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RELAXING CONVERGENCE CONDITIONS FOR THE JARRATT METHOD

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Abstract. We establish a convergence theorem for the Jarratt method by using recurrence relations. The purpose of this note is to relax convergence conditions and give an example where our convergence theorem can be applied but not other ones.

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Key Words and Phrases. Jarrat method, nonlinear equations in Banach spaces fourth-order method, recurrence relations.

1. INTRODUCTION.

Although multipoint iteration functions are not much used in practice, one interesting class of this type of functions, studied by Traub [4], is very efficient when the equation to be solved is such that the evaluation of the first derivative is rapid compared with the function. An example of this occurs when the function is defined by an integral.

In particular, Argyros, Chen and Qian [1] studied the Jarrat method, a two-point iterative process of order four and defined, for all $n \geq 0$, by

$$\begin{gathered} y_n = x_n - [F'(x_n)]^{-1}F(X_n) = x_n - \Gamma_n F(x_n), \\ z_n = x_n + \frac{2}{3}(y_n - x_n) = x_n - \frac{2}{3}\Gamma_n F(x_n) \\ H(x_n, y_n) = \Gamma_n[F'(z_n) - F'(x_n)], \\ x_{n+1} = y_n - \frac{3}{4}[I + \frac{3}{2}H(x_n, y_n)]^{-1}H(x_n, y_n)(y_n - x_n) \end{gathered} \tag{1}$$

where F is a nonlinear operator defined in an open convex domain Ω of a Banach space X with values in another Banach space Y. It has been assumed that F has a first order continuous Fréchet derivative on $\Omega, F'(x_0)^{-1}$ exists for $x_0 \in \Omega$.

These authors give in [1] a Kantorovich-like convergence theorem, in which the conditions are

$$\begin{gathered} \| F''(x) \| \leq L_1, \| F''(x) - F''(y) \| \leq L_2 \| x - y \|, x, y \in \Omega \\ \| \int_0^1 \left[F''[x + t(y-x)](1-t) - \frac{1}{2}F''[x + \frac{2}{3}t(y-x)] \right] dt \| \leq L_3 \| x - y \|^2 . \end{gathered} \tag{2}$$

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In this note, we give another convergence theorem for operator equations. In order to provide the convergence of (1), it is only assumed

$$\| F'(x) - F'(y) \| \leq K \| x - y \|, x, y, \in \Omega, \tag{3}$$

instead of (2). Observe that we can apply the method (1) under the same condition (3) than for the Newton method (see[5]). Finally, an example where the conditions (2) fail but not (3) is presented.

Denote

$$\overline{B(x,r)} = \{y \in X; \| y - x \| \leq r\} \qquad \text{and} \qquad B(x,r) = \{y \in X; \| y - x \| < r\}.$$

2. A CONVERGENCE THEOREM.

Theorem 2.1. *Let F be a nonlinear once Fréchet-differentiable operator in an open convex domain Ω of a Banach space X with values in another Banach space Y. Let $x_0 \in \Omega$ and suppose that $\Gamma_0 = F'(x_0)^{-1}$ exists. Let us assume*

$$\| \Gamma_0 \| \leq \alpha, \| y_0 - x_0 \| \leq \beta, \| F'(x) - F'(y) \| \leq K \| x - y \|, x, y \in \Omega.$$

If $a_0 = \alpha\beta K \leq s = 0.292246\ldots$ (s is the smallest positive root of polynomial $q(x) = 2x^4 - 17x^3 + 48x^2 - 40x + 8$) and $\overline{B(x_0,r)} \subset \Omega$, where

$$r = \left(1 + \frac{a_0}{2(1-a_0)}\right)\frac{1}{1-M_0} \| y_0 - x_0 \|,$$

$$M_0 = \frac{a_0(8 - 8a_0 + a_0^2)}{8(1-b_0)(1-a_0)^2} \qquad \textit{and} \qquad b_0 = a_0\left(1 + \frac{a_0}{2(1-a_0)}\right),$$

then the sequence $\{x_n\}$ given by (1) is well-defined, $x_n, y_n \in B(x_0, r)$, for all $n \geq 0$, and converges at least R-quadratically to a solution $x^* \in B(x_0, r)$ of the equation $F(x) = 0$. Moreover the solution x^* is unique in $B(x_0, \frac{2}{\alpha K} - r) \cap \Omega$.

Proof. Let us define, for all $n \geq 1$, the real sequences

$$a_0 = K\alpha\beta, b_n = a_n\left(1 + \frac{a_n}{2(1-a_n}\right), M_n = \frac{a_n(8 - 8a_n + a_n^2)}{8(1-n_n)(1-a_n)^2},$$

$$a_n = \frac{a_{n-1}^2(8 - 8a_{n-1} + a_{n-1}^2)}{8(1-b_{n-1})^2(1-an-1)^2} = a_{n-1}\frac{M_{n-1}}{1-b_{n-1}}.$$

Under the hypotheses mentioned above $\{a_n\}$ and $\{b_n\}$ are decreasing sequences, besides

$$\| \frac{3}{2}H(x_0, y_0) \| \leq K \| \Gamma_0 \| \| y_0 - x_0 \| \leq a_0 < 1.$$

From the last conditions, the existence of x_1 is guaranteed and

$$\| x - 1 - x_0 \| \leq \left(1 + \frac{a_0}{2(1-a_-)}\right) \| y_0 - x_0 \| .$$

Applying mathematical induction on n we can prove the following statements for all $n \geq 1$

$$(I_n) \qquad \begin{aligned} &a_{n+1} \leq \gamma^{2^n} a_n < \gamma^{2^{n+1}-1} a_0 \qquad \text{where} \qquad \gamma = a_1/a_0, b_{n+1} \leq \gamma^{2^n} b_n < \gamma^{2^{n+1}-1} b_0 \\ &\text{and} M_{n+1} \leq \gamma^{2^{n+1}-1} M_0, \end{aligned}$$

$$(II_n) \qquad \| \Gamma_n F'(x_{n-1}) \| \leq \frac{1}{1-b_{n-1}},$$

$$(III_n) \qquad \| \frac{3}{2} H(x_n, y_n) \| \leq K \| \Gamma_n \| \| y_n - x_n \| \leq a_n,$$

$$(IV_n) \qquad \| y_n - x_n \| \leq M_{n-1} \| y_{n-1} \| y_{n-1} - x_{n-1} \|,$$

$$(V_n) \qquad \| x - n + 1 - x_n \| \leq \left(1 + \frac{a_n}{2(1-a_n)}\right) \| y_n - x_n \|,$$

Is easy to prove that the function $g(x,y) = \frac{x(x^2-8x+8)}{8(1-y)(1-x)^2}$ is increasing in its first and second arguments for $x, y \in (0,1)$. Moreover $g(\gamma x, \gamma y) \leq \gamma g(x,y)$, then it follows that $M_1 \leq \gamma M_0$. Thus we have

$$\begin{aligned}
a_2 = a_1 \frac{M_1}{1-b_1} \leq \gamma a_0 \frac{\gamma M_0}{1-b_0} &= \gamma^2 a_1 < \gamma^3 a_0, \\
b_2 = a_2 \left(1 + \frac{a_2}{2(1-a_2)}\right) \leq \gamma^2 a_1 \left(1 + \frac{a_1}{2(1-a_1)}\right) &= \gamma^2 b_1 < \gamma^3 b_0, \\
M_2 = g(a_2, b_2) = g(\gamma^2 a_1, \gamma^2 b_1) < \gamma^2 g(a_1, b_1) &= \gamma^2 M_1 < \gamma^3 M_0.
\end{aligned}$$

This prove $[I_1]$. On the other hand Γ_1 exists, and $[II_1]$ is true by the Banach lemma. Taking into account the Taylor's formula

$$F(x_1) = -\frac{3}{4}\left[I + \frac{3}{2} H(x_0, y_0)\right]^{-1} H(x_0, y_0)(y_0 - x_0) + \int_{x_0}^{x_1} [F'(x) - F'(x_0)] dx.$$

So, it follows $[III_1]$,$[IV_1]$,$[IV_1]$ and $[V_1]$ from (1). Furthermore $x_1, y_1, z_1 \in \overline{B(X_0, r)}$. As the sequences $\{a_n\}$ and $\{b_n\}$ are decreasing, by induction we obtain $[I_{n+1}], [III_{n+1}], [IV_{n+1}]$ and $[V_{n+1}]$.

Besides,

$$\begin{aligned}
||x_{n+1} - x_n|| &\leq \left(1 + \frac{a_0}{2(1-a_0)}\right) M_{n-1} ||y_{n-1} - x_{n-1}|| \\
&\leq \cdots \leq \left(1 + \frac{a_0}{2(1-a_0)}\right) \prod_{i=0}^{n-1} M_j ||y_0 - x_0|| \\
\leq \left(1 + \frac{a_0}{2(1-a_0)}\right) \prod_{j=0}^{n-1} \gamma^{2^j - 1} M_0 ||y_0 - x_0|| &= \left(1 + \frac{a_0}{2(1-a_0)}\right) \gamma^{2^n} \frac{M_0^n}{\gamma^{n+1}},
\end{aligned}$$

and $\{x_n\}$, given by (1), is a Cauchy sequence, since $M_0 < 1$ and $\gamma < 1$. As a result, the sequence $\{x_n\}$ converges to $x^* \in \overline{B(x_0, r)}$. We now deduce that $F(x^*) = 0$ from the continuity of F and taking into account that $||y_n - x_n|| \leq M^n ||y_0 - x_0|| \to 0$ when $n \to \infty$.

On the other hand, as

$$
\begin{aligned}
||x_{n+m} - x_n|| &\le ||x_{n+m} - x_{n+m-1}|| + \cdots + ||x_{n+1} - x_n|| \\
&\le \left(1 + \frac{a_0}{2(1-a_0)}\right)\left[\gamma^{2^{n+m-1}} \frac{M_0^{n+m-1}}{\gamma^{n+m}} + \cdots + \gamma^{2^n} \frac{M_0^n}{\gamma^{n+1}}\right] ||y_0 - x_0|| \\
&\le \left(1 + \frac{a_0}{2(1-a_0)}\right)\gamma^{2^n - 1}\left[\left(\frac{M_0}{\gamma}\right)^{n+m-1} + \cdots + \left(\frac{M_0}{\gamma}\right)^n\right] ||y_0 - x_0|| \\
&\le \left(1 + \frac{a_0}{2(1-a_0)}\right)\frac{(1-b_0)^n[1-(1-b_0)^m]}{b_{0\gamma}}\gamma^{2^n} ||y_0 - x_0||,
\end{aligned}
$$

we have

$$
||x^* - x_n|| \le \left(1 + \frac{a_0}{2(1-a_0)}\right)\frac{(1-b_0)^n}{b_0\gamma}\beta\gamma^{2^n}
$$

therefore this process converges at least R-quadratically [3].

Finally, to show uniqueness, let us assume that $y^* \in B(x_0, \frac{2}{aK} - r) \cap \Omega$ is another solution of $F(x) = 0$. Following the technique given by Argyros and Chen in [2]

$$
0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^*))dt(y^* - x^*).
$$

3. EXAMPLE.

We present an example where the assumptions (2), fail but the conditions of Theorem 2.1 are fulfilled.

Let us consider the system of equations $F(x, y) = 0$ where $F : (-\frac{1}{2}, \frac{3}{2}) \times (-\frac{1}{2}, \frac{3}{2}) \to \mathbb{R}^2$ such that

$$
F(x, y) = (x^3 lnx^2 + 4y - 1/8, x(y-4)).
$$

If we choose $x_0 = (0, 0)$, the F does not satisfy the Lipschitz condition for F" given in (2).

On the other hand, we can apply Theorem 2.1, since

$$
\alpha = ||\Gamma_0|| = 1/4, b = ||y_0 - x_0|| = 1/32, K = 22.30,
$$

and consequently, $a_0 = K\alpha\beta = 0.1742187 \le s = 0.292246\ldots$. As a result, we can only study convergence for this system of equations under the hypotheses of Theorem 2.1.

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