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ALGEBRAS OF FUNCTIONS ON GROUPOID OF SOME SPECIAL FOLIATIONS

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Abstract. Here we study foliated manifold M, F. We assume that foliation F is given by locally free action of the commutative Lie group H. We also assume that M possess Ehresmann connection (i.e. a full connection transversal to the leaves of the foliation F).

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Introduction.

We obtain here certain invariants of functional algebras describing general properties of leaves and their mutual location on the manifold.

As soon as action of the commutative group H on M induces action of the universal covering $\widetilde{H}$ of H, say $\tilde{g}p = \rho(\tilde{g})p$, here $\tilde{g} \in \widetilde{H}$, $p \in M$, $\rho : \widetilde{H} \to H$ being the universal cover, we can let H be simply-connected, even R^n.

Let us point out some crucial definitions and facts.

We can associate groupoid of homotopy classes of paths to any leaf $L \subset M$ with fixed range and source. Let us denote its total space by $G(L)$. Projections on the beginning and the end are denoted respectfully s and $r : G(L) \to L$. There are certain spaces $G^x = \{\gamma \in G | r(\gamma) = x\}$, $G_y = \{\gamma \in G | s(\gamma) = y\}$, $G^x_y = \{\gamma \in G | r(\gamma) = x, s(\gamma) = y\}$.

Now we can associate holonomy groupoid to the foliated manifold M as in the Section 2 of [7].

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1. [1], [7] Groupoid G with base X is called Lie groupoid in case the following conditions are met:
1) G is a manifold,
2)s and $r : G \to X$ are surjective submersions,
3)multiplication of paths and embedding $\iota : X \to G$ are smooth functions.

Groupoids considered in this article are Lie groupoids due to the results from [8]. It is proved there that groupoid of the foliation generated by group action is in fact fibre bundle.

2. [1], [7] Left Haar system on locally compact groupoid G is the family of measures $(\lambda^x)_{x \in L}$ on G, such that
1) Support of λ^x is precisely G^x,
2) System of measures is invariant with respect to the action $L_\gamma : G^{s(\gamma)} \to G^{r(\gamma)}$ for any $\gamma \in G_x = \{\gamma \in G(L) | r(\gamma) = x\}$, peculiarly G^x_x
3) For all $\phi \in C_c(G(L))$ ($C_c(G(L))$ being variety of complex-valued functions with compact support) map

$$x \mapsto \lambda(\phi)(x) = \int \phi d\lambda^x$$

defines continuous function on L.

The equivalence relation on groupoid G^x given by: $\gamma_1 \equiv \gamma_2 \Leftrightarrow \exists \gamma \in G^x_x \gamma_1 = \gamma_2 \gamma$ is measurable [5] since groupoids under consideration are Lie groupoids, moreover its factor space $G^x / \equiv$ is precisely space of units of $G(L)$ being L itself. Hence we can define measure $\lambda^{x'}$ (called tangential measure) on the set of equivalence classes as factor-measure of the λ^x .

Thus we can associate algebra $F(L)$ of measurable functions for any leaf L of the foliation F.

We also assume the existence of so called invariant transverse measure ν [7] defined on the set of leaves of the foliated manifold M. One then can form measure μ on M with the help of the transversal and tangential measures given above.

Let us first take into consideration the following set of functions on groupoid G:

$$C(G) = \{f : G \to C | \int_M \int_{G^x} f(\gamma) \overline{f}(\gamma) d\lambda^{x'} d\mu < \infty\}$$

3. Let us consider variety of continuous, infinitely differentiable functions with compact on the total space of the groupoid G $C^\infty_c(G)$.

Let us define multiplication and involution on this set as follows:

$$f * g = \int_{G^{s(\gamma)}} f(\gamma_1 \gamma) g(\gamma_1^{-1}) d\lambda^{s(\gamma)}(\gamma_1);$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

Let us define $\|f\|_x = \|f\|_{L_2(G^{s(\gamma)})}$ and $\|f\| = \sup_{x \in M} \|f\|_x$

Closure of $C^\infty_c(G)$ in this norm is C^*-algebra which we will denote $C^*_r(G)$.

One can show that C^* algebra $C^*_r(G)$ can be represented as subalgebra of the algebra of bounded operators acting on H $B(H)$

Common algebra.

Let us take into consideration a manifold M with foliation F, generated by action of the commutative Lie group H. We suppose that there is also an integrable Ehresmann connection on (M, F).

Let L_1, L_2 be two (maybe the same) leaves of the foliation F and γ be a horizontal curve such that $x = \gamma(0) \in L_1$, $y = \gamma(1) \in L_2$.Then there is an isomorphism $A : G^x \leftrightarrow G^y$, which is given by Ehresmann connection. Namely, let δ be a representative of $\alpha \in G^x$. then by definition of the connection γ and δ give rise to rectangle $\Pi : [0,1] \times [0,1] \to M$, $\Pi(t,0) = \gamma(t)$,$\Pi(0,s) = \delta(t)$. Let us define $A([\delta]) = [\Pi(1,s) : [0,1] \to L_2]$. This definition is correct due to the properties of the Ehresmann connection.

Now we can define multivalent correspondence between two groupoids in the following way. Group action on any fixed leaf L_1 can be extended to the action of H on $G(L_1)$, $h([\beta]) = [h\beta]$. Let us denote by H_1 isotropy subgroup of the element $\alpha \in G^x$ (from the commutativity of H we infer that H_1 is isotropy subgroup for any element of $G(L_1)$). Thus we put in correspondence for any $\alpha \in G^x$ a subset $T(\alpha) = \{hA(\alpha) | h \in H_1\}$. Similarly for any $\beta \in G^y$ we take $T'(\beta) = \{hA^{-1}(\beta) | h \in H_2\}$, here H_2 is an isotropy subgroup for $\beta \in G^x$.

Let us introduce certain equivalence relation $\sim$ on the groupoid $G(L_1)$, generated by the relation $\equiv$, here $\alpha_1 \equiv \alpha_2$ in case $\alpha_1, \alpha_2 \in T'(\beta)$ for some β. The same relation maybe introduced on the $G(L_2)$.

We consider map $B : G(L_1)/\!\sim\, \to G(L_2)/\!\sim$. For $\alpha \in G(L_1)$, $r(\alpha) = x_1$ we take $B[\alpha] = [A(h^{-1}\alpha)]$, with $hx = x_1$.

Statement1. The map $B : G(L_1)/\!\sim\, \to G(L_2)/\!\sim$ is an isomorphism.

• It is enough to show injectivity of B. Let $[\alpha] \neq [\alpha']$, but $\beta_0 = A([\alpha]) = B([\alpha'])$. Then there $\exists \beta_i$, $i = \overline{1,k}$, $\beta_0 = B([\alpha])$, $\beta_k = B([\alpha'])$, $\beta_0 \equiv \ldots \equiv \beta_k$, $\beta_i, \beta_{i-1} \in T(\alpha_i)$. But in this case $\alpha \equiv \alpha_1 \equiv \ldots \equiv \alpha_k \sim \alpha'$ hence $\alpha \sim \alpha'$, what contradicts to the assumption. □

There is a subalgebra of A_1, let us denote it A_{12} consisting of functions constant on classes of the introduced equivalence relation, the same subalgebra may be considered in $A_{21} \subset A_2$. Clearly $A_{12} \cong A_{21}$.

Let L_1 be a fixed leaf. For any leaf L we can find subalgebra $A(L) \subset A_1$ in the given above way. Let $A_E = \cap A(L)$ (we take intersection over all leaves L).

A. Let foliation F be foliated bundle and horizontal curve γ be contractible. In this case algebra $A_E = A(F)$, F being standard leaf.

B. Foliation of the parallels on torus $T^2 = R^2/Z \times Z$ with the Ehresmann connection given by the vector fields tangent to the constant vector field $V_{(x,y)} = (a, b)$ $a/b \in R \setminus Q$. Factor of any leaf by equivalence relation from 1) won't be Hausdorff variety hence equivalence relation is not measurable.

C. For Seifert manifold M with standard leaf T^k [9] and Ehresmann connection compatible with group action. Let F be standard foliation on M. Then the following holds true:

1) Let foliation F have codimension 1 and singular leaves have Z_2 as isotropy subgroup. Then M is not oriented and $A_E \cong A_t/Z_2$, here A_t is algebra of measurable functions on the standart leaf T^k.

2) Let group T^k act locally free on M. Then in case $A_E = A_t/Z_3$ we have codimension of foliation greater than 1.

It is important to note that in case F being of codimension greater than 1 it is impossible to state that $A_E \cong A/Z_p$ for some p in general case [6].

3) $R \times C$. We can identify $(t, z) \equiv (t + m, e^{\frac{2\pi i m}{q}} z)$. Horizontal distribution is given by the condition $dt = 0$ with t coordinate on $[0, 1]$. Group $S^1 = R/Z$ acts on this manifold in natural way. $\alpha(t, z) = (t + \alpha, ze^{\frac{2\pi i \alpha}{q}})$.

We get maps $B_1 : S^1 \times R \to S^1 \times R$, $\phi(\alpha, x) = (q\alpha, qx)$ from groupoid of the typical leaf on the groupoid of the leaf which is q times covered by typical ones and $B_2 : S^1 \times R \to S^1 \times R$ vice versa. We have for $\phi \in A(F)$, with F being typical leaf, $\phi(\alpha, x) = (\frac{\alpha}{q} + \frac{2i\pi}{q}, \frac{x}{q} + \frac{2i\pi}{q})$, $i = \overline{0, q-1}$ from the groupoid of typical leaf onto groupoid of the singular one and back.

Thus in this case common subalgebra $A = A(S^1)$ coincides with algebra of the singular leaf and is subalgebra of $\frac{2\pi}{q}$-periodic functions on typical leaves.

D. Foliation on $R \times S^1$ given by integral curves of the vector field $(x, 1)$. Let us denote algebra on the leaves laying on the "left" side of the circle S^1 (let us denote them S_e) and on the "right" side by (S_i). We introduce Ehresmann connection E on $R^2 \setminus \{0\}$, in polar coordinates (ρ, φ) by equation $d\varphi = 0$. On $R^2 \setminus \{0\}$ acts S^1, $\phi(\rho, \alpha) = (\rho(\alpha + \phi), \alpha + \phi)$. Let us note that algebras of measured functions for two spirals inside S_i and outside S_e of the circle are isomorphic. Here $A^E(F) = A(S^1)$ (cm. Th. 1 [2]).

*Note*1. 1) We can extend equivalence relation to the abstract groupoids. At first we construct map from G^x to G^y then we slide along paths corresponding one to another at the first stage.

2) *(Partial case of given before equivalence relation in case Ehresmann connection is compatible with group action.) Let us build translation of one point $x \to y$, then we map path $\gamma : [0, 1] \to G\pi_{L_1} \longrightarrow L_1$ to $\gamma' : [0, 1] \to G\pi_{L_2} \longrightarrow L_2$, here first arrows coincide.*

Statement2. Let (M, F) be foliated manifold with Ehresmann connection, and L_1, L_2 be some leaves of F. In case $A_{12} = A_1$, $A_{21} = A_2$ we have L_1 diffeomorphic to L_2, moreover this diffeomorphism is compatible with group action. Inverse statement is clearly true as well the direct one.

• Let us construct map $f : L_1 \to L_2$. Let $x_0 \in L_1$, $y_0 \in L_2$ be points and γ be horizontal path, connecting x_0 and y_0, by which we construct algebra A_{12}.

For arbitrary $x = hx_0$ we define $f(x) = hy_0$. Let us show that f is diffeomorphism. By assumption $A_{12} = A_1$, $A_{21} = A_2$ and some properties of measurable functions we have in case $\alpha \sim \beta$ $\alpha = \beta$. Thus multivalent map T is here univalent moreover T' is also univalent and is inverse to T. Hence $T : G(L_1) \to G(L_2)$ is bijection of groupoids (T is isomorphism of groupoids only if Ehresmann connection is invariant with respect to the group action).

Then if $\gamma : [0, 1] \to L$, $\gamma(0) = x_0$, $\gamma(1) = hx_0$, there is $T(hx_0) = s(T(\gamma)) = hT(x_0) = hy_0$.

Differentiability of the map, induced by connection is due to the properties of the given connection. ▷

Conditions of this statement are met for example in partial case of foliations with Lie groupoids of leaves with system of smooth Haar measures.

Now let $A_1 \neq A_{12}$ or $A_2 \neq A_{21}$. For measures on each of the leaves L_1, L_2 induced by measures on groupoids $G(L_1)$ and $G(L_2)$ the following is true. Let us consider variety of equivalence classes on the groupoid $G(L_i)$. We can construct an equivalence relation on the leaf L_i itself in the way parallel to the way given on groupoids. Now let equivalence class of any point $[x]$, $x \in L_i$, $i = 1, 2$ is at most countable. We can define Borel maps $T : L_1 \to L_2/\sim$, $T' : L_2 \to L_1/\sim$. It is possible to take families of varieties K_1, K_2 consisting of points on leaves which belong to all different equivalence classes with measures and topologies, inherited from factors of leaves in case the latter have good structure and to consider maps of leaves to this varieties.

Corollary. There exists Borel measurable bijection (not a homeomorphism in general) of the elements from families $k_1 \in K_1$, $k_2 \in K_2$, $T : k_1 \leftrightarrow k_2$.

• By Theorem 4.5 from [7] there exists such a division of every leaf L_i, $i = 1, 2$ on Borel measurable subsets U_i^j that induced maps T and T' restricted to U_i^j are bijections. □

Statement3. Invariant subalgebra with respect to Ehresmann connection A_{12} does not depend on Ehresmann connection. This means that for any Ehresmann connection invariant subalgebras coincide up to isomorphism.

• Let there exist two points on the second leaf $y_1, y_2 \in L_2$. Then it is enough to take into consideration that $G^x \approx G^{y_1}$ $G^x \approx G^{y_2}$, hence $G^{y_1} \approx G^{y_2}$. The rest part of proof is true because of dependence of the last part of the construction only on the group action. □

Let us take into consideration partial case of manifold M with foliation induced by locally free action of the commutative group H and Ehresmann connection compatible with the group action.

Let $L_1, L_2 \in M$ be two leaves of the foliation. We define *covering subgroup of* L_2 *by* L_1 $H_{12} \equiv H_1/H_1 \bigcap H_2$, with H_i being isotropy subgroup of the leaf L_i, $i = 1, 2$. The same construction gives us H_{21}.

Statement4. Covering subgroup H_{12} is factor-group of the holonomy group J for the Ehresmann connection.

• Let us take the horizontal curve $\gamma : [0, 1] \to M$, $\gamma(0) = x \in L_1$, $\gamma(1) = y \in L_2$. It is enough to build surjective homomorphism $\rho : J \to H_{12}$.

Let $\alpha \in J$, $\exists a \in \pi_1(x, L_1)$, $[a] = \alpha$. There is a covering $\psi : H \to L_1$, $\psi(e) = x$. We will consider lift $\tilde{f} : [0, 1] \to H$, $\tilde{f}(0) = e$ of the path a. Let us define $g = \tilde{f}(1)$. Due to $a(0) = a(1)$ we have $g \in H_1$ and let $\phi \in H_{12}$ be its' equivalence class in $H_1/H_1 \bigcap H_2$. Now we define $\rho(\alpha) = \phi$.

If $h(t)$, $h(0) = e$ is some path in H, then thanks to invariance property of the Ehresmann connection we have a rectangle $h(t)\delta(s)$ for any horizontal curve $\delta(s)$. Thus parallel translation of the path γ along loop a is $\tilde{f}(t)\gamma$. So, if the loop a translate into itself every horizontal curve beginning in x and consequently γ then $\tilde{f}(1)\gamma(0) = \gamma(0)$ and $\tilde{f}(1)\gamma(1) = \gamma(1)$ or $\tilde{f}(1) \in H_1 \bigcap H_2$. That's because ρ is correctly defined

Surjectivity of the map is an immediate consequence of its construction. □

Corollary. For A_{12} isomorphic to A_1 but not isomorphic (by means of the connection) to A_2 leaf L_2 covers L_1 with covering group H_{21} and "standard fiber" from **Corollary 2.1**.

Structure of $C_c^\infty(G)$

Lemma1. Let M be a manifold with foliation given by action of the commutative group H. Let P be variety of points which can be connected with the given one point $x_0 \in M$ by the horizontal curve. [3]

Let P satisfy the following properties:

1) P is a manifold.

2) $\dim P = n-p$ with $n-p$ being codimension of the foliation.

3) There exists leaf S which has only one common point with P.

Then $S = H/span(\bigcup_{y \in M} H_y)$, where H_y is isotropy subgroup of the leaf L_y passing through y. By *span* we mean that generators of this group are nothing more but union of generators for H_y. Use of *span* is correct due to the fact that $H_y \subset R^n$ and group, generated by elements from the groups H_y can be considered as $span_Z H_y$

Let S' be connected subset of H such that universal cover $p : H \to S$, with $p(e) = x_0$ maps S' bijectively onto S being at the same time homeomorphism when restricted to the inner part of S'. Then there exists almost everywhere continuous bijection $\phi : M \to P \times S'$, $\phi(x) = (\pi(x), \pi'(x))$

• Let us show that for $h \in H_y$, $h \in H_{x_0}$. Due to $S \cap P = \{x_0\}$ and the property of h to map horizontal curve to horizontal one we can say that h maps horizontal curve connecting x_0 with y to horizontal curve connecting y with hx_0 but then hx_0 belongs to P. Thus $hx_0 = x_0$ and h is an element of the isotropy subgroup of the leaf S.

Without any loss in generality we can assume that $x_0 \in S$. Let us define projections $\pi_P : M \to P$, $\pi_S : M \to S$ as follows.

Let $x \in M$. Let us consider some curve connecting x_0 with x. It defines a rectangle lower side of which is a curve on P with the end point y and left side of which is a curve on S defining some point s on the universal covering space H.

Let S' be a connected subset of H such that universal cover $p : H \to S$, $p(e) = x_0$ maps S' bijectively onto S being at the same time homeomorphism restricted to the internal part of S'. Then there is in S' the only element s' with the property $s = s'h_0$ where $h_0 \in H_{x_0}$.

We define $\pi_P(x) = yh_0^{-1}$, $\pi_S(x) = s'$.

Let us take two curves $\gamma_1, \gamma_2 : [0,1] \to M$, $\gamma_1(0) = \gamma_2(0) = x_0$, $\gamma_1(1) = \gamma_2(1) = x$. They define two rectangles with lower sides $g_1^P(t)$ $g_2^P(t)$ and left sides $g_1^S(t)x_0$ and $g_2^S(t)x_0$ where $g_1^S, g_2^S : [0,1] \to H$, such that $g_1^P(0) = g_2^P(0) = x_0$, $g_1^S(0) = g_2^S(0) = e$, where e is unit element of H, $g_1^P(1) = y_1$, $g_2^P(1) = y_2$. Thanks to the uniqueness of the rectangle with given vertical and horizontal sides we have the following upper sides of them: $g_1' = g_1^P g_1^S(1)$ and $g_2' = g_2^P g_2^S(1)$ and we have $g_1'(1) = g_2'(1) = x$. So we get horizontal curve with the beginning in $g_1'(0)$ and end point in $g_2'(0)$ lying in S. After the action of $(g_1^S(1))^{-1}$, we get horizontal curve connecting x_0 with some point $x_1 \in S$. Due to 3) $x_1 = x_0$ henceforth $g_1'(0) = g_2'(0)$.

Thus $g_1^S(1)x_0 = g_1^S(1)g_1^P(0) = g_1'(0) = g_2'(0) = g_2^S(1)g_2^P(0) = g_2^S(1)x_0$ but then $g_1^S(1)^{-1}g_2^S(1)$ is an element of H_{x_0}. That is $y_1 = g_1^P(1) = g_1'(1)g_1^S(1)^{-1} = xg_1^S(1)^{-1} = g_2'(1)g_1^S(1)^{-1} = g_2^P(1)g_2^S(1)g_1^S(1)^{-1} = y_2 g_2^S(1)g_1^S(1)^{-1}$, or $y_1 g_1^S(1) = y_2 g_2^S(1)$.

So now we have a map $\phi : M \to P \times S'$

Continuity of this map almost everywhere is the consequence of bifoliated structure on M defined by P and F.

Really, let us show that inverse image of $U \times V$ $U = \circ U \subset S'$, $V = \circ V \subset P$ is open on the manifold M ($\circ V$ is an interior part of some set V).

It is enough to mention that for any cylinder $B_r(s) \times B_{r'}(p)$, $B_{r'}(p) \subset \circ S'$, $p \in P$, $s \in S'$, where $B_r(x)$ is an open ball of the radius r with the center in x there exists open ball $B_{min\{r,r'\}}(\phi^{-1}(p,s))$ in the inverse image of the cylinder.

There is no continuity only on the variety $\partial S' \subset \overline{S' \times P}$ which is the border of S'. $\square$

Corollary. Let M be a manifold with foliation given by action of the commutative group H. Let P be a variety of points which can be connected to the the given one $x_0 \in L_0 \subset M$ by horizontal curve. Let P satisfy the following conditions:

1) P is a manifold.

2)$\dim P = n - p$ $n - p$ being codimension of foliation.

Then $P \times L_0$ covers M with covering group L_0/S, $S = H/span(\bigcup_{y \in M} H_y)$, here H_y is isotropy group of the leaf L_y passing through the point y.

• It is clear that L_0 covers S and the action of L_0 on S meets $S(L_0/S) = L_0$. There exists variety of points $|L_0/S|$ from $P \times L_0$ defining the same point on M as defined in the beginning of the proof of the Lemma 1 as pairs $(hs, h^{-1}p)$, $s \in S$, $p \in P$, $h \in L_0/S$ represent different points from $P \times L_0$ but define the same curve on M. $\square$

E. Seifert manifold with the group S^1. Any manifold of this type bijectively corresponds to $C \times [0, 1)$.

F. Let us take ordinary torus T^2 with irrational foliation and connection given by parallels. Then we have covering $\phi : S^1 \times R \to T^2$

G. Spirals and circle on $R^2 \setminus 0 = S^1 \times R$ given by integral curves of the distribution $v_{r,\alpha} = (r, 1)$ in polar coordinate system. We can introduce connection E with lines $\alpha = const$ as horizontal curves. With the help of this construction we obtain isomorphism of algebras on external and internal spirals. Here everything depends on the leaf on which we choose point x if it lies on the circle we have a bijection $\phi : R \times [0, 1) \to S^1 \times R$ or in any other case covering $\phi : R \times R \to S^1 \times R$.

Statement5. Let all conditions of the previous Lemma be satisfied. Then algebra $C_c^\infty(G)$ can be embedded into $C(S') \otimes C(P) \otimes L_2(H)$.

• Let us mention that topology of the foliated bundle G over M with fiber H is weaker than the topology of the trivial bundle $P \times S' \times H$. It holds true because as it is shown in Lemma 2 M bijectively corresponds to $P \times S'$ and topology on M is stronger than that on $P \times S'$.

The rest is simple consequence of the results from [R]. $\square$

Let us consider algebra of functions $C_{r,e}^\infty$ from **Definition 2**.

Lemma2. Let all the conditions from Lemma 1 be satisfied.

Algebra $C(G_P^P)$ can be embedded into $C_{r,e}^\infty$.

• Since all paths from G_P^P are images of some paths on S, we can say that functions on them are elements from $C_{r,e}^\infty$. $\square$

We have $TM = F' \oplus F$, where F' is a transversal distribution. Let us take into consideration set Ω consisting of differential forms on M, then the ordinary differential operator $d = d' + d'' + \partial$. Here the members of the sum are the operators of the type $(1, 0)$, $(0, 1)$, $(2, -1)$, respectively.

4. Let us denote $H_F^{0,i} = \frac{Ker(d'' : \Omega^{0,i} \to \Omega^{0,i+1})}{Im(d'' : \Omega^{0,i-1} \to \Omega^{0,i})}$— leaf cohomologies [10].

Now we can formulate the following

Corollary. Let all the conditions of the Lemma 1 be met.

Group $H_F^{0,1}$ can be embedded into $C_{r,e}^*$ in case of the finite dimensional $H_1(L)$ $\forall L \in F$.

• Since all loops on each leaf are elements from isotropy subgroup of the leaf all homologies are described by loops on S in terms of Lemma 1. This means that all possible functions on homology group of any leaf are embedded into the set of functions on homology group of S.

It is enough to mention now, that for full description of the function from $H_F^{0,1}$, which is an ordinary commutative function on the homotopy class of the loops $f(\gamma_1\gamma_2) = f(\gamma_2\gamma_1)$ it is enough to find its' values on the generators, but for compactness of the latter set we must suppose to be finite. We will define function outside of this set by continuity in the way, which provides compact support. $\square$

We have now as a consequence from Theorem 6.14 of [7] and Lemma 2

Statement6. $C_r^*(G) \subset C(\pi_1(S)) \otimes C(P) \otimes L_2(H)$

Structure of the $C_r^*(G)$ with respect to the local convergence.

General notes.

5. Let $L_1 \subset M$ be some fixed leaf. Let us take restriction $C_r^*(L_1)$ ($\neq C_r^*(G(L_1))$ in general case) of the algebra $C_r^*(G)$ to this leaf. We can construct subalgebra $A(L_2) = A_{12} \subset C_r^*(L_1)$ for any other leaf L_2 by means of the equivalence relation given above. Let $C_{r,e}^*(L_1) = \bigcap\limits_{i \in I} A(L_i)$ (intersection over all leaves of the foliation).

This subalgebra is the first nontrivial invariant, defined for foliated manifold M and Ehresmann connection on the space of functions.

Of course here we do not suppose that the measure on the groupoids of the leaves is Haar one. This is a necessary condition for the non-triviality of the objects defined in this section.

H. Let us take into consideration the most trivial case of the foliations generated by product of manifolds. Then $C_{r,e}^*(M \times N)$ is $C_r^*(G)$. Here G is groupoid of any leaf N.

Let P be a manifold, defined as in the second section of this paper. Let us fix some so-called bifoliated metric g[Zh] on M which will generate one on P. Let us denote distance between points $x, y \in M$ as $d(x, y)$.

Now we can construct nondecreasing sequence of subalgebras of the algebra $C_r^*(G)$. Let us cover P by open balls of ε radii. So we can say that $P = \bigcup x \in PB_\varepsilon(x)$. We define algebra $C_{r,e}^*(B_\varepsilon(x))$ in any ball $B_\varepsilon(x)$ as above $C_{\varepsilon,r}^*$ for the entire P. The equivalence relation for any ball $B_\varepsilon(x)$ we define as equivalence relation on the manifold $sat(B_\varepsilon(x))$ with horizontal curves laying only in $P \bigcup B_\varepsilon(x)$.

6. Let us define algebra of functions $C_\varepsilon^* = \{f \in C_r^*(G) | \forall x \in P, f|_{B_\varepsilon(x)} \in C_{r,e,i}^*\}$.

It is easy to point out that $\lim\limits_{\varepsilon \longrightarrow \infty} C_\varepsilon^* = C_{r,e}^*$. Here lim means direct limit of algebras.

Now let us find necessary and sufficient conditions for the $\lim\limits_{\varepsilon \longrightarrow 0} C_\varepsilon^*$ to be $C_{r,e}^*$. We formulate them in two subsequent statements.

Statement7. Let foliated manifold M satisfy all the conditions from the introduction. $\lim\limits_{\varepsilon \longrightarrow 0} C_\varepsilon^* = C_r^*(G)$ in case the following conditions holds true:

1) $\forall x, y \in P$ $d(hx, hy) < const$ (action of H doesn't change distance between points on P).

2) there are only finitely many singular points on P.
3) All leaves of the foliation are closed.

• We can say that in this case set of functions $f_h : P \to P$ represented by isotropy group of P $H(P) = \{h \in H|hP = P\}$ is uniform continuous.

It holds true that $\forall \varepsilon > \varepsilon'$ $C^*_\varepsilon \subset C^*_{\varepsilon'}$. This is due to $satB_{\varepsilon'} \bigcap P \subset satB_\varepsilon(x) \bigcap P$ for any $x \in P$ which holds true by assumption.

This statement splits into two different cases.

$\exists \varepsilon' > 0$ $\forall \varepsilon < \varepsilon'$ $L(x) \bigcap B_\varepsilon(x) = x$

In this case it is enough to prove statement locally. Let $x \in P$ be some point lying on the singular leaf. Then $\exists \varepsilon' > 0$ $\forall \varepsilon < \varepsilon'$ $L(x) \bigcap B_\varepsilon(x) = x$. Let $B_\varepsilon(x)$ be some open ball which contains x. Now we can construct sequence of functions (f_n) such that $f_n(x) \to f(x)$ $(n \to \infty)$, $f_n \in C^\infty_\varepsilon$. Let us define this sequence as follows:

At first we must introduce two varieties $U_1(\varepsilon) = \{y \in P|\exists z \in B_\varepsilon(x) L(z) = L(y)\}$, $U_2(\varepsilon) = \{y \in P|\exists z \in P \setminus B_\varepsilon(x) L(z) = L(y)\}$. We have $\forall \varepsilon_1$ $\exists \varepsilon_2(\varepsilon_1)$ $U_2(\varepsilon_1) \bigcap U_1(\varepsilon_2) = \emptyset$ by assumption, moreover distance between these varieties is not 0.

So we define sequence $(f_n)_{n \in N}$ as follows:

$$f_n(\gamma) := 0$$

for $\gamma \in G(L(y))$ $\exists z \in P$ $L(y) = L(z)$, $z \in P \setminus B_{1/n}(x)$

$$f_n(\gamma) := f(\gamma')$$

whenever $\gamma \in G(L(y)) \simeq \gamma' \in G(L(x))$ for $y \in B_{\varepsilon_2(1/n)}(x)$.

$$f_n(\gamma) = n\rho f(\gamma')$$

for $\gamma \in G(L(y))$ $\rho = \min\{d(z, x)|L(y) = L(z)\}$.

Then we pass to the partition of unity defined by balls $B_{1/n}(x)$ $x \in P$ on the whole manifold M. $\square$

I. Seifert manifold [6]. Partial case with R^2 as the base and S^1 as leaf, passing through points $ze^{i\pi/n}$ in success. This is the example for the first case of the theorem.

J. Foliation on the cylinder $S^1 \times R$ given by circles up to some moment and spirals from that moment i.e. by vector field which has the following coordinates in natural coordinate chart $v_{(\alpha,x)} = (1, 0)$ for $x \leq 0$ $v_{(\alpha,x)} = (1, x)$ $x \geq 0$. This is a situation in which theorem doesn't hold true.

Statement8. Condition from the previous statement is satisfied for $h \in \bigcup_{x \in P} H_x$ in case $\lim_{\varepsilon \longrightarrow 0} C^*_{\varepsilon,x} = C^*_r(G)$

• Let us suppose the contrary, that is the condition of the above statement is violated.

Then there exist $x \in P$ and $\varepsilon > 0$ such that $\forall \delta > 0$ $\exists h(\delta) \in H$ $\exists y(\delta) \in B_\delta(x)$ $d(h(\delta)x, h(\delta), y) \geq \varepsilon$. This means that any function from $C^*_{\varepsilon,x}$ must take same values on the γ $s(\gamma) = y$ and γ' $s(\gamma') = hy$ but since for some relatively close to the unit e of H element $h' \in H$ (it exists due to the fact that action of H is locally free) and map $g : [0, 1] \to H$ with $g(0) = e$, $g(1) = h'$ we have different elements of

G given by the maps $g(t)x$, $g(t)y$ and $hg(t)y$ for every y. By general assumption G is Hausdorff so there exists an element from $C_r^*(G)$ corresponding to $g(t)x$ namely δ-function of $g(t)x$. Evidently, all the elements from $C_{\varepsilon,x}^*$ can't represent sequence converging to this characteristic function. $\square$

The last example from above doesn't answer this condition.

7. It is also natural to introduce algebras $C_{\varepsilon,x}^* = \{f \in C_r^*(G) | f|_{B_\varepsilon(x)} \in C_{r,e}^*(B_\varepsilon(x))\}$ Let us call them algebras of ε-neighborhood of the point x.

Of course we can apply aforesaid statements for this algebras $C_{\varepsilon,x}^*$ to converge to $C_r^*(G)$ by considering only local problem. So the statement can be reformulated in the following way:

$$d(hx, hy) = o(1)(\varepsilon \to 0), y \in B_\varepsilon(x)$$

Recent statements then holds true in case $\forall x \in P \lim_{\varepsilon \longrightarrow 0} C_{\varepsilon,x}^* = C_r^*(G)$

It is therefore interesting to find out what happens to foliated manifold in case the behavior of these algebras $C_{\varepsilon,x}^*$ is predictable.

It is easily seen that in case isotropy groups of all the leaves, passing through some ε'-neighborhood of some fixed point x coincide we will have sort of stability $C_{\varepsilon,x}^* = C_r^*(G)$.

The situation in which isotropy group of the leaf $L(x)$ is less than that of all the leaves, passing through the points in some neighborhood of x is impossible, since action of the group is assumed to be continuous.

So the most interesting situation is the situation for which we have sort of discontinuity in the point. This means that in any neighborhood of the point x there exists point y isotropy group of the leaf, passing through it is not equal to the isotropy group of the leaf

Statement9. If for almost all $x \in P$ $\exists \varepsilon > 0$ $C_{\varepsilon,x}^*$ stabilize to C_r^* then variety of points for which this convergence doesn't take place is nowhere dense or, equivalently, closure of this last variety of points in P doesn't contain subsets of dimension greater than $k-1$, $k = \dim P$.

• The question can be reduced to the investigation of the properties of the set T', consisting of points x in any neighborhood of which there exist points lying on the leaves with isotropy group different from one of the leaf, passing through x.

Reductio ad absurdum. Let us assume the contrary, then we take set of points $T' \subset P$ defined above such that $\overline{T'}$ contains at least one connected subset of the topological dimension k. Then by assumption we have in some neighborhood of any point from T' there exist points for which stabilization assumption is true. Then from the closed set $\overline{T'}$ we can extract nonsingular varieties, namely $B_\varepsilon(x)$ for some x. Of course, standard Borel measure of this closed set which clearly contains all points from the T' is less than measure of the $\overline{T'}$ so since this process can be extended to infinity and that it won't stop in case measure of the still decreasing set T' is greater than 0 we conclude that the rest of $\overline{T'}$ must be closed null set, which contradicts the assumption above. $\square$

Now it is useful to describe most suspicious subset of the set of points in which we won't have convergence of algebras $C_{\varepsilon,r}^*$.

Statement10. Set of points in every neighborhood of which we have points laying on leaves with different group of isotropy is at most $k-1$-dimensional (here k is dimension of P) in case $\overline{P} = P$.

• Let us take some subset $U \subset P$ then since action of the group H is free and $\overline{P} = P$ then group, generated by $\bigcup_{x \in U} H_x$ (sort of maximum group) must be discreet, or equivalently, finitely generated. Then there are only finalely many (say n) groups H_i, $i = \overline{1,n}$ of leaves with different isotropy groups. Now we can represent $U = \bigcup_{i=1}^{n} T_i$, here $T_i = \{x \in U | H_x = H_i\}$. Now by Baire theorem we have at least one T_j such that there exists some ball $B_\varepsilon(x) \subset \overline{T_j}$. If some other set T_i is dense in this ball then since it must be invariant under isotropy group H_j then $H_j \subset H_i$ and since inverse must be true then $H_i = H_j$ this contradicts the assumption that these sets are different. So $\forall i \in \overline{1,n} \setminus j$ T_i is nowhere dense in T_i. Thus the set $B_\varepsilon(x) \setminus \bigcup_{i=1, i \neq j}^{n} \overline{T_i}$ satisfies condition of the above statement. $\square$

Let us denote set of points, in which convergence doesn't take place P_s.

Corollary. If H is compact group then set P_s is nowhere dense.

• In this case there exists $K \in R^+$ such that $\forall h \in H$ $\forall x, y \in P$ $d(hx, hy) \leq Kd(x, y)$. Now we can apply statement 4.1 to some neighborhood of almost any point of P. $\square$

Construction of functions describing behavior of C^*_ε.

In this section we construct and consider the behavior of certain functions, which correspond to algebras C^*_ε and $C^*_{\varepsilon,x}$ in some cases. This can help one to estimate "speed" of convergence of algebras $C^*_{\varepsilon,x}$ to C^*_r if there is one. Here we must assume that $\dim P_s \leq k - 1$.

The first variant.

Let us take into consideration space of functions $C(G)$ from the introductory section, Let us recall that norm in this space is given by scalar product as follows:

$$(f, g) = \int\limits_M \int\limits_{G^y} f(\gamma)\overline{g(\gamma)} d\lambda^y d\mu$$

Now we can apply here method used for construction of C^*_ε as in the beginning of the preceding section.

8. As above we can define subspaces of $C(G)$ and $F(G(L)) = \{f : G \to C | f \text{ is measurable }\}$, $C_{12} \subset F(G(L_1))$, $C_{r,e}(B_\varepsilon(x)) \subset C(G)$, $C_\varepsilon(G) = \{f \in C(G) | \forall x \in P f|_{B_\varepsilon(x)} \in C_{r,e}(B_\varepsilon(x))\} \subset C(G)$, $C_{\varepsilon,x}(G) \subset C(G)$. Let us also consider subspaces of $C(G)$ defined as

Since space $C(G)$ is Hilbert one can construct orthoprojectors onto the spaces defined above. Let us denote them P_ε (projector onto C_ε), $P_{\varepsilon,x}$ (projector onto $C_{\varepsilon,x}$). By corollary 10.1 we have $P_{0,x} = Id$ for almost all $x \in P$.

Here we assume that these projectors commute. Later we will find the condition that is enough to ensure it at least in some cases.

Specter of the algebra generated by projectors from above is at most $P/\simeq$ (note that topology on specter doesn't coincide with one on factor-manifold) due to invariance of the projectors under action of the group H. Here the equivalence relation $\simeq$ is defined as follows:

$$p, q \in P, p \simeq q \Leftrightarrow \exists L (leaf of the foliation) p, q \in L$$

Now we are in situation described by Gelfand-Naimark theorem [4]. We can consider projectors as functions on the specter of this algebra. Let us denote isomorphism built in this theorem by U. Since it can be embedded into P we can use functions defined in Gelfand-Naimark theorem functions with P as domain.

Now we can define functions $f_{\varepsilon,x} : P \to R$, corresponding to projectors $P_{\varepsilon,x}$ as $f_{\varepsilon,x}(y) = U(P())$.

Statement11. $\lim\limits_{\varepsilon \longrightarrow 0} C_\varepsilon = C(G) \Leftrightarrow$ images under Gelfand isomorphism U of the projectors $P_{\varepsilon,x}$ converge to identity (pointwise convergence). $\forall x \in P/\simeq$.

• $\Rightarrow$ Let us assume that on the contrary. $\exists x \in P/\simeq$ and $\exists \varepsilon' > 0\ \forall \varepsilon < \varepsilon'$ $\exists y \in B_\varepsilon(x)\ U(P_\varepsilon)(y) = U(P_\varepsilon)(x) = 0$.

Now we must consider subspaces of $C_(G)$ which contain functions, that separate some neighborhood $B_\delta(x)$ of x for $\delta < \varepsilon'$ from any y. Since images of projectors doesn't separate points in which functions from the given spaces may take different values then the projectors which can be represented also as multiplicators by characteristic functions of the varieties $P_{B_\varepsilon(x)/\simeq} \bigcup P \setminus B/\simeq$ won't contain in their image spaces defined in the beginning of this paragraph.

$\Leftarrow$ Is evident. $\square$

Statement12. In case $x \in P$ is a singular point and $C_\infty \neq 0$ then function $U(P_{\infty,x}) = U(P_\infty)$ must be characteristic function of the set of singular points.

• Since in any ε-neighborhood of this point we can find $B_{\varepsilon'(\varepsilon,x)}(x'(\varepsilon, x))$ for which we have $\forall y \in B_{\varepsilon'}(x')\ P_{0,y} = Id$ then by structure of the projectors this function $U(P_{\infty,x})$ must satisfy $U(P_{\infty,x}) = (U(P_{\infty,x}))^2 = U(P_{\infty,x})U(P_{0,x'}) \neq 1$ then it can be only 0 whenever $x \notin P_s$. Since it can't vanish by assumption then there is only one possibility left. $\square$

K. Fibre bundles. Function f_0 which corresponds to $\lim\limits_{\varepsilon \longrightarrow 0} C_\varepsilon$ is constant and equals 1.

L. Seifert manifold described in example H. Function $f_\infty(z) = \chi_0$, $z \in C$.

M. Spirals and circle given by vector field $(|x|, 1)$ as in Example 10. $f_0(x) = f_\infty(x) = 0$, $x \in R$.

*Note*2. In case projectors, constructed above don't generate commutative algebra we can consider only local problem.

Particularly the following holds true

Statement13. If singular point $x \in P$ is an isolated one and again all leaves of the foliation are proper then we can again formulate and prove previous statements in some neighborhood of x.

• Let ε be such that $P_s \bigcap B_\varepsilon(x) = \{x\}$ then we can take projectors $P_{\varepsilon',y}$ for $y \in B_\varepsilon(x)\ \varepsilon' \leq \varepsilon - d(x, y)$. $\square$

Second variant. Here we introduce structure of Hilbert space on subset of $C^*_r(G)$.

Let us consider subset R_ε of algebra $C^*_\varepsilon(G)$ consisting of $f \in C^*_\varepsilon(G)\ f * f \geq 0$. It will consist only of functions, depending only on the range of the γ since $f(\gamma\gamma_1) = \overline{f}(\gamma_1^{-1})$. Now we define scalar product as $(f, g) = \int\limits_M f * g$.

Under these assumptions we can prove all the statements from the first subsection.

Also we obviously can reformulate the first one as

Statement14. $\lim\limits_{\varepsilon \longrightarrow 0} R^*_\varepsilon = R^*_r(G) \Leftrightarrow P_\varepsilon$ is a decomposition of unity for D_0.

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