varepsilon(a) \cap -B_\delta(b))$. Hence $(\tilde{L}, \tilde{\Delta}, \tilde{<})$ is a metric linear kinematics.

To construct an immersion $\phi : \tilde{L} \to L$ we must fix some choice of representatives $\tilde{a}\alpha\, a =: \phi(\tilde{a})$. It is an injection. Continuity of ϕ follows from the equality $\phi^{-1}(B_\varepsilon(a) \cap -B_\delta(b)) = \tilde{B}_\varepsilon(\tilde{a}) \cap -\tilde{B}_\delta(\tilde{b})$. Let us consider the abbreviation of $\phi, ab\,\phi : \tilde{L} \to\ im\phi$. The inverse mapping $(ab\,\phi)^1 : im\,\phi \to \tilde{L}$ is continuous, because $(ab\,\phi)^{-1} = \pi|im\,\phi$. Therefore ϕ is an immersion.

Now let $\phi : \tilde{L} \to L$ be an arbitrary topological immersion. Then homomorphism $\psi : L \to \phi(\tilde{L}) \times \tilde{0}$ can be defined as $\psi(a) := (\phi(\tilde{a}), a - \phi(\tilde{a})), a \in L$. The base of topology in L is identical with the base of topology in $\phi(\tilde{L}) \times \tilde{0}$. Also there is a homomorphism $L \to \tilde{L} \times \tilde{0}$. It can be defined by $\tilde{a}\alpha(\tilde{a}, a - \phi(\tilde{a}))$.

Due to 4.3.1 all open (and closed) sets of the topology of a metric kinematics consist of complete classes of equivalence. In particular on the subset $\tilde{0} \subset L$ the topology is anti-discrete. Indeed this set is closed (4.3.4) and consists of the only class of equivalence. Therefore it does not contain non-trivial closed subsets. Theorem 7 has been proved.

5.6.2. Corollary. *Let (L, f) be a finite dimensional metric linear kinematics. Then L is a Cartesian product of two linear spaces, $L + E \times A$, where A has anti-discrete and E Euclidean topology.*

Proof. Due to Theorem 7 and 4.4 $L = H \times A$, where H, A are linear spaces with Housedorff (H) and anti-discrete ($A = 0$) topology. Due to Theorem 1 and 5.4.3 $L = E \times D \times A$, where E, D, A are linear spaces with Euclidean, discrete and anti-discrete topology, respectively. Therefore $H = E \times D$. But due to $\grave{I}_1$ discrete space cannot be presented in this decomposition, i.e. $dimD = 0$. Let on the contrary $dimD > 0$. Then either there is a set from the base Δ so that aff $(B_\varepsilon(a) \cap -B_\delta(b)) \neq L$, or the cone Q_0^+ contains rays belonging to ∂Q_0^+ (where the frontier is taken in Euclidean topology of the linear space L). In the second case contradiction with M_1 is obvious. In the first case there are points $x \in B_{\varepsilon,\delta}(a,b)$, $y \in L \setminus \text{aff}B_{\varepsilon,\delta}(a,b)$. Due to 5.2.2 and 5.2.3, $\exists \gamma \in \mathbb{R}_+\ x + \gamma(y - x) \in B_{\varepsilon,\delta}(a,b)$, but it is impossible because $y \notin \text{aff}B_{\varepsilon,\delta}(a,b)$.

Thus factor-topology of a finite dimensional metric linear kinematics is always Euclidean. Euclidean spaces with partial ordering were also studied by A.D. Alexandrov [1, 2].

5.6.3. The case of "parallel worlds" can be partially described in terms of metric linear kinematics if the condition of axiom $\grave{I}_1$ is changed:

$\grave{\mathbf{I}}_4$. $\forall a \in Q_0^+ \quad \forall v \in F \quad \exists \alpha \in \mathbb{R}_+ \quad a + \alpha b \in Q_0^+$, or in a functional form.

$\grave{\mathbf{I}}_4$. $\forall a \in F \quad f(a) > 0 \Rightarrow \forall b \in F \quad \exists \alpha \in \mathbb{R}_+ \quad a + \alpha b \in F \& f(a + \alpha b) > 0$. In this statement condition "$\forall b \in L$"is replaced by the weaker one "$\forall b \in F$". Corollaries from 5.2 can be easily reformulated considering this change (replacing "$b \in L$"with "$b \in F$"). Corollary from 5.6.2 will be changed:

Proposition. *Let (L, f) be a finite dimensional metric linear kinematics, satisfying axioms $\grave{I}_4, \grave{I}_2, \grave{I}_3$ Then L is a Cartesian product of three linear spaces, $L = E \times D \times A$ with Euclidean, discrete and anti-discrete topology, respectively.*

It should be noted that layers $E \times A$ are not linked by the relation of ordering, as it is in 3.9.

5.6.4. Example.. The most well-known example of a metric linear kinematics is a Minkovskii space. Let us consider the space $L = \mathbb{R}^4$ and the function $f(x, y, z, t) = sgn(t)\sqrt{t^2 - x^2 - y^2 - z^2}\,(sgn(0) := 0)$. Designate $F := \mathrm{Supp} f \cup \{0\}$. It is a cone and function f is a time-like norm (the cone Q_0^+ is a set where f is positive, axioms $\grave{I}_{1-3}$ are fulfilled). Function f determines in L a Euclidean topology.

5.7. Question.. It is obvious that under the conditions of axioms $M_1 - M_3$ the unction f is concave on both Q_0^+ and Q_0^-. Therefore the following question arises: *let f be a strictly positive concave function on Q_0^+ defined on a cone $F = Q_0^+ \cup Q_0^- \cup \{0\}$ where Q_0^+ satisfies the axiom M_1. Is it true that f determines a base of a topology in a linear space L?*

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A FACTORED FORM OF COMPLEX ANALYTIC FUNCTIONS AND ITS APPLICATION TO FINDING PARTIAL FRACTION DECOMPOSITIONS

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Abstract. A fast algorithm is introduced to find the coefficients of the partial fraction decomposition of proper rational functions. Instead of solving a large-scale system of linear algebraic equations with full coefficient matrix or differentiating repeatedly rational functions, a special system of linear algebraic equations is solved, where the coefficient matrix is lower triangular. Therefore, there is no need for elimination, the solutions are obtained by forward substitution making the procedure very effective. The method is based on a special representation of analytic functions which has further applications in constructing iteration methods for solving nonlinear equations.

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1. INTRODUCTION.

Special function representations have been shown to be very useful in all kinds of applications. This paper introduces a new factored form of analytic complex functions, which can be applied to derive a fast algorithm to find partial fraction decompositions for rational functions with high multiplicity poles as well as in constructing iteration schemes for solving nonlinear equations. Integrating rational functions and solving time-invariant linear systems by the transformation method require the determination of the partial fraction decomposition of proper rational functions ([2],[3],[4]). In addition, several computer methods for solving polynomial equations are based on manipulations of rational functions ([1]). Partial fraction decomposition make the application of such methods easier to apply.

In the literature two methods are prevalent for performing partial fraction decompositions. In the case of first method, known as the algebraic method, a large system of linear equations is solved, which can be very difficult, unless the coefficient matrix has a special structure. In the second method, called the residue method, repeated derivatives of rational functions have to be determined, which is sometimes very complicated. The algorithm presented in this paper also requires the solution of a large system of linear algebraic equations; however, the coefficient matrix is of lower-triangular type and therefore there is no need for elimination, the solution can be easily obtained by forward substitution.

This paper develops as follows. In section 2 a unique representation theorem will be proved for analytic functions. This will be the basis for determining the partial fraction coefficients. The detailed algorithm will be presented in section 3, and section 4 concludes the paper with an illustrative example.

2. A REPRESENTATION THEOREM.

Let $n \geq 1$ be an integer, and assume that functions P and Q are analytic on an open complex domain D. Let $\eta \in D$ be a given point and assume that for $k = 0, 1, \dots, n-1, Q^{(k)}(\eta) \neq 0$.

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Theorem 1. *There exists a unique sequence $a_0, A_1, \ldots, A_{n-1}$ of complex numbers and a unique analytic complex function P_1 such that*

$$P(s) = Q(s) \cdot [A_0 + A_1(s-\eta) + \cdots + A_{n-1}(s-\eta)^{n-1}] + P_1(s) \cdot (s-\eta)^n. \tag{1}$$

Proof. Define $R(s) = A_0 + A_1(s-\eta) + \cdots + A_{n-1}(s-\eta)^{n-1}$, and assume that decomposition (1) exists. For $k = 0, 1, \ldots, n-1$, differentiate both sides of equation (1)$k-$ times and substitute $s = \eta$ to have

$$\begin{aligned} P^{(k)}(\eta) &= \sum_{i=0}^{k} \binom{k}{i} Q^{(k-i)}(\eta) \cdot \frac{d^i}{ds^i}[A_0 + A_1(s-\eta) + \cdots + A_{n-1}(s-\eta)^{n-1}]_{s=\eta} \\ &+ \sum_{i=0}^{k} \binom{k}{i} P_1^{k-i}(\eta) \cdot \frac{d^i}{ds^i}[(s-\eta)^n]_{s=\eta} \\ &= \sum_{i=0}^{k} \binom{k}{i} Q^{k-i}(\eta) \cdot A_i \cdot i!. \end{aligned}$$

Introducing the quantities $t_{ki} = \frac{k!}{(k-i)!} Q^{(k-i)}(\eta)$ the above equations can be rewritten as a system of linear algebraic equations

$$\underline{p} = \underline{T} \cdot \underline{a} \tag{2}$$

where

$$\underline{T} = \begin{bmatrix} t_{00} & 0 & \ldots & 0 \\ t_{10} & t_{11} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ t_{n-10} & t_{n-11} & \ldots & t_{n-1n-1} \end{bmatrix}, \underline{p} = \begin{bmatrix} P(\eta) \\ P^{(1)}(\eta) \\ \vdots \\ P^{(n-1)}(\eta) \end{bmatrix}$$

and

$$\underline{a} = \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{n-1} \end{bmatrix}$$

The assumption $Q^{(k)}(\eta) \neq 0 \text{for} k = 0.1. \ldots, n-1$ quarantees that the diagonal elements of $\underline{T}$ are nonzero, therefore system (2) has a unique solution which can be obtained by simple forward substitution.

For a given pair P, Q of analytic functions consider the coefficients $A_0, A_1, \ldots, A_{n-1}$ obtained by solving system (2). The construction of these constants implies that for $k = 0, 1, \ldots, n-1$

$$P^{(k)}(\eta) = \frac{d^k}{ds^k}[Q(s) \cdot (A_0 + A_1(s-\eta) + \cdots + A_{n-1}(s-\eta)^{n-1})]_{s=\eta}$$

Therefore function $P(s) - Q(s) \cdot R(s)$ is also analytic and η is its root with multiplicity at least n. Hence, with some analytic function P_1,

$$P(s) - Q(s) \cdot R(s) = P_1(s) \cdot (s-\eta)^n, \tag{3}$$

which completes the proof.♠

Remark 1. From equation (2) we have

$$A_0 = \frac{P(\eta)}{t_{00}}, \tag{4}$$

and for $k = 0, 1, \ldots, n-1$

$$A_k = \frac{1}{t_{kk}} \left(P^{(k)}(\eta) - \sum_{i=0}^{k-1} t_{ki} \cdot A_i \right). \tag{5}$$

In addition, from equation (3) we conclude that

$$p_1(s) = \frac{P(s) - Q(s) \cdot R(s)}{(s-\eta)^n}. \tag{6}$$

Remark 2.. Assume now that P and Q are polynomials of degree p and q, respectively. Then the degree of $P - Q \cdot R$ is at most $m = max\{p, q+n-1\}$, and η is its root with multiplicity of at least n. Therefore P_1 is also a polynomial with degree at most $m - n$. Furthermore, if we introduce the notation $f_n = P - Q \cdot R$, then

$$P_1(s) = \sum_{k=0}^{p_1} \frac{f_n^{(n+k)}(\eta)}{(n+k)}! \cdot (s-\eta)^k, \tag{7}$$

where p_1 is the degree of P_1.

Notice that in Theorem 1, functions P, Q, and P_1 do not need to be polynomials. Representation (1) has many applications, and one of them will be elaborated in the next section. Other applications in constructing Newton- like methods and other iteration schemes of fast convergence rates will be discussed in a future paper.

3. PARTIAL FRACTION DECOMPOSITION.

Any proper rational function can be written as

$$\frac{P(s)}{(s-\eta)^n Q(s)} = \frac{B_0}{(s-\eta)} + \frac{B_1}{(s-\eta)^2} + \cdots + \frac{B_{n-1}}{(s-\eta)^n} + \frac{P_1(s)}{Q(s)} \tag{8}$$

where $Q(\eta) \neq 0$ and P_1 is a polynomial of degree less than that of Q. Multiply both sides of equation (8) by the common denominator to have

$$P(s) = Q(s) \cdot \left(A_0 + A_1(s-\eta) + \cdots + A_{n-1}(s-\eta)^{n-1} \right) + P_1(s) \cdot (s-\eta)^n,$$

which is exactly the same as (1) with $A_i = B_{n-l-i} (i = 0, 1, \ldots, n-1)$. If function (8) is proper rational, then $p < n + q$. Hence, the algorithm of the previous section is applicable to find $A_0, A_1, \ldots, A_{n-1}$ and P_1. The algorithm has to be then repeated for a pole of the decompostion remainder rational function $\frac{P_1(s)}{Q(s)}$, and so on, until all poles of the original rational function have been exhausted.

4. AN ILLUSTRATIVE EXAMPLE.

Consider the rational function

$$\frac{4s^8 - 33s^7 + 101s^6 - 146s^5 + 94s4 + 94s^3 - 178s^2 - 166s + 66}{(s-3)^4 \cdot (s^5 + 2s^3 + 4s^2 + 2s + 3)}$$

where we have

$$P(s) = 4s^8 - 33s^7 + 101s^6 - 146s^5 + 94s^3 - 178s^2 - 166s + 66,$$
$$Q(s) = s^5 + 2s^3 + 4s^2 + 2s + 3,$$

$\eta = 3, n = 4, q = 5$ and $p = 8$. Simple differentiation shows that $Q^{(0)}(3) = 342, Q^{(1)}(3) = 485, Q^{(2)}(3) = 584, Q^{(3)}(3) = 552$, and $P^{(0)}(3) = 342, P^{(1)}(3) = 1169, P^{(2)}(3) = 4576, P^{(3)}(3) = 20994$. Therefore

$$\underline{T} = \begin{pmatrix} 342 & o & o & o \\ 485 & 342 & 0 & 0 \\ 584 & 970 & 684 & 0 \\ 552 & 1752 & 2910 & 2052 \end{pmatrix}, and \underline{p} = \begin{bmatrix} 342 \\ 1169 \\ 4576 \\ 20994 \end{bmatrix}$$

which implies that the solution of equation (2) is given by

$$\underline{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

so $A_0 = 1, A_1 = 2, A_2 = 3$, and $A_3 = 4$. Hence,

$$R(s) = 1 + 2(s-3) + 3(s-3)^2 + 4(s-3)^3 = 4s^3 - 33s^2 + 92s - 86,$$

and therefore

$$f_4(s) = P(s) - Q(s) \cdot R(s) = s^6 - 10s^5 - 48s^3 + 81s^2 - 270s + 324.$$

The degree of P_1 is $6 - 4 = 2$, and simple differentiation shows that

$$f_4^{(4)}(3) = 456, f_4^{(5)}(3) = 960, f_4^{(6)}(3) = 720.$$

From equation (7) we conclude that

$$\begin{aligned} P_1(s) &= \frac{456}{4!}(s-3)^0 + \frac{960}{5!}(s-3)^1 + \frac{720}{6!}(s-3)^2 \\ &= 19 + 8(s-3) + (s-3)^2 = s^2 + 2s + 4, \end{aligned}$$

that is, the partial fraction decomposition (8) is as follows:

$$\frac{4}{(s-3)} + \frac{3}{(s-3)^2} + \frac{2}{(s-3)^3} + \frac{1}{(s-3)^4} + \frac{s^2+2s+4}{s^5+2s^3+4s^2+2s+3},$$

since $B_0 = A_3 = 4, B_1 = A_2 = 3, B_2 = A_1 = 2$, and $B_3 = A_0 = 1$.

We finally mention that a subsequent paper we will provide more numerical examples and applications to validate our results and compare them to those of earlier methods.

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RELAXING CONVERGENCE CONDITIONS FOR THE JARRATT METHOD

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Abstract. We establish a convergence theorem for the Jarratt method by using recurrence relations. The purpose of this note is to relax convergence conditions and give an example where our convergence theorem can be applied but not other ones.

A.M.S. (MOS) Subject Classification Codes. 65J15, 47H17.

Key Words and Phrases. Jarrat method, nonlinear equations in Banach spaces fourth-order method, recurrence relations.

1. INTRODUCTION.

Although multipoint iteration functions are not much used in practice, one interesting class of this type of functions, studied by Traub [4], is very efficient when the equation to be solved is such that the evaluation of the first derivative is rapid compared with the function. An example of this occurs when the function is defined by an integral.

In particular, Argyros, Chen and Qian [1] studied the Jarrat method, a two-point iterative process of order four and defined, for all $n \geq 0$, by

$$\begin{gathered} y_n = x_n - [F'(x_n)]^{-1}F(X_n) = x_n - \Gamma_n F(x_n), \\ z_n = x_n + \frac{2}{3}(y_n - x_n) = x_n - \frac{2}{3}\Gamma_n F(x_n) \\ H(x_n, y_n) = \Gamma_n[F'(z_n) - F'(x_n)], \\ x_{n+1} = y_n - \frac{3}{4}[I + \frac{3}{2}H(x_n, y_n)]^{-1}H(x_n, y_n)(y_n - x_n) \end{gathered} \tag{1}$$

where F is a nonlinear operator defined in an open convex domain Ω of a Banach space X with values in another Banach space Y. It has been assumed that F has a first order continuous Fréchet derivative on $\Omega, F'(x_0)^{-1}$ exists for $x_0 \in \Omega$.

These authors give in [1] a Kantorovich-like convergence theorem, in which the conditions are

$$\begin{gathered} \| F''(x) \| \leq L_1, \| F''(x) - F''(y) \| \leq L_2 \| x - y \|, x, y \in \Omega \\ \| \int_0^1 \left[F''[x + t(y-x)](1-t) - \frac{1}{2}F''[x + \frac{2}{3}t(y-x)] \right] dt \| \leq L_3 \| x - y \|^2 . \end{gathered} \tag{2}$$

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In this note, we give another convergence theorem for operator equations. In order to provide the convergence of (1), it is only assumed

$$\| F'(x) - F'(y) \| \leq K \| x - y \|, x, y, \in \Omega, \tag{3}$$

instead of (2). Observe that we can apply the method (1) under the same condition (3) than for the Newton method (see[5]). Finally, an example where the conditions (2) fail but not (3) is presented.

Denote

$$\overline{B(x,r)} = \{y \in X; \| y - x \| \leq r\} \qquad \text{and} \qquad B(x,r) = \{y \in X; \| y - x \| < r\}.$$

2. A CONVERGENCE THEOREM.

Theorem 2.1. *Let F be a nonlinear once Fréchet-differentiable operator in an open convex domain Ω of a Banach space X with values in another Banach space Y. Let $x_0 \in \Omega$ and suppose that $\Gamma_0 = F'(x_0)^{-1}$ exists. Let us assume*

$$\| \Gamma_0 \| \leq \alpha, \| y_0 - x_0 \| \leq \beta, \| F'(x) - F'(y) \| \leq K \| x - y \|, x, y \in \Omega.$$

If $a_0 = \alpha\beta K \leq s = 0.292246\ldots$ (s is the smallest positive root of polynomial $q(x) = 2x^4 - 17x^3 + 48x^2 - 40x + 8$) and $\overline{B(x_0, r)} \subset \Omega$, where

$$r = \left(1 + \frac{a_0}{2(1-a_0)}\right)\frac{1}{1-M_0} \| y_0 - x_0 \|,$$

$$M_0 = \frac{a_0(8 - 8a_0 + a_0^2)}{8(1-b_0)(1-a_0)^2} \qquad \text{and} \qquad b_0 = a_0\left(1 + \frac{a_0}{2(1-a_0)}\right),$$

then the sequence $\{x_n\}$ given by (1) is well-defined, $x_n, y_n \in B(x_0, r)$, for all $n \geq 0$, and converges at least R-quadratically to a solution $x^* \in B(x_0, r)$ of the equation $F(x) = 0$. Moreover the solution x^* is unique in $B(x_0, \frac{2}{\alpha K} - r) \cap \Omega$.

Proof. Let us define, for all $n \geq 1$, the real sequences

$$a_0 = K\alpha\beta, b_n = a_n\left(1 + \frac{a_n}{2(1-a_n}\right), M_n = \frac{a_n(8 - 8a_n + a_n^2)}{8(1-n_n)(1-a_n)^2},$$

$$a_n = \frac{a_{n-1}^2(8 - 8a_{n-1} + a_{n-1}^2)}{8(1-b_{n-1})^2(1-an-1)^2} = a_{n-1}\frac{M_{n-1}}{1-b_{n-1}}.$$

Under the hypotheses mentioned above $\{a_n\}$ and $\{b_n\}$ are decreasing sequences, besides

$$\| \frac{3}{2}H(x_0, y_0) \| \leq K \| \Gamma_0 \| \| y_0 - x_0 \| \leq a_0 < 1.$$

From the last conditions, the existence of x_1 is guaranteed and

$$\| x - 1 - x_0 \| \leq \left(1 + \frac{a_0}{2(1-a_-)}\right) \| y_0 - x_0 \| .$$

Applying mathematical induction on n we can prove the following statements for all $n \geq 1$

$$(I_n) \qquad \begin{array}{l} a_{n+1} \leq \gamma^{2^n} a_n < \gamma^{2^{n+1}-1} a_0 \qquad \text{where} \qquad \gamma = a_1/a_0, b_{n+1} \leq \gamma^{2^n} b_n < \gamma^{2^{n+1}-1} b_0 \\ \text{and} M_{n+1} \leq \gamma^{2^{n+1}-1} M_0, \end{array}$$

$$(II_n) \qquad \| \Gamma_n F'(x_{n-1}) \| \leq \frac{1}{1-b_{n-1}},$$

$$(III_n) \qquad \| \frac{3}{2} H(x_n, y_n) \| \leq K \| \Gamma_n \| \| y_n - x_n \| \leq a_n,$$

$$(IV_n) \qquad \| y_n - x_n \| \leq M_{n-1} \| y_{n-1} \| y_{n-1} - x_{n-1} \|,$$

$$(V_n) \qquad \| x - n + 1 - x_n \| \leq \left(1 + \frac{a_n}{2(1-a_n)}\right) \| y_n - x_n \|,$$

Is easy to prove that the function $g(x,y) = \frac{x(x^2-8x+8)}{8(1-y)(1-x)^2}$ is increasing in its first and second arguments for $x, y \in (0,1)$. Moreover $g(\gamma x, \gamma y) \leq \gamma g(x,y)$, then it follows that $M_1 \leq \gamma M_0$. Thus we have

$$\begin{aligned}
a_2 = a_1 \frac{M_1}{1-b_1} \leq \gamma a_0 \frac{\gamma M_0}{1-b_0} &= \gamma^2 a_1 < \gamma^3 a_0, \\
b_2 = a_2 \left(1 + \frac{a_2}{2(1-a_2)}\right) \leq \gamma^2 a_1 \left(1 + \frac{a_1}{2(1-a_1)}\right) &= \gamma^2 b_1 < \gamma^3 b_0, \\
M_2 = g(a_2, b_2) = g(\gamma^2 a_1, \gamma^2 b_1) < \gamma^2 g(a_1, b_1) &= \gamma^2 M_1 < \gamma^3 M_0.
\end{aligned}$$

This prove $[I_1]$. On the other hand Γ_1 exists, and $[II_1]$ is true by the Banach lemma. Taking into account the Taylor's formula

$$F(x_1) = -\frac{3}{4}\left[I + \frac{3}{2}H(x_0, y_0)\right]^{-1} H(x_0, y_0)(y_0 - x_0) + \int_{x_0}^{x_1} [F'(x) - F'(x_0)] dx.$$

So, it follows $[III_1]$,$[IV_1]$,$[IV_1]$ and $[V_1]$ from (1). Furthermore $x_1, y_1, z_1 \in \overline{B(X_0, r)}$. As the sequences $\{a_n\}$ and $\{b_n\}$ are decreasing, by induction we obtain $[I_{n+1}], [III_{n+1}], [IV_{n+1}]$ and $[V_{n+1}]$.

Besides,

$$\begin{aligned}
||x_{n+1} - x_n|| &\leq \left(1 + \frac{a_0}{2(1-a_0)}\right) M_{n-1} ||y_{n-1} - x_{n-1}|| \\
&\leq \cdots \leq \left(1 + \frac{a_0}{2(1-a_0)}\right) \prod_{i=0}^{n-1} M_j ||y_0 - x_0|| \\
&\leq \left(1 + \frac{a_0}{2(1-a_0)}\right) \prod_{j=0}^{n-1} \gamma^{2^j - 1} M_0 ||y_0 - x_0|| = \left(1 + \frac{a_0}{2(1-a_0)}\right) \gamma^{2^n} \frac{M_0^n}{\gamma^{n+1}},
\end{aligned}$$

and $\{x_n\}$, given by (1), is a Cauchy sequence, since $M_0 < 1$ and $\gamma < 1$. As a result, the sequence $\{x_n\}$ converges to $x^* \in \overline{B(x_0, r)}$. We now deduce that $F(x^*) = 0$ from the continuity of F and taking into account that $||y_n - x_n|| \leq M^n ||y_0 - x_0|| \to 0$ when $n \to \infty$.

On the other hand, as

$$
\begin{aligned}
||x_{n+m} - x_n|| &\le ||x_{n+m} - x_{n+m-1}|| + \cdots + ||x_{n+1} - x_n|| \\
&\le \left(1 + \frac{a_0}{2(1-a_0)}\right)\left[\gamma^{2^{n+m-1}}\frac{M_0^{n+m-1}}{\gamma^{n+m}} + \cdots + \gamma^{2^n}\frac{M_0^n}{\gamma^{n+1}}\right]||y_0 - x_0|| \\
&\le \left(1 + \frac{a_0}{2(1-a_0)}\right)\gamma^{2^n-1}\left[\left(\frac{M_0}{\gamma}\right)^{n+m-1} + \cdots + \left(\frac{M_0}{\gamma}\right)^n\right]||y_0 - x_0|| \\
&\le \left(1 + \frac{a_0}{2(1-a_0)}\right)\frac{(1-b_0)^n[1-(1-b_0)^m]}{b_{0\gamma}}\gamma^{2^n}||y_0 - x_0||,
\end{aligned}
$$

we have

$$||x^* - x_n|| \le \left(1 + \frac{a_0}{2(1-a_0)}\right)\frac{(1-b_0)^n}{b_0\gamma}\beta\gamma^{2^n}$$

therefore this process converges at least R-quadratically [3].

Finally, to show uniqueness, let us assume that $y^* \in B(x_0, \frac{2}{aK} - r) \cap \Omega$ is another solution of $F(x) = 0$. Following the technique given by Argyros and Chen in [2]

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^*))dt(y^* - x^*).$$

3. EXAMPLE.

We present an example where the assumptions (2), fail but the conditions of Theorem 2.1 are fulfilled.

Let us consider the system of equations $F(x, y) = 0$ where $F : (-\frac{1}{2}, \frac{3}{2}) \times (-\frac{1}{2}, \frac{3}{2}) \to \mathbb{R}^2$ such that

$$F(x, y) = (x^3 lnx^2 + 4y - 1/8, x(y-4)).$$

If we choose $x_0 = (0, 0)$, the F does not satisfy the Lipschitz condition for F" given in (2).

On the other hand, we can apply Theorem 2.1, since

$$\alpha = ||\Gamma_0|| = 1/4, b = ||y_0 - x_0|| = 1/32, K = 22.30,$$

and consequently, $a_0 = K\alpha\beta = 0.1742187 \le s = 0.292246\ldots$. As a result, we can only study convergence for this system of equations under the hypotheses of Theorem 2.1.

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COINCIDENCE POINTS OF COMPATIBLE MULTIVALUED MAPPINGS

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ABSTRACT. In this paper we establish a coincidence theorem for a generalized contractive type multivalued mappings in complete metric spaces generalizing some results of Constantin [1]. Two examples are given to support our generalization.

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Key words and phrases. Coincidence points, compatible mappings, complete metric spaces.

1. INTRODUCTION

In [2,3,4,5,9,11,12,13,15,16], the authors proved some coincidence point theorems for multivalued mappings in complete metric spaces under various contractive conditions.

Recently Constantin [1] proved some coincidence point theorems for multivalued mappings in complete metric spaces generalizing some results in [4, 5, 11, 12] and others.

In this paper we consider a generalized contraction condition involving two multivalued mappings which are compatible with two single-valued mappings and we establish a coincidence theorem which improves, extends some results of Constantin [1].

Let (X, d) be a metric space. We denote by $CB(X)$ the set of all nonempty closed bounded subsets of (X, d) and by H the Hausdorff metric on $CB(X)$

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\},$$

where $A, B \in CB(X)$ and

$$d(x, A) = \inf_{y\in A} d(x, y).$$

Our main theorem is prefaced with the following lemma due to Nadler [10].

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Lemma 1.1. Let $A, B \in CB(X)$ and $k > 1$. Then for each $a \in A$, there is a point $b \in B$ such that

$$d(a, b) \leq kH(A, B).$$

For single-valued mappings the notion of the weak commutativity is introduced by S. Sessa [14].

Definition 1.1. The mappings $S, I : X \to X$ are said to be weakly commuting if

$$d(SIx, ISx) \leq d(Ix, Sx)$$

for all x in X.

It is clear that two commuting mappings are weakly commuting but the converse is not true as is shown in [14].

G.Jungck [6] extended the concept of weak commutativity in the following way:

Definition 1.2. Let S and I be mappings from a metric space (X, d) into itself. The mappings S and I are said to be *compatible* if

$$\lim_{n\to\infty} d(SIx_n, ISx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ix_n = z$$

for some $z \in X$.

It is obvious that two *weakly commuting mappings* are *compatible*, but the converse is not true as is shown in [6].

H. Kaneko and S. Sessa [8] extended the concept of compatibility for single-valued mappings to the setting of single-valued and multi-valued mappings in the following way:

Definition 1.3. The mappings $A : X \to CB(X)$ and $S : X \to X$ are *compatible* iff $SAx \in CB(X)$ for all $x \in X$ and $d(SAx_n, ASx_n) \to 0$ whenever $\{x_n\} \subset X$ such that $Ax_n \to M \in CB(X)$ and $Sx_n \to t \in M$.

Definition 1.3 extends Definition 1.2 above as well as the Definition 1.1 above, where one may replace $I : X \to X$ with $A : X \to CB(X)$, that is $S : X \to X$ and $A : X \to CB(X)$ are said to be weakly commuting if for all $x \in X, SAx \in CB(X)$ and $H(SAx, ASx) \leq d(Sx, Ax)$, where H is the Hausdorff metric defined on $CB(X)$.

On the other hand, O. Hadzic [4] introduced the following definition:

Definition 1.4. The mappings $A : X \to CB(X)$ and $S : X \to X$ are said to be *weakly commuting* iff for every $x \in X$ and $y \in X$ such that $y \in Ax$ we have

$$d(Sy, SAx) \leq d(Sx, Ax).$$

It is obvious that *a commutative* pair of mappings (S, A) is also *weakly commutative* in the sense of Definition 1.4 but the converse is not true as is shown by Constantin [1].

We now introduce the following new definition which is slight variant of Kaneko-Sessa's [8] definition in the light of Definition 1.4.

Definition 1.5. The mappings $A : X \to CB(X)$ and $S : X \to X$ are said to be *compatible* if

$$\lim_{n\to\infty} d(Sy_n, ASx_n) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} y_n = z$$

for some z in X where $y_n \in Ax_n$ for $n = 1, 2, ...$

Clearly every weakly commuting pair of mappings (S, A) are compatible but the converse is not true. To see this, let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $y_n \in Ax_n$ for $n = 1, 2, ...$ and

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} y_n = z$$

for some z in X . Suppose S and A are weakly commuting. Since $d(Sx_n, Ax_n) \leq d(Sx_n, y_n)$, it follows that $\lim_{n\to\infty} d(Sx_n, Ax_n) = 0$. Since S and A are weakly commuting, we have

$$\lim_{n\to\infty} H(SAx_n, ASx_n) = 0.$$

Again, since $d(Sy_n, ASx_n) \leq H(SAx_n, ASx_n)$, we have

$$\lim_{n\to\infty} d(Sy_n, ASx_n) = 0,$$

proving that S and A are compatible.

Example 1.1. Let $X = [1, \infty)$ with the Euclidean metric d and define

$$Sx = 4x^4 - 3, \quad Ax = [1, x^2]$$

for all $x \geq 1$.

Clearly S and A are continuous and $A(X) = S(X) = X$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X defined by $x_n = 1 + \frac{1}{n}, y_n = 1$ for $n = 1, 2, ...$, we then have

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} y_n = 1 \in X, \quad y_n \in Ax_n.$$

Moreover,

$$\lim_{n\to\infty} H(SAx_n, ASx_n) = 0 \quad iff \quad x_n \to 1 \quad as \quad n \to \infty,$$

and since

$$d(Sy_n, ASx_n) \leq H(SAx_n, ASx_n),$$

we have

$$\lim_{n\to\infty} d(Sy_n, ASx_n) = 0.$$

Therefore S and A are compatible, but S and A are not weakly commuting at $x = 2$.

A point z in X is said to be a coincidence point of S and A if $Sz \in Az$ and is a fixed point of A if $z \in Az$.

2. A COINCIDENCE THEOREM

Following Constantin [1], we consider the set G of all continuous functions $g : [0,\infty)^5 \longrightarrow [0,\infty)$ satisfying the following conditions:

(i) $g(1,1,1,2,0) = g(1,1,1,0,2) = h \in (0,1)$;

(ii) g is subhomogenuous nondecreasing in each variable;

(iii) $g(x,y,z,t,0) < g(x_1,y_1,z_1,t,0)$ and $g(x,y,z,0,t) < g(x_1,y_1,z_1,0,t)$ if $x \leq x_1, y \leq y_1, z \leq z_1, \max\{x,y,z\} < \max\{x_1,y_1,z_1\}$ and $t \leq \max\{x,y,z\} + \min\{x,y,z\}$;

(iv) if $u \leq g(v,v,u,u+v,0)$ or $u \leq g(v,v,u,0,u+v)$ or $u \leq g(v,u,v,u+v,0)$ or $u \leq g(v,u,v,0,u+v)$ then $u \leq hv$.

The following Lemma due to Constantin [1] is useful in the sequel.

Lemma 2.1. If $g \in G$ and $k > 1$ is such that $kh < 1$ we have that $kg \in G$.

Adrian Constantin [1] proved the following Theorem.

Theorem 2.1. Let (X,d) be a complete metric space, S and T continuous mappings from X into X, A and B H-continuous mappings from X into $CB(SX \cap TX)$ such that the pairs (A,S) and (B,T) are weakly commutative in the sense of Definition 1.4 and

$$H(Ax,By) \leq g(d(Sx,Ty),d(Sx,Ax),d(Ty,By),d(Sx,By),d(Ty,Ax)), \tag{2.1}$$

for every $x,y \in X$, where $g \in G$. Then there exists a sequence $\{x_n\}_{n\in N}$ such that

1. For every $n \in N, Sx_{2n+1} \in Bx_{2n}, Tx_{2n} \in Ax_{2n-1}$;
2. There exists $z = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1}$;
3. $Sz \in Az, Tz \in Bz$.

Now replacing weakly commuting mappings with compatible mappings (Definition 1.5) in Theorem 2.1, we prove the following generalization of Theorem 2.1.

Theorem 2.2. Let (X,d) be a complete metric space, S and T continuous mappings from X into X, A and B H-continuous mappings from X into $CB(SX \cap TX)$ such that the pairs (A,S) and (B,T) are compatible in the sense of Definition 1.5 and condition (2.1) holds. Then there exists a sequence $\{x_n\}_{n\in N}$ such that

1. For every $n \in N, Sx_{2n+1} \in Bx_{2n}, Tx_{2n} \in Ax_{2n-1}$;
2. There exists $z = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1}$;
3. $Sz \in Az, Tz \in Bz$.

Proof. Let $k > 1$ be such that $kh < 1$. Let $x_0 \in X$. Since $Bx_0 \subset SX$ there exists $x_1 \in X$ such that $Sx_1 \in Bx_0$. Then there exists an element $u_1 \in Ax_1$ such that

$$d(u_1, Sx_1) \leq kH(Ax_1, Bx_0)$$

and $u_1 = Tx_2$ for some $x_2 \in X$ because $Ax_1 \subset TX$. Continuing in this fashion we define a sequence $\{x_n\}_{n\in N}$ in X where $Tx_{2n} \in Ax_{2n-1}$ and $Sx_{2n+1} \in Bx_{2n}$ are such that

$$d(Tx_{2n}, Sx_{2n-1}) \leq kH(Ax_{2n-1}, Bx_{2n-2}) \tag{2.2}$$

$$d(Sx_{2n+1}, Tx_{2n}) \leq kH(Bx_{2n}, Ax_{2n-1}) \tag{2.3}$$

for every $n \geq 1$.

We observe that by (2.1) and (2.3) we have that

$$d(Sx_{2n+1}, Tx_{2n}) \le kH(Bx_{2n}, Ax_{2n-1}) \le kg(d(Sx_{2n-1}, Tx_{2n}), d(Sx_{2n-1}, Ax_{2n-1}),$$
$$d(Tx_{2n}, Bx_{2n}), d(Sx_{2n-1}, Bx_{2n}), d(Tx_{2n}, Ax_{2n-1})) \le kL,$$

where

$$L = g(d(Sx_{2n-1}, Tx_{2n}), d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n}, Sx_{2n+1}),$$
$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, Sx_{2n+1}), 0).$$

By Lemma 2.1 we have that $kg \in G$ and so we deduce that

$$d(Tx_{2n}, Sx_{2n+1}) \le khd(Sx_{2n-1}, Tx_{2n}).$$

In a similar way it follows that

$$d(Tx_{2n}, Sx_{2n-1}) \le khd(Sx_{2n-1}, Tx_{2n-2}),$$

and so we deduce that

$$d(Tx_{2n}, Sx_{2n+1}) \le (kh)^{2n} d(Sx_1, Tx_0). \tag{2.4}$$

and

$$d(Tx_{2n}, Sx_{2n-1}) \le (kh)^{2n} d(Sx_1, Tx_0). \tag{2.5}$$

for every $n \ge 1$.

We deduce that $\{Sx_{2n-1}\}_{n \in N}$ is a Cauchy sequence in the complete metric space X and so $\lim_{n\to\infty} Sx_{2n-1} = z$ for some z in X. By (2.5), we also have $\lim_{n\to\infty} Tx_{2n} = z$.

We shall now prove that $Sz \in Az$, i.e., z is a coincidence point of S and A. For every $n \in N$, we have

$$d(STx_{2n}, Az) \le d(STx_{2n}, ASx_{2n-1}) + H(ASx_{2n-1}, Az). \tag{2.6}$$

By H-continuity of A and the fact that $\lim_{n\to\infty} Sx_{2n-1} = z$, we have

$$\lim_{n\to\infty} H(ASx_{2n-1}, Az) = 0.$$

Since S and A are compatible mappings and $\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} y_n = z$, where $y_n = Tx_{2n} \in Ax_{2n-1}$ and $z_n = x_{2n-1}$, we have

$$\lim_{n\to\infty} d(STx_{2n}, ASx_{2n-1}) = \lim_{n\to\infty} d(Sy_n, ASz_n) = 0.$$

Thus the inequality (2.6) yields $\lim_{n\to\infty} d(STx_{2n}, Az) = 0$ and so from

$$d(Sz, Az) \le d(Sz, STx_{2n}) + d(STx_{2n}, Az),$$

as $n \to \infty$ and the continuity of S, it follows that $d(Sz, Az) = 0$. Hence $Sz \in Az$, Since Az is a closed subset of $SX \cap TX$. We can similarly prove that $Tz \in Bz$, that is z is a coincidence point of T and B. This completes the proof.

Now , we give two examples to support our generalization.

Example 2.1. Let $X = [1, \infty)$ be with the Euclidean metric d. Define $A, B : X \to CB(SX \cap TX)$ and $S, T : X \to X$ by

$$\begin{aligned} Ax &= [1, x^2], \quad Bx = [1, x^4] \\ Sx &= 2x^4 - 1, \quad Tx = 2x^8 - 1 \end{aligned}$$

for all $x \in X, n \geq 1$.

Clearly A, B, S and T are continuous and $A(X) = B(X) = S(X) = T(X) = X$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X defined by $x_n = 1 + \frac{1}{n}, y_n = 1$ for $n = 1, 2, ...$, we then have

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} y_n = 1 \in X, y_n \in Ax_n.$$

Moreover,

$$\begin{aligned} \lim_{n\to\infty} H(SAx_n, ASx_n) &= \lim_{n\to\infty} |(2x_n^4 - 1)^2 - 2x_n^8 + 1| \\ &= 0 \quad iff \quad x_n \to 1 \quad as \quad n \to \infty, \end{aligned}$$

and since $d(Sy_n, ASx_n) \leq H(SAx_n, ASx_n)$, we have

$$\lim_{n\to\infty} d(Sy_n, ASx_n) = 0.$$

Therefore S and A are compatible . Similarly, one can show that B and T are compatible.

Assuming $g \in G$ as

$$g(t_1, t_2, t_3, t_4, t_5) = \frac{t_1}{2}$$

for every $t_i \in R^+, i = 1, 2, 3, 4, 5$.

We have for any $x, y \in X$

$$\begin{aligned} H(Ax, By) &= |x^2 - y^4| \\ &\leq |x^2 - y^4||x^2 + y^4| \\ &= |x^4 - y^8| \\ &= \frac{1}{2}|(2x^4 - 1) - (2y^8 - 1)| \\ &= \frac{d(Sx, Ty)}{2} \\ &= g(d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)). \end{aligned}$$

All the assumptions of Theorem 2.2 are verified and 1 is the coincidence point of S and A and of T and B.

Note that Theorem 2.1 is not applicable even if $A = B$ and $S = T$ because A and S are not weakly commuting mappings at $x = 2$.

Example 2.2. Let $X = [0, \infty)$ be with the Euclidean metric d. Define $A, B : X \to CB(SX \cap TX)$ and $S, T : X \to X$ by

$$\begin{aligned} Ax &= [0, x^2], \quad Bx = [0, 3x^2] \\ Sx &= 4x^4, \quad Tx = 36x^4 \end{aligned}$$

for all $x \in X$.

Clearly A, B, S and T are continuous and $A(X) = B(X) = S(X) = T(X) = X$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X defined by $x_n = \frac{1}{n^2}, y_n = \frac{1}{n^3}$ for $n = 1, 2, ...$, we then have

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} y_n = 0 \in X, y_n \in Ax_n.$$

Moreover,

$$\begin{aligned} \lim_{n\to\infty} H(SAx_n, ASx_n) &= \lim_{n\to\infty} |4x_n^4 - 16x_n^4| \\ &= 0 \quad iff \quad x_n \to 0 \quad as \quad n \to \infty, \end{aligned}$$

and since $d(Sy_n, ASx_n) \leq H(SAx_n, ASx_n)$, we have

$$\lim_{n\to\infty} d(Sy_n, ASx_n) = 0.$$

Therefore S and A are compatible . Similarly, one can show that B and T are compatible.

Assuming $g \in G$ as

$$g(t_1, t_2, t_3, t_4, t_5) = \frac{t_1}{4}$$

for every $t_i \in R^+, i = 1, 2, 3, 4, 5$.

We have for any $x, y \in X$

$$\begin{aligned} H(Ax, By) &= |x^2 - 3y^2| \\ &\leq |x^2 - 3y^2||x^2 + 3y^2| \\ &= |x^4 - 9y^4| \\ &= \frac{1}{4}|4x^4 - 36y^4| \\ &= \frac{d(Sx, Ty)}{4} \\ &= g(d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)). \end{aligned}$$

All the assumptions of Theorem 2.2 are verified and 0 is the coincidence point of S and A and of T and B.

Note that Theorem 2.1 is not applicable even if $A = B$ and $S = T$ because A and S are not weakly commuting mappings at $x = 2$.

Corollary 2.1 ([9]). Let (X,d) be a complete metric space, S and T continuous mappings from X into X, A and B closed mappings from X into $CB(SX\cap TX)$ such that $ASx=SAx, BTx=TBx$ for every $x\in X$ and

$$H(Ax,By)\leq q\max\{d(Sx,Ty),d(Sx,Ax),d(Ty,By),\frac{1}{2}[d(Sx,By)+d(Ty,Ax)]\},$$

for every $x,y\in X$, where $q\in(0,1)$. Then there exists a sequence $\{x_n\}_{n\in N}$ in X such that

1. For every $n\in N, Sx_{2n+1}\in Bx_{2n}, Tx_{2n}\in Ax_{2n-1}$;
2. There exists $z=\lim_{n\to\infty}Tx_{2n}=\lim_{n\to\infty}Sx_{2n+1}$;
3. $Sz\in Az, Tz\in Bz$.

Corollary 2.2 ([3]). Let (X,d) be a complete metric space, S and T continuous mappings from X into X, A a closed mapping from X into $CB(SX\cap TX)$ such that $ASx=SAx, ATx=TAx$ for every $x\in X$ and

$$H(Ax,Ay)\leq qd(Sx,Ty),$$

for every $x,y\in X$, where $q\in(0,1)$. Then there exists a sequence $\{x_n\}_{n\in N}$ in X such that

1. For every $n\in N, Sx_{2n+1}\in Ax_{2n}, Tx_{2n}\in Ax_{2n-1}$;
2. There exists $z=\lim_{n\to\infty}Tx_{2n}=\lim_{n\to\infty}Sx_{2n+1}$;
3. $Sz\in Az, Tz\in Az$.

Corollary 2.3 ([7]). Let (X,d) be a complete metric space, $A:X\to CB(X)$ and S a continuous mapping from X into X, and

$$H(Ax,Ay)\leq hd(Sx,Sy),$$

for every $x,y\in X$, where $0\leq h<1$ and $ASx=SAx$ for every $x\in X$. If $A(X)\subseteq S(X)$, then there exists $t\in X$ such that $St\in At$.

Corollary 2.4 ([11]). Let (X,d) be a complete metric space, S and T continuous mappings from X into X, A and B closed mappings from X into $CB(SX\cap TX)$ such that $ASx=SAx, BTx=TBx$ for every $x\in X$ and

$$\begin{aligned}&H^2(Ax,By)\leq q\max\{d(Sx,Ax)d(Ty,By),d(Sx,By)d(Ty,Ax),\\&\frac{1}{2}[d(Sx,Ax)d(Sx,By)+d(Ty,Ax)d(Ty,By)],d^2(Sx,Ty)\},\end{aligned}$$

for every $x,y\in X$, where $q\in(0,1)$. Then there exists a sequence $\{x_n\}_{n\in N}$ in X such that

1. For every $n\in N, Sx_{2n+1}\in Bx_{2n}, Tx_{2n}\in Ax_{2n-1}$;
2. There exists $z=\lim_{n\to\infty}Tx_{2n}=\lim_{n\to\infty}Sx_{2n+1}$;
3. $Sz\in Az, Tz\in Bz$.

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EIGENVALUE DISTRIBUTION OF INTEGRAL OPERATORS DEFINED BY BESOV–ORLICZ KERNELS

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Abstract. The real interpolation method has been generalized to the interpolation method with function parameter $(X_0, X_1)_{\rho,p}$ instead of $(X_0, X_1)_{\theta,p}$, replacing t^θ by some concave function ρ, see Gustavsson-Peetre [GP] and Fernandez-Garcia [FG]. The new method is particularly well-suited to interpolation of Besov-Orlicz spaces $B^s_{\Phi,q}$. The main aim of this paper is to combine the new interpolation method with König's eigenvalue theorem of integral operators defined by kernels in Besov and Sobolev spaces to find the optimal results for the asymptotic distribution of the eigenvalues of integral operators defined by kernel in Besov-Orlicz and Sobolev-Orlicz spaces. We derive some lemmas and theorems using the new interpolation method (e.g., the interpolation of L_p-spaces, Schatten classes and Besov spaces). Finally, we give a good estimate for the eigenvalues of integral operators defined by kernels belonging to vector-valued Besov-Orlicz spaces $B^s_{\Phi,q}(B^\sigma_{\Psi,v})$. We show that the eigenvalues of the integral operators defined by kernels $k \in B^s_{\Phi,q}(B^\sigma_{\Psi,v})$ belong to some Lorentz-Orlicz space $l_{\Phi,q}$. Moreover, we derive some special cases, e.g., $k \in W^s_\Phi(W^\sigma_\Psi)$ and $k \in B^s_{\Phi,q}(L_\Psi)$.

Keywords: Interpolation, Eigenvalue distribution, integral operators.

Mathematics subject classification numebrs: 46B20, 47B10

1. The Fernandez-Garcia interpolation method.

We now introduce a variant of the Gustavsson-Peetre interpolation method. It will depend on function parameters. The function parameter will be taken in Peetre's class P^{+-} , i.e., the class of pseudo-concave positive functions ρ on $\mathbb{R}^+$ which satisfy

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$$\bar{\rho}(t) \quad = \quad \sup_{s>0} \frac{\rho(st)}{\rho(s)} \quad = \quad \circ(\max(1,t)). \tag{1}$$

For a function parameter $\rho \in P^{+-}$, we can replace (1) by

$$\bar{\rho}(t) \quad = \quad \circ(\max(t^{\varepsilon}, t^{1-\varepsilon})), \tag{2}$$

provided $\varepsilon > 0$ is small enough. Consequently, we have

$$\int_0^{\infty} \min(1, 1/t) \quad \bar{\rho}(t) \frac{dt}{t} < \infty,$$

which implies

$$\sum_{n=0}^{\infty} \frac{\rho(2^n)}{2^n} + \sum_{n=0}^{\infty} \rho(2^{-n}) < \infty. \tag{3}$$

We shall also consider the sequence $(\tilde{r_n}(.))_{n\in\mathbb{Z}}$ defined in $[0,1]$ by

$$\tilde{r_n} \quad = \quad \begin{cases} r_{2n} & \text{if} \quad n \geq 0 \\ & \\ r_{2|n|-1} & \text{if} \quad n < 0, \end{cases}$$

where $(r_n(.))_{n\in\mathbb{Z}}$ is the sequence of Rademacher functions.

Definition 1.1. *Let (X_0, X_1) be a Banach couple. The space $\langle X_0, X_1\rangle_{\rho,p}$ is the linear space of all $x \in X_0 + X_1$ such that there is a sequence $(u_n)_{n\in\mathbb{Z}}$ in $X_0 \cap X_1$ which satisfies*

$$x = \sum_{n=-\infty}^{\infty} u_n \qquad (\textit{convergence in} \quad X_0 + X_1), \tag{4}$$

$$\sup_J \int_0^1 \left\| \sum_{n\in J} \tilde{r_n}(t) \frac{2^{kn}}{\rho(2^n)} u_n \right\|_{X_k}^p dt < \infty, \tag{5}$$

where the supremum is taken over all finite subsets J in $\mathbb{Z}$. Moreover, we shall also assume that, for $k = 0, 1$, the sequence $(\tilde{r_n}(.)\frac{2^{kn}}{\rho(2^n)} u_n)_{n\in\mathbb{Z}}$ is $L_p([0,1], X_k)$-summable.

We equip the space $\langle X_0, X_1\rangle_{\rho,p}$ with the norm

$$\|x\|_{\langle X_0,X_1\rangle_{\rho,p}} := \inf \max_{k=0,1} \sup_J \left(\left\| \sum_{n\in J} \tilde{r_n}(.) \frac{2^{kn}}{\rho(2^n)} u_n \right\|_{L_p([0,1],X_k)} \right), \tag{6}$$

where the infimum is taken over all admissible sequence $(u_n)_{n\in\mathbb{Z}}$.

Theorem 1.2. *Let (X_0, X_1) be a Banach couple, $\rho \in P^{+-}$ and $1 \leq p < \infty$. Then the space $\langle X_0, X_1\rangle_{\rho,p}$ is a Banach space.*

Proof : See [FG p. 25] □

To show that $\langle X_0, X_1\rangle_{\rho,p}$ is an intermediate space between X_0 and X_1, we need the following two lemmas.

Lemma 1.3. *For all Banach spaces X, we have $\langle X, X\rangle_{\rho,p} \quad = \quad X$.*

Proof : The imbedding $X \hookrightarrow \langle X, X\rangle_{\rho,p}$ is immediate. On the other hand, we have $\langle X, X\rangle_{\rho,p} \subset X$. It remains to prove that the imbedding is bounded.
Let $x \in \langle X, X\rangle_{\rho,p}$ and let $(u_n)_{n\in\mathbb{Z}}$ be an admissible sequence in X for x, such that

$$\sup_J \left\| \sum_{n\in J} \tilde{r_n}(.) \frac{2^{kn}}{\rho(2^n)} u_n \right\|_{L_p([0,1],X)} \quad \leq \quad 2\|x\|_{\langle X,X\rangle_{\rho,p}}.$$

Hence, for each $n \in \mathbb{Z}$ and $k = 0, 1$, we have

$$\left\| \frac{2^{kn}}{\rho(2^n)} \quad u_n \right\|_X \quad \leq \quad 2\|x\|_{\langle X,X\rangle_{\rho,p}},$$

and recalling (3), it follows that

$$\begin{aligned} \|x\|_X \quad \leq \quad \sum_{n=-\infty}^{\infty} \|u_n\|_X \quad &= \quad \sum_{n=0}^{\infty} \|u_n\|_X + \sum_{n=-\infty}^{-1} \|u_n\|_X \\ &\leq \quad 2\|x\|_{\langle X,X\rangle_{\rho,p}} \left(\sum_{n=0}^{\infty} \frac{\rho(2^n)}{2^n} + \sum_{n=-\infty}^{-1} \rho(2^n) \right) \\ &\leq \quad c\|x\|_{\langle X,X\rangle_{\rho,p}}. \end{aligned}$$

□

Lemma 1.4. *Let (X_0, X_1) and (Y_0, Y_1) be Banach couples such that $X_k \subset Y_k, (k = 0, 1)$. Then*

$$\langle X_0, X_1\rangle_{\rho,p} \hookrightarrow \langle Y_0, Y_1\rangle_{\rho,p}.$$

Proof : Immediate. □

Theorem 1.5. *If (X_0, X_1) is a Banach couple we have*

$$X_0 \cap X_1 \hookrightarrow \langle X_0, X_1\rangle_{\rho,p} \hookrightarrow X_0 + X_1,$$

i.e., $\langle X_0, X_1\rangle_{\rho,p}$ is an intermediate space between X_0 and X_1,

Proof : The left imbedding is clear. Now since $X_k \hookrightarrow X_0 + X_1$ we have

$$\langle X_0, X_1\rangle_{\rho,p} \hookrightarrow \langle X_0 + X_1, X_0 + X_1\rangle_{\rho,p} \quad = \quad X_0 + X_1,$$

using the above two lemmas. □

One has the fact is that the functor $(X_0, X_1) \mapsto \langle X_0, X_1\rangle_{\rho,p}$ is an interpolation functor.

Theorem 1.6. *Let (X_0, X_1) and (Y_0, Y_1) be Banach couples, $\rho \in P^{+-}$ and $1 \leq p < \infty$. Then, for all $T \in L(X_k, Y_k), (k = 0, 1)$, we have*

$$T \in L(\langle X_0, X_1\rangle_{\rho,p}, \langle Y_0, Y_1\rangle_{\rho,p}).$$

Moreover,

$$\|T\| \quad \leq \quad \max(\|T\|_0, \|T\|_1).$$

Proof : Let $x \in \langle X_0, X_1\rangle_{\rho,p}$, let $(u_n)_{n\in\mathbb{Z}}$ be an admissible sequence of x, and set $M_k \quad = \quad \|T\|_{L(X_k,Y_K)}, (k = 0, 1)$. Since the convergence in (4) is in $X_0 + X_1$, we have $Tx = \sum_{n\in\mathbb{Z}} Tu_n$.
Hence, setting $v_n = Tu_n$, we see that $\sum_{n\in\mathbb{Z}} v_n$ converges in $Y_0 + Y_1$ and is a representation of Tx.
Next, for $k = 0, 1$, we get

$$\begin{aligned}\left\|\sum_{n\in J} \tilde{r_n}(.)\frac{2^{kn}}{\rho(2^n)}v_n\right\|_{L_p([0,1],Y_k)} &= \left\|T\left(\sum_{n\in J} \tilde{r_n}(.)\frac{2^{kn}}{\rho(2^n)}u_n\right)\right\|_{L_p([0,1],X_k)} \\ &\leq M_k \left\|\sum_{n\in J} \tilde{r_n}(.)\frac{2^{kn}}{\rho(2^n)}u_n\right\|_{L_p([0,1],X_k)},\end{aligned}$$

and consequently $\|T\|_{\langle Y_0,Y_1\rangle_{\rho,p}} \quad \leq \quad c\|x\|_{\langle X_0,X_1\rangle_{\rho,p}}$.
The proof is complete. □

Theorem 1.7. *(Kahane, see [FG]) For every $1 < r < \infty$, for any Banach space X and every finite sequence $(u_j)_{1 \ \leq \ j \ \leq \ n}$ in X, we have*

$$\int_0^1 \left\|\sum_{j=1}^n r_j(t)u_j\right\|_X dt \approx \left(\int_0^1 \left\|\sum_{j=1}^n r_j(t)u_j\right\|_X^r dt\right)^{1/r}.$$

Theorem 1.8. *Let (X_0, X_1) and (Y_0, Y_1) be Banach couples, $\rho \in P^{+-}$ and $1 \leq p, q < \infty$. Then $\langle X_0, X_1\rangle_{\rho,p} \quad = \quad \langle X_0, X_1\rangle_{\rho,q}$, with norm equivalence.*

Proof : This follows at once from Kahane's theorem. □

2. Interpolation of $L_p(X)$ spaces

We shall show that L_p-spaces commute with the interpolation space $\langle X_0, X_1\rangle_{\rho,p}$.

Theorem 2.1 (FG)**.** *Let (X_0, X_1) be a Banach couple of U.M.D. spaces (see FG p. 23). Then the interpolation space $\langle X_0, X_1\rangle_{\rho,p}$ is also a U.M.D. space.*

Theorem 2.2. *Let L_Φ be an Orlicz space such that the Orlicz function Φ and the conjugate function Φ^* satisfy the Δ_2-condition. Then L_Φ has the U.M.D. property.*

Corollary 2.3. *The space $Y(X)$ is U.M.D. if and only if Y and X are U.M.D. spaces.*

Lemma 2.4 (FG)**.** *Let X be a Banach space, $\rho \in P^{+-}$ and $1 \leq p < \infty$. Then if $(u_j)_{1 \leq j \leq n}$ is a finite sequence in X, then*

$$\left\| \sum_{j=1}^{n} |u_j| \right\|_X \leq cR \left(\left\| (\sum_{j=1}^{n} (\frac{|u_j|}{\rho(2^j)})^p)^{1/p} \right\|_X , \left\| (\sum_{j=1}^{n} (\frac{2^j |u_j|}{\rho(2^j)})^p)^{1/p} \right\|_X \right),$$

where $R(s,t) = s\rho(t/s)$.

Theorem 2.5. *Let (X_0, X_1) be a U.M.D Banach couple. Then for all $p, 1 \leq p < \infty$, we have*

$$\langle L_p(X_0), L_p(X_1) \rangle_{\rho,p} = L_p(\langle X_0, X_1 \rangle_{\rho,p}), \tag{7}$$

with equivalent norms.

Proof : See [FG]

Remark 2.6. *Using the fact that l_q^s-spaces are special L_q-spaces and the previous theorem, we have*

$$\langle l_q^s(X_0), l_q^s(X_1) \rangle_{\rho,p} = l_q^s(\langle X_0, X_1 \rangle_{\rho,p}), \tag{8}$$

with the same assumption as in theorem (2.5).

The following theorem shows that reflexive Orlicz spaces can be characterized as interpolation spaces between L_p-spaces. This is a converse of Gustavsson-Peetre's theorem [GP theorem (7.3)].
By an Orlicz function we mean a convex and continuous function
$\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$.

Theorem 2.7 (FG)**.** *Let Φ be an Orlicz function and let $L_\Phi(E)$ be the corresponding E-valued Orlicz space on a σ-finite measure space, where E is a U.M.D. Banach space. Suppose that Φ and the conjugate Φ^* satisfy the Δ_2-condition. Then there exist p_0 and p_1, with $1 < p_1 < p_0 < \infty$,*

$$\langle L_{p_0}(E), L_{p_1}(E) \rangle_{\rho,p} = L_\Phi(E) \qquad \text{,where} \tag{9}$$

$$\rho(t) = \begin{cases} t^{\frac{p_1}{p_1 - p_0}} \quad \Phi^{-1}(t^{\frac{p_0 p_1}{p_0 - p_1}}) & ,t > 0 \\ \\ 0 & t = 0 \end{cases} \tag{10}$$

and $1 < p < \infty$.

3. Besov-Orlicz and Sobolev-Orlicz spaces

Definition 3.1. *Suppose $s \in \mathbb{R}$ and let (ϕ_n) be a system of functions(see T p. 171). Let X be a Banach space and Φ be an Orlicz function satisfy the Δ_2-condition,we define for $1 \leq q \leq \infty$, the Besov-Orlicz spaces*

$$B^s_{\Phi q}(\mathbb{R}^N; X) = \{f \in S' \quad : (\phi_n * f)_n \in l^s_q(L_\Phi(\mathbb{R}^N; X))\} \tag{11}$$

We equip the spaces $B^s_{\Phi q}(\mathbb{R}^N; X)$ with the norm

$$\|f\|_{B^s_{\Phi q}} = \|(\phi_n * f)_n\|_{l^s_q(L_\Phi(\mathbb{R}^N;X))} = \|(\|\phi_n * f\|_{L_\Phi(X)})_n\|_{l^s_q}, \tag{12}$$

Remark 3.2 (FG). *The spaces $B^s_{\Phi q}$ do not depend on the particular system of functions used to define them.*

We shall prove that the Besov-Orlicz space $B^s_{\Phi q}(\mathbb{R}^N; X)$ is a retract of the space $l^s_q(L_\Phi(\mathbb{R}^N; X))$ using the composition

$$B^s_{\Phi q}(\mathbb{R}^N; X) \xrightarrow{I} l^s_q(L_\Phi(\mathbb{R}^N; X)) \xrightarrow{P} B^s_{\Phi q}(\mathbb{R}^N; X).$$

Theorem 3.3. *Let Φ be an Orlicz function and let $L_\Phi(X)$ be the corresponding X-valued Orlicz space. Then the space $B^s_{\Phi q}(\mathbb{R}^N; X)$ is a retract of $l^s_q(L_\Phi(\mathbb{R}^N; X)) \quad (s \in \mathbb{R}, 1 \leq q \leq \infty)$.*

Proof : Consider the mappings I and P are defined by

$$(If)_j = \phi_j * f \qquad j = 1, 2, \ldots \qquad , (If)_0 = \psi * f,$$

$$P\alpha = \sum_{j=0}^{\infty} \tilde{\phi_j} * \alpha_j \qquad (\text{convergence in} S^*) \qquad \text{where} \quad \alpha = (\alpha_j)_{j=0}^{\infty},$$

$$\alpha_j \in S^*, \quad j = 1, 2, \ldots \qquad \text{and} \qquad \tilde{\phi_0} = \psi + \phi_1$$

$$\tilde{\phi_j} = \sum_{l=-1}^{1} \phi_{j+l} \quad j = 1, 2, \ldots \qquad .$$

Clearly $PIf = f \quad , f \in S^*$, since $\tilde{\phi_j} * \phi_j = \phi_j \quad j = 1, 2, \ldots$ and $\tilde{\phi_0} * \psi = \psi$.

(i) First,we note that

$$\|f\|_{B^s_{\Phi q}(\mathbb{R}^N;X)} \sim \left(\sum_{j=0}^{\infty} (2^{js} \quad \|\phi_j * f\|_{L_\Phi(\mathbb{R}^N;X)})^q\right)^{1/q}$$

$$= \quad \left(\sum_{j=0}^{\infty} (2^{js} \quad \|(If)_j\|_{L_\Phi(\mathbb{R}^N;X)})^q\right)^{1/q}$$

$$= \quad \|If\|_{l^s_q(L_\Phi(\mathbb{R}^N;X))}.$$

Thus $I : B^s_{\Phi q}(\mathbb{R}^N;X) \to l^s_q(L_\Phi(\mathbb{R}^N;X))$.

(ii) Since $\phi_j * P\alpha \;=\; \phi_j * \alpha_j \quad (j \geq 1)$, we have

$$\|P\alpha\|_{B^s_{\Phi q}(\mathbb{R}^N;X)} = \left(\sum_{j=1}^{\infty}(2^{js} \quad \|\phi_j * \alpha_j\|_{L_\Phi(\mathbb{R}^N;X)})^q\right)^{1/q}$$

$$\leq \quad c \quad \left(\sum_{j=1}^{\infty}(2^{js} \quad \|\alpha_j\|_{L_\Phi(\mathbb{R}^N;X)})^q\right)^{1/q}$$

$$\leq \quad \|\alpha\|_{l^s_q(L_\Phi(\mathbb{R}^N;X))}.$$

Thus $B^s_{\Phi q}(\mathbb{R}^N;X)$ is a retract of $l^s_q(L_\Phi(\mathbb{R}^N;X))$. □

Definition 3.4. *Assume that $s \in \mathbb{R}$ and let Φ be an Orlicz function satisfy the Δ_2-condition. We define the Sobolev-Orlicz spaces $W^s_\Phi(\mathbb{R}^N;X)$ by*

$$W^s_\Phi(\mathbb{R}^N;X) \quad = \quad \{f \in S' \quad : J^s f \in L_\Phi(\mathbb{R}^N;X)\}. \tag{13}$$

Here J^s is the Bessel operator .

We equip the space $W^s_\Phi(\mathbb{R}^N;X)$ with the norm

$$\|f\|_{W^s_\Phi(\mathbb{R}^N;X)} \quad = \quad \|J^s f\|_{L_\Phi(\mathbb{R}^N;X)}.$$

Definition 3.5. *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary domain.Further,let $-\infty < s < \infty, 1 < p < \infty$ and $1 \leq q \leq \infty$. Then $B^s_{pq}(\Omega;X)$ is the restriction of $B^s_{pq}(\mathbb{R}^N;X)$ to Ω,*

$$\|f\|_{B^s_{pq}(\Omega;X)} = \inf_{g|_\Omega = f} \|g\|_{B^s_{pq}(\mathbb{R}^N;X)}, \quad g \in B^s_{pq}(\mathbb{R}^N;X).$$

And $W^s_p(\Omega;X)$ is the restriction of $W^s_p(\mathbb{R}^N;X)$ to Ω,

$$\|f\|_{W^s_p(\Omega;X)} = \inf_{g|_\Omega = f} \|g\|_{W^s_p(\mathbb{R}^N;X)}, \quad g \in W^s_p(\mathbb{R}^N;X).$$

We shall now introduce two mappings S and R.The mapping S (coretraction) maps $B^s_{pq}(\Omega;X)$ to $B^s_{pq}(\mathbb{R}^N;X)$.It is defined by

$$S(f) = \lim_j f\psi_j, \qquad f \in B^s_{pq}(\Omega;X)$$

The mapping R (retraction) is given by

$$R(\sum_j \beta_j) = \sum_j \psi_j * f_j,$$

where the mapping I (coretraction) maps $B^s_{pq}(\mathbb{R}^N;X)$ to $l^s_q(L_p(\mathbb{R}^N;X))$ is defined by

$$I(g) = \{g * \phi_j\}_{j=1}^{\infty} = (g_j)_{j=1}^{\infty}, \qquad g \in B^s_{pq}(\mathbb{R}^N;X),$$

and the mapping P (retraction) is given by

$$P(\alpha) = \sum_j \psi_j * g_j, \qquad \alpha = (g_j)_{j=1}^{\infty} = \sum_j \beta_j.$$

Therefore, the mapping IS (coretraction) maps $B^s_{pq}(\Omega; X)$ to $l^s_q(L_p(\mathbb{R}^N; X))$ and the mapping RP is the corresponding retraction.

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain.*
If $-\infty < s < \infty, 1 < p < \infty$ and $1 \le q \le \infty$, then the restriction operator from $B^s_{pq}(\mathbb{R}^N; X)$ onto $B^s_{pq}(\Omega; X)$ is a retraction.
If $r \in \mathbb{N}$, then there exists a corresponding coretraction independent of $1 < p < \infty, 1 \le q \le \infty$ and $|s| < r$.

Proof : See H.Trieble [T p. 311] □

Remark 3.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain, $s \in \mathbb{R}$,*
$1 \le q \le \infty$ and Φ be an Orlicz function satisfy the Δ_2-condition. Then $B^s_{\Phi q}(\Omega; X)$ is a retract of $l^s_q(L_\Phi(\Omega; X))$

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain, Φ be an Orlicz function which satisfies, together with its conjugate Φ^*, the Δ_2-condition. Then there exist p_0 and p_1, with $1 < p_0, p_1 < \infty$, such that*

$$\langle B^s_{p_0 q}(\Omega; X), B^s_{p_1 q}(\Omega; X)\rangle_{\rho,p} \quad = \quad B^s_{\Phi q}(\Omega; X), \tag{14}$$

where ρ is given by (10) .

Proof : Using remark (3.7), remark (2.6) and theorem (2.7), we have

$$\begin{aligned}\langle B^s_{p_0 q}(\Omega; X), B^s_{p_1 q}(\Omega; X)\rangle_{\rho,p} &= \langle RP(l^s_q(L_{p_0}(\Omega; X))), RP(l^s_q(L_{p_1}(\Omega; X)))\rangle_{\rho,p} \\ &= RP(l^s_q(\langle L_{p_0}(\mathbb{R}^N; X), L_{p_1}(\Omega; X)\rangle_{\rho,p})) \\ &= RP(l^s_q(L_\Phi(\Omega; X))) \quad = \quad B^s_{\Phi q}(\Omega; X).\end{aligned}$$

□

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain ,*
$1 < p, q < \infty$, and $s \in \mathbb{R}$. Then we have, for Banach spaces X_0, X_1

$$\langle B^s_{pq}(X_0), B^s_{pq}(X_1)\rangle_{\rho,p} \quad = \quad B^s_{pq}(\langle X_0, X_1\rangle_{\rho,p}). \tag{15}$$

Proof : Using remark (3.7) , remark (2.6) and theorem (2.5), we have

$$\begin{aligned}\langle B^s_{pq}(X_0), B^s_{pq}(X_1)\rangle_{\rho,p} &= \langle RP(l^s_q(L_p(X_0))), RP(l^s_q(L_p(X_1)))\rangle_{\rho,p}) \\ &= RP(l^s_q(\langle L_p(X_0), L_p(X_1)\rangle_{\rho,p}) \\ &= RP(l^s_q(L_p(\langle X_0, X_1\rangle_{\rho,p})))\end{aligned}$$

$$= \quad B^s_{pq}(\langle X_0, X_1\rangle_{\rho,p}).$$

□

Theorem 3.10. *Let $1 < p, q, u_0, u_1, v < \infty$ and $s \in \mathbb{R}$. Then we have*

$$\langle B^s_{pq}(B^\sigma_{u_0 v}(\Omega; X)), B^s_{pq}(B^\sigma_{u_1 v}(\Omega; X))\rangle_{\rho,q} \quad = \quad B^s_{pq}(B^\sigma_{\Phi v}(\Omega; X)), \tag{16}$$

with,

$$\rho(\tau) \quad = \quad \tau^{\frac{u_1}{u_1 - u_0}} \quad \Phi^{-1}(\tau^{\frac{u_0 u_1}{u_0 - u_1}}).$$

Proof : Using theorem (3.9) and theorem (3.8) ,we have

$$\langle B^s_{pq}(B^\sigma_{u_0 v}(\Omega; X)), B^s_{pq}(B^\sigma_{u_1 v}(\Omega; X))\rangle_{\rho,q}$$

$$= \quad B^s_{pq}(\langle B^\sigma_{u_0 v}(\Omega; X), B^\sigma_{u_1 v}(\Omega; X))\rangle_{\rho,q}) \quad = \quad B^s_{pq}(B^\sigma_{\Phi v}(\Omega; X)).$$

□

4. Schatten-Orlicz classes

Definition 4.1. *Let Φ be an Orlicz function as before. We define for $1 \le q < \infty$, the Lorentz-Orlicz spaces $l_{\Phi q}$ as follows*

$$l_{\Phi q} = \left\{ x = (x_n)_n : \|x\|_{l_{\Phi q}} := \inf \left(c : (\sum_{n=1}^{\infty} \Phi(\frac{|x_n|}{c})^q)^{1/q} \le 1 \right) < \infty \right\}, \tag{17}$$

and,

$$l_{\Phi\infty} = \left\{ x = (x_n)_n \quad : \|x\|_{l_{\Phi\infty}} := \inf \left(c : \sup_n \Phi(\frac{|x_n|}{c}) \le 1 \right) < \infty \right\}. \tag{18}$$

Definition 4.2. *Let Φ be an Orlicz function . We define for $1 \le q \le \infty$, the Schatten-Orlicz classes $S^s_{\Phi q}$ as follows*

$$S^s_{\Phi q}(X) \quad := \quad \{T \in L(X) \quad : (s_n(T))_n \in l_{\Phi q}\}. \tag{19}$$

Theorem 4.3. *Let $1 < p_0, p_1 < \infty$. Then we have*

$$\langle S^s_{p_0 q}(X), S^s_{p_1 q}(X)\rangle_{\rho,p} \subset S^s_{\Phi q}(X), \tag{20}$$

where ρ is given by (10).

Proof : Let $T \in \langle S^s_{p_0 q}(X), S^s_{p_1 q}(X)\rangle_{\rho,p} \subset \langle S^s_{p_0\infty}(X), S^s_{p_1\infty}(X)\rangle_{\rho,p}$ with $\|T\|_{\langle S^s_{p_0\infty}(X), S^s_{p_1\infty}(X)\rangle_{\rho,p}} \le 1$ and let $(T_n)_n$ be an admissible sequence for T such that

$$\max_{k=0,1} \sup_J \left\| \sum_{n \in J} \tilde{r_n}(.) \frac{2^{kn}}{\rho(2^n)} T_n \right\|_{L_p([0,1], S^s_{p_k\infty}(X))} < 1.$$

Since the sequence $(\tilde{r_n}(.)\frac{2^{kn}}{\rho(2^n)}T_n)_{n\in\mathbb{Z}}$ is $L_p([0,1], S^s_{p_k\infty})$-summable , given $\varepsilon > 0$, there exist a finite subset $J_\varepsilon \subset \mathbb{N}$ such that for all $J' \subset \mathbb{N}$ with $J' \cap J_\varepsilon \;=\; \emptyset$ we have

$$\left\| \left(\sum_{n\in J'} (\frac{2^{kn}}{\rho(2^n)} T_n)^2 \right)^{1/2} \right\|_{S^s_{p_k\infty}(X)} \;\leq\; \varepsilon^{1/p_k},$$

Now setting

$$A_k(m) \;=\; \max_n n^{1/p_k} \left\| \left(\sum_{n\in J'} (\frac{2^{kn}}{\rho(2^n)} \;\; s_m(T_n))^2 \right)^{1/2} \right\|_X \;\leq\; \varepsilon^{1/p_k} \quad \text{and}$$

$$A(m) \;=\; \max_{k=0,1} \left(\frac{A_k(m)}{2\varepsilon} \right)^{p_k}.$$

From the fact, $\frac{\rho(t)}{t}$ is non-increasing and lemma (2.4)

$$\frac{1}{2\varepsilon} \| \sum_{n\in J'} s_m(T_n) \|_X \;\leq\; c\frac{A_0(m)}{2\varepsilon} \;\; \rho\left(\frac{A_1(m)}{A_0(m)} \right)$$

$$\leq\; cA(m)^{\frac{1}{p_0}} \;\; \rho(A(m)^{\frac{1}{p_1}-\frac{1}{p_0}}) \;\leq\; c\Phi^{-1}(A(m)).$$

Therefore,

$$\Phi\left(\frac{1}{2c\varepsilon} \| \sum_{n\in J'} s_m(T_n) \|_X \right) \;\leq\; A(m)$$

$$\leq\; \frac{1}{2}\left((\frac{A_0(m)}{\varepsilon})^{p_0} + (\frac{A_1(m)}{\varepsilon})^{p_1} \right) \;\leq\; 1.$$

Therefore,$\| \sum_{n\in J'} T_n \|_{S_{\Phi\infty}(X)} \;\leq\; 2c\varepsilon$, the sequence $(T_n)_n$ satisfies the Cauchy condition in $S_{\Phi\infty}(X)$.
Then, we have $\quad T \in S_{\Phi\infty}(X) \Rightarrow T \in S_{\Phi q}(X)$. Hence,

$$\langle S^s_{p_0q}(X), S^s_{p_1q}(X)\rangle_{\rho,q} \subset S^s_{\Phi q}(X) \quad \text{with} \quad \Phi^{-1}(t) \;=\; t^{\frac{1}{p_0}} \;\; \rho(t^{\frac{1}{p_1}-\frac{1}{p_0}})$$

which equivalent to the relation (10). □

5. Eigenvalue distribution of integral operators

Now, we derive our main results using the eigenvalue theorem for Besov spaces and the new interpolation results (interpolation of S^s_{pq}-spaces, and B^s_{pq}-spaces). We will prove that the eigenvalues of integral operators defined by kernels $k \in B^s_{\Phi q}(B^\sigma_{\Psi v})$ belong to some Lorentz-Orlicz space $l_{\Phi q}$.

Theorem 5.1. *(Eigenvalue theorem I) [K] Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain, $1 \le p, q, u, v \le \infty$ and $s, \sigma > 0$ with $s + \sigma > N(\frac{1}{p} + \frac{1}{u} - 1)$. Define t by $\frac{1}{t} := \frac{s+\sigma}{N} + \frac{1}{\max(2,u')}$. Then the eigenvalues of any kernel $k \in B^s_{pq}(\Omega; B^\sigma_{uv}(\Omega))$ belong to the Lorentz sequence space $l_{t,q}$ with*

$$\|(\lambda_n(k))_{n\in\mathbb{N}}\|_{l_{t,q}} \le c\|k\|_{B^s_{pq}(B^\sigma_{uv})}. \tag{21}$$

The constant c depend only on the indices and the domain Ω.

Theorem 5.2 (K)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain, $1 \le p, q, u \le \infty$ and $s > 0$ with $s > N(\frac{1}{p} - \frac{1}{u'})$. Define t by $\frac{1}{t} := \frac{s}{N} + \frac{1}{\max(2,u')}$. Then the eigenvalues sequence of any kernel $k \in B^s_{pq}(\Omega; L_u(\Omega))$ belong to the Lorentz sequence space $l_{t,q}$ with*

$$\|(\lambda_n(k))_{n\in\mathbb{N}}\|_{l_{t,q}} \le c_{p,q,u,s,\Omega}\|k\|_{B^s_{pq}(L_u)}. \tag{22}$$

Theorem 5.3 (K)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain , $1 \le p, u, v \le \infty$ and $\sigma > 0$ with $\sigma > N(\frac{1}{u} - \frac{1}{\max(p',v')})$. Define t by $\frac{1}{t} := \frac{\sigma}{N} + \frac{1}{\max(2,u')}$. Then the eigenvalues of any kernel $k \in L_p(\Omega; B^\sigma_{uv}(\Omega))$ belong to the Lorentz sequence space $l_{t,v}$ with*

$$\|(\lambda_n(k))_{n\in\mathbb{N}}\|_{l_{t,v}} \le c_{p,u,v,\sigma,\Omega}\|k\|_{L_p(B^\sigma_{uv})}. \tag{23}$$

Remark 5.4. *(i) As shown by Besov [B] , the Sobolev spaces $W^s_u(\Omega)$ of integer order $s \in \mathbb{N}$ satisfy the inclusion $W^s_u(\Omega) \subset B^s_{uv}(\Omega)$ with $v := \max(2, u)$. Hence the eigenvalues of the kernel operator defined in $L_p(\Omega)$ by any kernel $k \in L_p(\Omega; W^s_u(\Omega))$ belong to l_{tv} for $\frac{1}{t} := \frac{s}{N} + \frac{1}{\max(2,u')}$ provided that $s > N(\frac{1}{u} - \frac{1}{\max(p',v')})$.*
(ii) For $v \ge p, r \in \mathbb{N}$,we have

$$W^r_p(\Omega; X_{\theta,v}) \subset (W^r_p(\Omega; X_0), W^r_p(\Omega; X_1))_{\theta,v}.$$

This yields that the eigenvalues of kernels $k \in W^r_p(\Omega; B^\sigma_{uv}(\Omega))$ belong to l_t, v, $\frac{1}{t} := \frac{\sigma+r}{N} + \frac{1}{\max(2,u')}$, provided that $\sigma + r$ satisfies $\sigma + r > N(\frac{1}{u} - \frac{1}{\max(p',v')})$.
Using the above inclusion $W^s_p(\Omega) \subset B^s_{uv}(\Omega)$ for $v := \max(u, 2)$, one finds:

If $r, s \in \mathbb{N}$ with $\quad r + s > N(\dfrac{1}{u} - \dfrac{1}{\max(p', v')}) \quad$ *and* $\; k \in W^r_p(\Omega; W^s_u(\Omega))$,

the eigenvalues of T_k (e.g., in $L_p(\Omega)$ if $p \ge u'$) belong to $l_{t,v}$ for

$$\frac{1}{t} := \frac{r+s}{N} + \frac{1}{\max(2, u')} \quad \text{and} \quad v := \max(2, u).$$

Theorem 5.5. *(The eigenvalue theorem II.) Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain, $1 \le p, q, u_0, u_1, v \le \infty$, $u_1 < u_0$ and $s + \sigma > 0$ with $s + \sigma > N(\frac{1}{p} + \frac{1}{u_1} - 1)$. Define*

$$\Psi^{-1}(\tau) = \tau^{\frac{s+\sigma}{N} + \frac{1}{u_1'} - \frac{1}{\min(2,u_0)}} \quad \Phi^{-1}(\tau). \tag{24}$$

Then the eigenvalues of any kernel $k \in B^s_{pq}(\Omega; B^\sigma_{\Phi v}(\Omega))$ *belong to the Lorentz-Orlicz space* $l_{\Psi q}$ *with*

$$\|(\lambda_n(k))_{n\in\mathbb{N}}\|_{l_{\Psi,q}} \le c\|k\|_{B^s_{pq}(B^\sigma_{\Phi v})}. \tag{25}$$

Proof : Using the eigenvalue theorem (5.1), we have, for $s+\sigma > N(\frac{1}{p}+\frac{1}{u_1}-1), i=0,1$

$$B^s_{pq}(B^\sigma_{u_i v}) \subset S^s_{t_i,q} \quad \text{with} \quad \frac{1}{t_i} := \frac{s+\sigma}{N} + \frac{1}{\max(2,u_i')}. \tag{26}$$

By theorem (3.10), we have

$$\langle B^s_{pq}(B^\sigma_{u_0 v}(X)), B^s_{pq}(B^\sigma_{u_1 v}(X))\rangle_{\rho,q} = B^s_{pq}(B^s_{\Phi v}(X)), \tag{27}$$

with,

$$\rho(\tau) = \tau^{\frac{u_1}{u_1-u_0}} \quad \Phi^{-1}(\tau^{\frac{u_0 u_1}{u_0-u_1}}). \tag{28}$$

(i) The case $u_1 < u_0 < 2 \quad (2 < u_0' < u_1')$:

Using theorem (4.3),for $\dfrac{1}{t_i} := \dfrac{s+\sigma}{N} + \dfrac{1}{u_i'}$, we have,

$$\langle S^s_{t_0 q}(X), S^s_{t_1 q}(X)\rangle_{\rho,q} = S^s_{\Psi q}(X), \tag{29}$$

with,

$$\rho(\tau) = \tau^{\frac{t_0}{t_0-t_1}} \quad \Psi^{-1}(\tau^{\frac{t_0 t_1}{t_1-t_0}}). \tag{30}$$

Let $\beta = \dfrac{t_0 t_1}{t_1 - t_0} \Rightarrow \dfrac{1}{\beta} = \dfrac{1}{t_0} - \dfrac{1}{t_1} = \dfrac{u_1'-u_0'}{u_0' u_1'} \Rightarrow \qquad \beta = \dfrac{u_0' u_1'}{u_1'-u_0'}$,

and $\gamma = \dfrac{t_0}{t_0-t_1} = -\dfrac{1}{t_1}\beta = (\dfrac{s+\sigma}{N} + \dfrac{1}{u_1'})(\dfrac{u_0' u_1'}{u_0'-u_1'})$.

Consequently, we have

$$\rho(\tau) = \tau^\gamma \quad \Psi^{-1}(\tau^\beta). \tag{31}$$

From (27) , (28) , (29) , (31) and the fact, that if $X_k \subset Y_k, (k=0,1) \Rightarrow \langle X_0, X_1\rangle_{\rho,p} \subset \langle Y_0, Y_1\rangle_{\rho,p}$, we have

$$B^s_{pq}(B^s_{\Phi v}(X)) \subset S^s_{\Psi q}(X) \quad \text{with} \quad \Psi^{-1}(\tau^\beta) = \tau^{\frac{u_1}{u_1-u_0}-\gamma} \quad \Phi^{-1}(\tau^{\frac{u_0 u_1}{u_0-u_1}}).$$

But,$\beta = \frac{u_0' u_1'}{u_1'-u_0'} = \frac{u_0 u_1}{u_0-u_1}$ and

$$\frac{u_1}{u_1-u_0} - \gamma = \frac{-\beta}{u_0} + \beta(\frac{s+\sigma}{N} + \frac{1}{u_1'}) = \beta(\frac{s+\sigma}{N} + \frac{1}{u_1'} - \frac{1}{u_0}).$$

Then, $\qquad \Psi^{-1}(\tau^\beta) = \tau^{\beta(\frac{s+\sigma}{N}+\frac{1}{u_1'}-\frac{1}{u_0})} \quad \Phi^{-1}(\tau^\beta).$

Take $\tau^\beta = \xi$, we find

$$\Psi^{-1}(\xi) = \xi^{\frac{s+\sigma}{N}+\frac{1}{u_1'}-\frac{1}{u_0}} \quad \Phi^{-1}(\xi). \tag{32}$$

(ii) The case $u_1 < 2 < u_0 \quad (u_0' < 2 < u_1')$:
Using the same method as above, we can find

$$\Psi^{-1}(\xi) = \xi^{\frac{s+\sigma}{N}+\frac{1}{u_1'}-\frac{1}{2}} \quad \Phi^{-1}(\xi). \tag{33}$$

Combining (32) and (33), we have

$$\Psi^{-1}(\tau) = \tau^{\frac{s+\sigma}{N}+\frac{1}{u_1'}-\frac{1}{\min(2,u_0)}} \quad \Phi^{-1}(\tau).$$

Then, we readily get the desired results. □

Theorem 5.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain, $1 \leq p, q, u_0, u_1 \leq \infty, u_1 < u_0$ and $s > 0$ with $s > N(\frac{1}{p} + \frac{1}{u_1} - 1)$. Define*

$$\Psi^{-1}(\tau) = \tau^{\frac{s}{N}+\frac{1}{u_1'}-\frac{1}{\min(2,u_0)}} \quad \Phi^{-1}(\tau). \tag{34}$$

Then the eigenvalues of any kernel $k \in B^s_{pq}(\Omega; L_\Phi(\Omega))$ belong to the Lorentz-Orlicz space $l_{\Psi q}$ with

$$\|(\lambda_n(k))_{n\in\mathbb{N}}\|_{l_{\Psi,q}} \leq c\|k\|_{B^s_{pq}(L_\Phi)}. \tag{35}$$

Proof : Replacing $s+\sigma$ by s in theorem (5.5) and using the same method, we get directly the desired results □

Theorem 5.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain, $1 \leq p, u_0, u_1, v \leq \infty, u_1 < u_0$ and $\sigma > 0$ with $\sigma > N(\frac{1}{u_1} - \frac{1}{\max(p',v')})$. Define*

$$\Psi^{-1}(\tau) = \tau^{\frac{\sigma}{N}+\frac{1}{u_1'}-\frac{1}{\min(2,u_0)}} \quad \Phi^{-1}(\tau). \tag{36}$$

Then the eigenvalues of any kernel $k \in L_p(\Omega; B^\sigma_{\Phi v}(\Omega))$ belong to the Lorentz-Orlicz space $l_{\Psi v}$ with

$$\|(\lambda_n(k))_{n\in\mathbb{N}}\|_{l_{\Psi,v}} \leq c\|k\|_{L_p(B^\sigma_{\Phi v})}. \tag{37}$$

Proof : Using theorem (5.3) and the proof of theorem (5.5) . □

Remark 5.8. *(i) As shown by Besov [B], The Sobolev spaces W^σ_u of integer order $\sigma \in \mathbb{N}$ satisfy the inclusion $W^\sigma_u \subset B^\sigma_{uv}$ with $v := \max(2, u)$.*
Hence, using theorem (5.5) together with remark (5.4), we can prove that, the eigenvalues of any kernel $k \in W^s_p(\Omega; W^\sigma_\Phi(\Omega))$ belong to the Lorentz-Orlicz space $l_{\Psi,v}$ with

$$\Psi^{-1}(\tau) = \tau^{\frac{s+\sigma}{N}+\frac{1}{u_1'}-\frac{1}{\min(2,u_0)}} \quad \Phi^{-1}(\tau),$$

provided that $s+\sigma > N(\frac{1}{u_1} - \frac{1}{\max(p',v')})$.
(ii) Also, we can prove that, $k \in L_p(\Omega; W_\Phi^\sigma(\Omega))$ *belong to the Lorentz-Orlicz space* $l_{\Psi,v}$ *with*

$$\Psi^{-1}(\tau) = \tau^{\frac{s+\sigma}{N} + \frac{1}{u_1'} - \frac{1}{\min(2,u_0)}} \quad \Phi^{-1}(\tau),$$

provided that $s > N(\frac{1}{u_1} - \frac{1}{\max(p',v')})$.

Corollary 5.9. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded* C^∞*-domain,* $1 \le p, q, u_0, u_1, v \le \infty, u_1 < u_0$ *and* $s+\sigma > 0$ *with* $s+\sigma > N(\frac{1}{p} + \frac{1}{u_1} - 1)$*. Define* t *by*

$$\frac{1}{t} := \frac{s+\sigma}{N} + \frac{1}{u_1'} - \frac{1}{\min(2,u_0)} + \frac{1}{u} \quad \textit{with} \quad \frac{1}{u} = \frac{1-\theta}{u_0} + \frac{\theta}{u_1},$$

$0 < \theta < 1$*. Then the eigenvalues of any kernel* $k \in B_{pq}^s(\Omega; B_{uv}^\sigma(\Omega))$ *belong to the Lorentz-Orlicz space* l_{tq} *with*

$$\|(\lambda_j(k))_{j\in\mathbb{N}}\|_{l_{tq}} \le c \quad \|k\|_{B_{pq}^s(B_{uv}^\sigma)}. \tag{38}$$

Proof : Consider $\rho(t) = t^\theta$.Then using the prove of theorem (5.5),we get the desired result. ☐

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APPLICATIONS OF A SPECIAL REPRESENTATION OF ANALYTIC FUNCTIONS

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Abstract. A special factored representation of analytic functions is applied to obtain iteration methods for solving nonlinear equations. As particular examples, the Newton method, and the Halley method are derived.

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1. INTRODUCTION.

In our earlier papers (Mahrt and Szidarovszky, 1997) we have introduced a special representation of analytic functions and have applied it to the efficient computation of the partial fraction decompositions of rational functions with poles of high multiplicity. The representation theorem can be stated as follows.

Theorem. *Let $n \geq 1$ be a positive integer and assume that functions P and Q are analytic on an open complex domain D. Let $\eta \in D$ be a given point such that for $k = 0, 1, \dots, n-1$, $Q^{(k)}(\eta) \neq 0$. Then there exists a unique sequence A_0, $A_1, \dots, A_{n-1}$ of complex numbers such that*

$$(1) \quad P(s) = Q(s)\big[A_0 + A_1 \cdot (s-\eta) + \cdots + A_{n-1} \cdot (s-\eta)^{n-1}\big] + P_1(s) \cdot (s-\eta)^n.$$

We have also shown that this representation can be obtained as the unique solution of the triangular system of linear algebraic equations

$$(2) \qquad \underline{P} = \underline{T}\,\underline{a}$$

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX where

$$\underline{T} = \begin{pmatrix} t_{00} & 0 & \cdots & 0 \\ t_{10} & t_{11} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ t_{n-1,0} & t_{n-1,1} & \cdots & t_{n-1,n-1} \end{pmatrix}$$

with

$$t_{ki} = \frac{k!}{(k-i)!} Q^{(k-i)}(\eta),$$

furthermore

$$\underline{P} = \begin{pmatrix} P(\eta) \\ P'(\eta) \\ \vdots \\ P^{(n-1)}(\eta) \end{pmatrix} \quad \text{and} \quad \underline{a} = \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{n-1} \end{pmatrix}.$$

In this short paper we will show how this representation can be used to construct iteration schemes for solving nonlinear equations.

2. CONSTRUCTION OF ITERATION METHODS.

In this paper we assume that we want to find a locally unique solution of equation

$$P(s) = 0 \tag{3}$$

where P is a function satisfying the conditions of the Theorem.

We will first show how the well known Newton method results from representation (1). Select $n = 2$ and $Q(x) \equiv 1$. In order to use equation (2) we need first to compute the corresponding matrix elements t_{ki}:

$$t_{00} = \frac{0!}{0!} Q^{(0)}(\eta) = 1,$$
$$t_{10} = \frac{1!}{1!} Q'(\eta) = 0,$$

and

$$t_{11} = \frac{1!}{0!} Q^{(0)}(\eta) = 1,$$

therefore equation (2) has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} P(\eta) \\ P'(\eta) \end{pmatrix}$$

showing that

$$A_0 = P(\eta) \quad \text{and} \quad A_1 = P'(\eta).$$

Therefore the representation shows that

$$P(s) = P(\eta) + P'(\eta)(s-\eta) + P_1(s)(s-\eta)^2 = 0.$$

Let $\eta = x_m$, then from this equation

$$s = x_m - \frac{P(x_m)}{P'(x_m)} - \frac{P_1(s)}{P'(x_m)}(s - x_m)^2. \tag{4}$$

By omitting the last term we get the approximation

$$x_{m+1} = x_m - \frac{P(x_m)}{P'(m_m)} \tag{5}$$

which is the well known Newton method.

Select next $n = 2$ and $Q(x) = [P'(x)]^{1/2}$. Then we have

$$\begin{aligned} t_{00} &= \frac{0!}{0!} \cdot P'(\eta)^{1/2}, \\ t_{10} &= \frac{1!}{1!} \cdot \frac{1}{2} P'(\eta)^{-1/2} P''(\eta), \end{aligned}$$

and

$$t_{11} = \frac{1!}{0!} \cdot P'(\eta)^{1/2},$$

therefore equation (2) can be written as

$$\begin{pmatrix} P'(\eta)^{1/2} & 0 \\ \frac{1}{2}P'(\eta)^{-1/2}P''(\eta) & P'(\eta)^{1/2} \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} P(\eta) \\ P'(\eta) \end{pmatrix}$$

which implies that

$$A_0 = P(\eta)P'(\eta)^{-1/2}$$

and

$$A_1 = \left(P'(\eta) - \frac{1}{2} P(\eta) P'(\eta)^{-1} P''(\eta) \right) P'(\eta)^{-1/2}.$$

Therefore representation (1) has the form

$$\begin{aligned} P(s) = P(\eta) &+ \left(P'(\eta) - \frac{1}{2} P(\eta) P'(\eta)^{-1} P''(\eta) \right) (s - \eta) \\ &+ \; P_1(s)(s - \eta)^2 = 0. \end{aligned}$$

Let $\eta = x_m$, then from this equation

$$\begin{aligned} s = x_m &- \frac{P(x_m)P'(x_m)^{-1}}{1 - \frac{1}{2}P(x_m)P'(x_m)^{-2}P''(x_m)} \\ &- \frac{P_1(s)}{P'(x_m) - \frac{1}{2}P(x_m)P'(x_m)^{-1}P''(x_m)}(s - x_m)^2. \end{aligned} \tag{6}$$

By omitting the last term we get the approximation

$$x_{m+1} = x_m - \frac{P(x_m)P'(x_m)^{-1}}{1 - \frac{1}{2}P(x_m)P'(x_m)^{-2}P''(x_m)}, \tag{7}$$

which is the method of tangent hyperbolas or the Halley method.

Select finally $n = 2$ and $P(x) = [P'(x)]^\alpha$ where $\alpha > 0$ is a real number. Then

$$
\begin{aligned}
t_{00} &= \frac{0!}{0!}\,P'(\eta)^\alpha,\\
t_{10} &= \frac{1!}{1!}\,\alpha P'(\eta)^{\alpha-1}P''(\eta),
\end{aligned}
$$

and

$$t_{11} = \frac{1!}{0!}\,P'(\eta)^\alpha,$$

therefore equation (2) has the form

$$\begin{pmatrix} P'(\eta)^\alpha & 0 \\ \alpha P'(\eta)^{\alpha-1}P''(\eta) & P'(\eta)^\alpha \end{pmatrix}\begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} P(\eta) \\ P'(\eta) \end{pmatrix}$$

from which we have

$$A_0 = P(\eta)P'(\eta)^{-\alpha}$$

and

$$A_1 = \big(P'(\eta) - \alpha P(\eta)P'(\eta)^{-1}P''(\eta)\big)P'(\eta)^{-\alpha}.$$

Therefore representation (1) can be written as

$$
\begin{aligned}
P(s) = P(\eta) &+ \big(P'(\eta) - \alpha P(\eta)P'(\eta)^{-1}P''(\eta)\big)(s-\eta)\\
&+\ P_1(s)(s-\eta)^2 = 0.
\end{aligned}
$$

By selecting $\eta = x_m$ we obtain

$$
\begin{aligned}
s = x_m &- \frac{P(x_m)P'(x_m)^{-1}}{1 - \alpha P(x_m)P'(x_m)^{-2}P''(x_m)}\\
&- \frac{P_1(s)}{P'(x_m) - \alpha P(x_m)P'(x_m)^{-1}P''(x_m)}(s-x_m)^2.
\end{aligned}
\tag{8}
$$

By omitting the last term we obtain the generalized Halley method:

$$x_{m+1} = x_m - \frac{P(x_m)P'(x_m)^{-1}}{1 - \alpha P(x_m)P'(x_m)^{-2}P''(x_m)}. \tag{9}$$

The convergence rate of these methods can be analyzed by using equations (4), (6), and (8). The details are not given here, since they are standard techniques. Note however that as it can immediately be seen from (4), (6) and (8), methods (5), (7) and (9) are of order at least 2. Moreover if η is a zero of multiplicity β for function P_1 then the order of convergence of these methods becomes at least $\beta + 2$. In particular in Argyros (1998) it was shown that method (9) for $\alpha = \frac{1}{2}$, is of order at least 3 and for quadratic equations of order at least 4. Furthermore the results mentioned here hold in the more general setting of a Banach space. We mention finally that by different selection of $Q(x)$ different methods can be obtained. A summary of the most popular iteration methods can be found, for example, in Argyros and Szidarovszky (1993).

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