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A LOCAL CONVERGENCE ANALYSIS AND APPLICATIONS OF NEWTON'S METHOD UNDER WEAK ASSUMPTIONS

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Abstract. We provide a local convergence theorem for Newton's method in a Banach space setting. We use Lipschitz, center Lipschitz conditions and small majorizing sequences. It turns out that this way our convergence radius (ball) is larger and the error bounds finer than before. This observation is very important in computational mathematics, and finds several applications such as the construction of efficient mesh independence refinement strategies.

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1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1}$$

where, F is defined on the closed ball $\overline{U}(x^*, R) = \{x \in X \mid \|x - x^*\| \leq R\}$ $(R \geq 0)$ which is a subset of a Banach space X with values in a Banach space Y.

The famous Newton–Kantorovich method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0) \tag{2}$$

has been used extensively to generate a sequence converging to x^* under various assumptions. For a survey of such results we refer the reader to [3], [7], [8] and the references there. The assumptions used involve Lipschitz conditions on the Fréchet derivative $F'(x)$ of $F(x)$, Lipschitz constants [7], [8] and or small majorizing functions [2], [3].

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Here we use Lipschitz as well as center Lipschitz conditions. The center Lipschitz conditions are used for the computation of the upper bounds on $\|F'(x)^{-1}F'(x^*)\|$.

This way we manage to obtain more precise error bounds and a larger convergence radius than before [8]. This observation is very important in computational mathematics, and finds many applications such as the construction of efficient mesh independence refinement strategies [1], [3], [5], [6].

Finally, we complete this study with a numerical example to show that our convergence radius (ball) and error bounds are finer than before.

2. Local Convergence Analysis of Newton's Method

We present the following local result for the convergence of Newton's method:

Theorem 1. *Let $F\colon \overline{U}(x^*, R) \subseteq X \to Y$ be a Fréchet-differentiable operator for some $R \geq 0$, x^* such that $F(x^*) = 0$, and $F'(x^*)^{-1} \in L(Y, X)$. Moreover, assume*:

$$\|F'(x^*)^{-1}[F'(x) - F'(y)]\| \leq w(\|x - y\|), \tag{3}$$

$$\|F'(x^*)^{-1}[F'(x) - F'(x^*)]\| \leq w_0(\|x - x^*\|) \tag{4}$$

for all $x, y \in \overline{U}(x^*, R)$, and

$$\overline{U}(x^*, r_0) \subseteq \overline{U}(x^*, R), \tag{5}$$

where, functions $w, w_0\colon [0, R] \to [0, \infty)$ are monotonically increasing with

$$\lim_{r \to 0} w(r) = 0, \tag{6}$$

$$0 \leq w_0(r) \leq w(r) \quad (0 \leq r \leq R), \tag{7}$$

and r_0 is the positive zero of function f given by

$$f(r) = \int_0^1 w(tr)dt + w_0(r) - 1. \tag{8}$$

Then, Newton's method $\{x_n\}$ $(n \geq 0)$ generated by (2) is well defined, remains in $\overline{U}(x^*, r_0)$ for all $n \geq 0$, and converges to x^* provided that $x_0 \in U(x^*, r_0)$. Moreover the following error bounds hold for all $n \geq 0$:

$$\|x_{n+1} - x^*\| \leq \frac{\int_0^1 w(t\|x_n - x^*\|)dt}{1 - w_0(\|x_n - x^*\|)} \|x_n - x^*\|. \tag{9}$$

Proof. Let $x \in U(x^*, r_0)$. It follows from (4) that

$$\begin{aligned} \|F'(x^*)^{-1}[F'(x) - F'(x^*)]\| &\leq w_0(\|x - x^*\|) \\ &< w_0(r_0) \leq 1 \quad \text{(by (4) and (8))}. \end{aligned} \tag{10}$$

It follows from (10) and the Banach Lemma on invertible operators [7] $F'(x)^{-1}$ $F'(x^*)$ exists and

$$\|F'(x)^{-1}F'(x^*)\| \leq [1 - w_0(\|x - x^*\|)]^{-1}. \tag{11}$$

In particular we have: $x_0 \in U(x^*, r_0)$. Assume $x_k \in U(x^*, r_0)$ for all $k \leq n$. Using (2), (3), (8) and (11) we get in turn:

$$\begin{aligned}\|x_{k+1} - x^*\| &= \Bigg\| [F'(x_k)^{-1} F'(x^*)] \\ &\quad \cdot \left[F'(x^*)^{-1} \int_0^1 (F'(x_k + t(x^* - x_k)) - F'(x_k))(x^* - x_k) dt \right] \Bigg\| \\ &\leq \frac{\int_0^1 w(\|x_k - x^*\| + t\|x_k - x^*\| - \|x_k - x^*\|) dt}{1 - w_0(\|x_k - x^*\|)} \|x_k - x^*\| \\ &< \frac{\int_0^1 w(tr_0)}{1 - w_0(r_0)} \|x_k - x^*\| = \|x_k - x^*\|,\end{aligned} \tag{12}$$

which shows (9), $x_{k+1} \in U(x^*, r_0)$, and $\lim_{k \to \infty} x_k = x^*$.

That completes the proof of Theorem 1. □

Remark 1. Condition (4) can be replaced by

$$\|F'(x)^{-1} F'(x^*)\| \leq [1 - w_0(\|x - x^*\|)]^{-1}, \tag{13}$$

for all $x \in \overline{U}(x^*, R)$ [2], [3]. Similarly we can show a local result using function $\overline{w}$ instead of w defined by

$$\overline{w}(r) = \sup\{w(v_1) + w_0(v_2) \colon v_1 + v_2 = r\}. \tag{14}$$

Theorem 2. *Under the hypotheses of Theorem 1, denote by r_1 the positive zero of function g, given by*

$$g(r) = \int_0^1 [w((1+t)r) - w_0(r)] dt + w_0(r) - 1, \tag{15}$$

and further assume:

$$\overline{U}(x^*, r_1) \subseteq \overline{U}(x^*, R). \tag{16}$$

Then Newton's method $\{x_n\}$ $(n \geq 0)$ generated by (2) is well defined, remains in $\overline{U}(x^*, r_1)$ for all $n \geq 0$, and converges to x^* provided that $x_0 \in U(x^*, r_1)$. Moreover the following hold:

$$\begin{aligned}&\|x_{n+1} - x^*\| \leq \\ &\frac{\int_0^1 [\overline{w}(\|x_n - x^*\| + t\|x_n - x^*\|) - w_0(\|x_n - x^*\|)] dt}{1 - w_0(\|x_n - x^*\|)} \|x_n - x^*\| \ (n \geq 0)\end{aligned} \tag{17}$$

and

$$r_0 \leq r_1. \tag{18}$$

Proof. The proof is exactly the same with the corresponding one in Theorem 1 but to arrive at (12) we use

$$
\begin{aligned}
(19)\qquad & \|x_{k+1} - x^*\| \\
& \leq \frac{\int_0^1 w(\|x_k - x^*\| + t\|x_k - x^*\| - \|x_k - x^*\|)dt}{1 - w_0(\|x_k - x^*\|)}\|x_k - x^*\| \\
& \leq \frac{\int_0^1 [\overline{w}(\|x_k - x^*\| + t\|x_k - x^*\|) - w_0(\|x_k - x^*\|)]dt}{1 - w_0(\|x_k - x^*\|)}\|x_k - x^*\| \\
(20)\qquad & < \|x_k - x^*\|.
\end{aligned}
$$

Finally estimate (18) follows from the observation that

$$0 = f(r_0) \leq g(r_0). \tag{21}$$

That completes the proof of Theorem 2. □

3. Application

In order to compare Theorems 1 and 2 with earlier results we state the following theorem. The proof of the following local result can be found in [8].

Theorem 3. *Let $F\colon D \subseteq X \to Y$ be a Fréchet-differentiable operator. Assume: there exist a simple zero x^* of equation $F(x) = 0$, a parameter $p \geq 0$ such that*

$$\|F'(x^*)^{-1}[F'(x) - F'(y)]\| \leq p\|x - y\| \quad \textit{for all } x, y \in D \tag{22}$$

and

$$\overline{U}(x^*, q) \subseteq D, \tag{23}$$

where,

$$q = \frac{2}{3p}\,. \tag{24}$$

Then, Newton's method $\{x_n\}$ $(n \geq 0)$ generated by (2) is well defined, remains in $\overline{U}(x^*, q)$ for all $n \geq 0$ and converges to x^*, provided that $x_0 \in \overline{U}(x_0, q)$. Moreover the following error bounds holds for all $n \geq 0$

$$\|x_{n+1} - x^*\| \leq \frac{p\|x^* - x_n\|^2}{2[1 - p\|x^* - x_n\|]} = \alpha_n. \tag{25}$$

Remark 2. As noted in [1], [3], [5], [6], [7] the local results obtained here can be used for projection methods such as Arnoldi's, the generalized minimum residual method (GMRES), the generalized conjugate residual method (GCR), for combined Newton/finite-difference projection methods and in connection with the mesh independence principle in order to develop the cheapest mesh refinement strategies.

Remark 3. The local results obtained here can also be used to solve equations of the form $F(x) = 0$, where F' satisfies the autonomous differential equation [3], [5], [7]:

$$F'(x) = T(F(x)), \tag{26}$$

where, $T\colon Y \to X$ is a known continuous operator. Since $F'(x^*) = T(F(x^*)) = T(0)$, we can apply the results obtained here without actually knowing the solution x^* of equation (1).

We complete this section with a numerical example.

Example 1. Let $X = Y = \mathbf{R}$, $D = U(0,1)$, $R = 1$, and define function F on D by

$$F(x) = e^x - 1. \tag{27}$$

Set $w(r) = pr$, $w_0(r) = p_1 r$, $r \in [0,1]$. Then it can easily be seen that we can set $T(x) = x + 1$ in (26). Using (3), (4), (22), we get $p = e$. Moreover since $x^* = 0$ we obtain in turn

$$\begin{aligned} F'(x) - F'(x^*) &= e^x - 1 = x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \\ &= \left(1 + \frac{x}{2!} + \cdots + \frac{x^{n-1}}{n!} + \cdots\right)(x - x^*) \end{aligned}$$

and for $x \in U(0,1)$, $\|F'(x) - F'(x^*)\| \leq (e-1)\|x - x^*\|$. That is, $p_1 = e - 1$. Using (8) and (24) we obtain respectively:

$$q = .245252961 \tag{28}$$

and

$$r_0 = .254028662. \tag{29}$$

That is, our convergence radius r_0 is larger than the corresponding one q due to Rheinboldt [8]. This observation is very important in computational mathematics (see also Remark 2). Moreover the corresponding error bounds (9) (see also (25)) are given by

$$\|x_n - x^*\| \leq \beta_n = \frac{p\|x_n - x^*\|^2}{2[1 - p_1\|x_n - x^*\|]} \quad (n \geq 0), \tag{30}$$

and

$$\beta_n < \alpha_n \quad (n \geq 0), \tag{31}$$

since $p_1 < p$. Hence our claims made at the introduction are justified.

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