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A MONOTONE CONVERGENCE THEOREM FOR NEWTON-LIKE METHODS USING HYPOTHESES ON DIVIDED DIFFERENCES OF ORDER TWO

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Abstract. In this study we approximate a solution of a nonlinear operator equation on a partially ordered topological space by enclosing it by two monotonically convergent Newton-like sequences. In contrast to earlier results we use hypotheses on bilinear instead of on linear operators. This way we establish different sufficient convergence conditions for Newton-like methods than before. Hence, our results have theoretical and practical value.

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1. Introduction

In this study we are concerned with the problem of approximating a solution x^* of the equation

$$F(x) = 0, \tag{1}$$

where F is a nonlinear operator defined on a closed convex subset D of a partially ordered topological space E_1 (POTL-space) with values in a (POTL-space) E_2. We refer the reader to [5], [10] for definitions of the concepts introduced here.

So far Newton-like methods have been used to enclose monotonically x^*, under hypotheses on linear operators (see (20) and (21)). For a survey of such results we refer the reader to [5].

Here, we provide new sufficient convergence conditions for the monotone convergence of Newton-like methods, but we use hypotheses on bilinear (see (3) and (4)) as well as linear operators. This way our convergence conditions differ from earlier ones [1]–[8], [10]. Hence, our results have theoretical and practical value.

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Theorem 1. *Consider an operator $F: D \subset E_1 \to E_2$, where E_1 is a regular POTL-space, and E_2 is a POTL-space. Let x_0, y_0, y_{-1} be three points of D such that*

$$x_0 \le y_0 \le y_{-1}, \ [x_0, y_{-1}] \subseteq D, \ F(x_0) \le 0 \le F(y_0). \tag{2}$$

Denote by D_1, D_2, D_3 the sets

$$D_1 = \{(x,y) \in E_1^2; x_0 \le x \le y \le y_0\},$$
$$D_2 = \{(z, y_{-1}) \in E_1^2; x_0 \le z \le y_0\},$$

and $D_3 = D_1 \cup D_2$.

Assume:

(a) *operator F has nonnegative, symmetric, and continuous divided differences of order one and two, denoted by $[x,y;F] \in L(E_1, E_2)$ and $[x,y,z;F] \in L(E_1, L(E_1, E_2))$ at the points (x,y) and (x,y,z) $(x,y,z \in D_3)$ respectively;*

(b) *there exists operator $A: D_3 \to L(E_1, E_2)$ having a continuous nonsingular nonnegative left subinverse for all $(x,y) \in D_3$; and an operator $B \in L(E_2, L(E_1, E_2))$ such that*

$$0 \le B \le [x,y,z;F] \tag{3}$$

and

$$[y,z;F] - B(z-x) - A(y,w) \le 0 \tag{4}$$

for all $x,y,z,w \in D_3$ with $x \le y \le z \le w$.

Then, there exist two sequences $\{x_n\}$ $(n \ge 1)$, $\{y_n\}$ $(n \ge 1)$, and two points $x^*, y^* \in E_2$ such that for all $n \ge 0$, and for $A_n := A(y_n, y_{n-1})$ $(n \ge 0)$:

$$F(y_n) + A_n(y_{n+1} - y_n) = 0, \tag{5}$$
$$F(x_n) + A_n(x_{n+1} - x_n) = 0, \tag{6}$$
$$F(x_n) \le 0 \le F(y_n), \tag{7}$$

$$x_0 \le x_1 \le \cdots \le x_n \le x_{n+1} \le y_{n+1} \le y_n \le \cdots \le y_1 \le y_0, \tag{8}$$

and

$$\lim_{n\to\infty} x_n = x^*, \quad \lim_{n\to\infty} y_n = y^*, \tag{9}$$

with

$$x^* \le y^*. \tag{10}$$

Moreover, if operators A_n $(n \ge 0)$ are inverse nonnegative, then any solution of the equation $F(x) = 0$ from the interval $[x_0, y_0]$ belongs to the interval $[x^*, y^*]$.

Proof. Let L_0 be a continuous nonsingular nonnegative left subinverse of A_0. Define operator $P: [0, y_0 - x_0] \to E_1$ by

$$P(x) = x - L_0(F(x_0) + A_0(x)). \tag{11}$$

Since L_0, F and A_0 are continuous, so is operator P. From $x \le y$, and $L_0 A_0 \le I$, we can get $(I - L_0 A_0)(y - x) \ge 0$, which shows $P(x) \le P(y)$. That is operator P is also isotone. By (2) and (5) we have respectively:

$$
\begin{aligned}
&P(0) = -L_0 F(x_0) \ge 0,\\
&P(y_0 - x_0) = y_0 - x_0 - L_0 F(y_0) + L_0(F(y_0) - F(x_0) - A_0(y_0 - x_0)).
\end{aligned} \tag{12}
$$

But using (3) and (4), we get

$$
\begin{aligned}
&F(y_0) - F(x_0) - A_0(y_0 - x_0)\\
&\quad = F(y_0) - F(x_0) - [y_{-1}, y_0; F](y_0 - x_0)\\
&\qquad + [y_{-1}, y_0; F](y_0 - x_0) - A_0(y_0 - x_0)\\
&\quad = ([x_0, y_0; F] - [y_{-1}, y_0; F])(y_0 - x_0)\\
&\qquad + ([y_{-1}, y_0] - A_0)(y_0 - x_0)\\
&\quad = -([y_{-1}, y_0, x_0; F] - B)(y_{-1} - x_0)(y_0 - x_0)\\
&\qquad + [[y_{-1}, y_0; F] - A_0 - B(y_{-1} - x_0)]\,(y_0 - x_0) \le 0.
\end{aligned} \tag{13}
$$

Approximation (12) because of (2) and (12) gives

$$P(y_0 - x_0) \le y_0 - x_0.$$

According to the famous Kantorovich theorem concerning fixed points on regular POTL-spaces [5], [6], operator P has a fixed point $w \in [0, y_0 - x_0]$. Set $x_1 = x_0 + w$ to obtain

$$F(x_0) + A_0(x_1 - x_0) = 0, \quad x_0 \le x_1 \le y_0. \tag{14}$$

Using (5), (6) and (14) we obtain

$$
\begin{aligned}
F(x_1) &= F(x_1) - F(x_0) - A_0(x_1 - x_0)\\
&= ([x_0, x_1; F] - A_0)(x_1 - x_0)\\
&= ([x_1, x_0; F] - [y_0, x_1; F])(x_1 - x_0) + ([x_1, y_0; F] - A_0)(x_1 - x_0)\\
&= ([y_0, x_1, x_0; F] - B)(x_0 - y_0)(x_1 - x_0)\\
&\quad + [[x_1, y_0; F] - A_0 - B(y_0 - x_0)]\,(x_1 - x_0) \le 0.
\end{aligned} \tag{15}
$$

Similarly, we define operator $Q\colon [0, y_0 - x_1] \to E_1$ by

$$Q(x) = x + L_0(F(y_0) - A_0(x)).$$

Arguing as above we conclude that Q is continuous and isotone.

By (2) and (6) we have respectively,

$$
\begin{aligned}
&Q(0) = L_0 F(y_0) \ge 0,\\
&Q(y_0 - x_1) = y_0 - x_1 + L_0 F(x_1) + L_0(F(y_0) - F(x_1) - A_0(y_0 - x_1)).
\end{aligned} \tag{16}
$$

But using (3) and (4) we get

$$
\begin{aligned}
(17) \qquad F(y_0) - F(x_1) - A_0(y_0 - x_1) &= [[y_0, x_1; F] - [y_{-1}, y_0; F]]\,(y_0 - x_1) \\
&\quad + ([y_{-1}, y_0; F] - A_0)(y_0 - x_1) \\
&= ([y_{-1}, y_0, x_1; F] - B)(x_1 - y_{-1})(y_0 - x_1) \\
&\quad + [[y_0, y_{-1}; F] - A_0 - B(y_{-1} - x_1)]\,(y_0 - x_1) \le 0.
\end{aligned}
$$

Using (2) and (16), approximation (17) gives

$$Q(y_0 - x_1) \le y_0 - x_1.$$

By Kantorovich's theorem we deduce the existence of a point $z \in [0, y_0 - x_1]$ such that $z = Q(z)$. Set $y_1 = y_0 - z$ to obtain

$$(18) \qquad F(y_0) + A_0(y_1 - y_0) = 0, \quad x_1 \le y_1 \le y_0.$$

Using (18) and (9) we get

$$
\begin{aligned}
(19) \qquad F(y_1) &= F(y_1) - F(y_0) - A_0(y_1 - y_0) \\
&= ([y_0, y_1; F] - A_0)(y_1 - y_0) \\
&= [y_{-1}, y_0, y_1; F](y_1 - y_{-1})(y_1 - y_0) \ge 0.
\end{aligned}
$$

Proceeding by induction on $n \ge 0$, we can show that there exist two sequences $\{x_n\}$ $(n \ge 1)$ and $\{y_n\}$ $(n \ge 1)$ satisfying (5)–(8), in a regular space E_1. It follows that there exist $x^*, y^* \in E_1$ such that $x^* = \lim_{n\to\infty} x_n$, and $y^* = \lim_{n\to\infty} y_n$. We obviously have $x^* \le y^*$.

If $x_0 \le u \le y_0$ and $F(u) = 0$, we can have

$$
\begin{aligned}
A_0(y_1 - u) = A_0(y_0) - F(y_0) - A_0(u) &= A_0(y_0 - u) - (F(y_0) - F(u)) \\
&= [[y_0, y_{-1}; F] - [u, y_0; F]]\,(y_0 - u) \\
&= [u, y_0, y_{-1}; F](y_{-1} - u)(y_0 - u) \ge 0,
\end{aligned}
$$

and

$$
\begin{aligned}
A_0(x_1 - u) = A_0(x_0) - F(x_0) - A_0(u) &= A_0(x_0 - u) - (F(x_0) - F(u)) \\
&= [([y_0, y_{-1}; F] - [y_0, x_0; F]) + ([y_0, x_0; F] - [x_0, u; F])]\,(x_0 - u) \\
&= [[x_0, y_0, y_{-1}; F](y_{-1} - x_0) + [u, x_0, y_0; F](y_0 - u)]\,(x_0 - u) \le 0.
\end{aligned}
$$

Hence, if operator A_0 is inverse nonnegative it follows $x_1 \le u \le y_1$. Proceeding by induction, we deduce $x_n \le u \le y_n$ $(n \ge 0)$. Hence, we conclude, $x^* \le u \le y^*$.

That completes the proof of the Theorem. □

Remark 1. A possible choice for operator B can be $B = [x, y, z; F]$, in which case (3) is satisfied, whereas (4) can give

$$[y, z; F] \le [x, y, z; F](z - x) + A(y, w) = [y, z; F] - [x, y; F] + A(y, w),$$

or

$$[x, y; F] \le A(y, w). \tag{20}$$

In the special case, when $A(y, w) = [y, w; F]$ $(y, w \in D_3)$ (secant method), it follows from (20) that (4) is satisfied if

$$[x, y; F] \le [w, y; F] \text{ for all } x, y, w \in D_1 \text{ with } x \le y \le w. \tag{21}$$

Monotone convergence results have been proved in [1]–[8], [10] for Newton-like methods based on the assumption

$$[x, y; F] \le A(w, z) \text{ for all } (x, y),\ (y, w) \in D_1,\ (w, z) \in D_3,$$

which is condition (20), in this case.

Natural conditions and their proofs under which the points x^* and y^* are solutions of the equation $F(x) = 0$ can be found in [1]–[8], [10]. However, here for completion we list some of them.

Proposition. *Assume*:

(a) *hypotheses of Theorem 1 hold*;
(b) *operator F is continuous at x^* and y^*.*

If, one of the following conditions hold:
(i) $x^* = y^*$,
(ii) E_1 *is normal and there exists an operator* $H\colon E_1 \to E_2$, *with* $H(0) = 0$ *having an isotone continuous inverse at the origin such that* $A_n \le H$ *for sufficiently large* n;
(iii) *operators* A_n $(n \ge 0)$ *are equicontinuous*;
(iv) E_2 *is normal and there exists an operator* $G\colon E_1 \to E_2$ *with* $G(0) = 0$, *continuous at the origin with* $A_n \le G$ *for sufficiently large* n.

Then, we deduce

$$F(x^*) = F(y^*) = 0.$$

The regularity of the space E_1 assumed in Theorem 1 is essentially used to show sequences $\{x_n\}$, $\{y_n\}$ $(n \ge 0)$ are well defined and convergence can be dropped. Instead of iterative procedure (5)–(6) consider the explicit form of it

$$y_{n+1} = y_n - L_n F(y_n), \quad (n \ge 0), \tag{22}$$

$$x_{n+1} = x_n - L'_n F(x_n), \quad (n \ge 0), \tag{23}$$

where L_n and L'_n are nonnegative subinverses of A_n $(n \ge 0)$. By dropping the regularity assumption on E_1 we cannot prove convergence. However, we can prove in the theorem that follows that for any solution $u \in [x_0, y_0]$ of equation $F(u) = 0$, the following enclosure estimates hold for all $n \ge 0$:

$$x_n \le x_{n+1} \le u \le y_{n+1} \le y_n \quad (n \ge 0).$$

Such a result is important in cases where the existence of the solution can be proved by other means and we just want to enclose it monotonically [1], [2]–[5], [7].

Theorem 2. *Let $F: D \subseteq E_1 \to E_2$ be an operator between two POL-spaces E_1 and E_2. Assume:*

(a) *there exist points $x_0, y_0, y_{-1} \in D$ satisfying (2);*
(b) *hypotheses (a) of Theorem 1 hold;*
(c) *define D_1, D_2, D_3 by Theorem 1 and assume there exists an operator $A: D_3 \to L(E_1, E_2)$ having a nonnegative subinverse for any $(v, w) \in D_3$; and an operator $B \in L(E_1, L(E_1, E_2))$ such that hypotheses (3) and (4) hold.*

Then, iterative algorithm (22)–(23) generates two sequences having properties (5)–(18).

Moreover for any solution $u \in [x_0, y_0]$ of the equation $F(x) = 0$, we have

$$x_n \le u \le y_n \quad (n \ge 0). \tag{24}$$

Proof. We show that (24) is true using induction on $n \ge 0$. First for $n = 0$, we have $x_0 \le y_0$, $F(x_0) \le 0 \le F(y_0)$,

$$\begin{aligned} &L_0 \ge 0,\ L_0' \ge 0,\ I \ge A_0 L_0,\ I \ge L_0 A_0, \\ &I \ge A_0 L_0',\ \text{and } I \ge L_0' A_0. \end{aligned} \tag{25}$$

We can write in turn

$$\begin{aligned} y_0 - y_1 &= L_0 F(y_0) \ge 0, \\ y_1 - x_0 &= y_0 - x_0 - L_0 F(y_0) \ge y_0 - x_0 - L_0(F(y_0) - F(x_0)) \\ &\ge L_0(A_0(y_0 - x_0) - (F(y_0) - F(x_0))) \ge 0, \quad \text{(by (13))}, \\ x_1 - x_0 &= -L_0' F(x_0) \ge 0, \\ y_0 - x_1 &= y_0 - x_0 + L_0' F(x_0) \ge y_0 - x_0 - L_0'(F(y_0) - F(x_0)) \\ &\ge L_0'[A_0(y_0 - x_0) - (F(y_0) - F(x_0))] \ge 0, \quad \text{(by (13))}. \end{aligned}$$

That is points $x_1, y_1 \in [x_0, y_0]$. Moreover using (3), (4), (22), (23), (25) and the proof of Theorem 1 we can also have in turn

$$\begin{aligned} F(y_1) &= F(y_1) + A_0(y_0 - y_1 - L_0 F(y_0)) = F(y_1) - A_0 L_0 F(y_0) + A_0(y_0 - y_1) \\ &\ge F(y_1) - F(y_0) + A_0(y_0 - y_1) \ge 0, \quad \text{(by (19))}, \\ F(x_1) &= F(x_1) - A_0(x_1 - x_0 + L_0' F(x_0)) = F(x_1) - A_0 L_0' F(x_0) - A_0(x_1 - x_0) \\ &\le F(x_1) - F(x_0) - A_0(x_1 - x_0) \le 0, \quad \text{(by (15))}, \\ y_1 - x_1 &\ge y_1 - x_1 + L_0 F(x_1) = y_0 - x_1 - L_0(F(y_0) - F(x_1)) \\ &\ge L_0(A_0(y_0 - x_1) - (F(y_0) - F(x_1)) \ge 0, \quad \text{(by (19))}. \end{aligned}$$

Hence, we obtain $x_0 \le x_1 \le y_1 \le y_0$, and $F(x_1) \le 0 \le F(y_1)$.

Let $u \in [x_0, y_0]$ with $F(u) = 0$. We get

$$\begin{aligned} y_1 - u &= y_0 - u - L_0 F(y_0) + L_0 F(u) \ge L_0(A_0(y_0 - u) - (F(y_0) - F(u))) \ge 0, \\ u - x_1 &= u - x_0 + L_0' F(x_0) - L_0' F(u) \ge L_0'(A_0(u - x_0) - (F(u) - F(x_0)) \ge 0. \end{aligned}$$

That is, we get $x_1 \leq u \leq y_1$. Proceeding by induction it follows that $x_n \leq u \leq y_n$.
That completes the proof of the Theorem. □

We state the following proposition for completion whose proof can be found in [5], [7], [10].

Proposition 2. *Under the hypotheses of Theorem 2, assume*:

(a) E_2 *is a POTL-space and* E_1 *is a normal POTL-space*;
(b) $\lim_{n\to\infty} x_n = x^*$, *and* $\lim_{n\to\infty} y_n = y^*$;
(c) F *is continuous at* x^* *and* y^*, *and*
(d) *there exist two continuous, nonnegative, nonsingular operators* L *and* L' *such that* $L_n \geq L$ *and* $L'_n \geq L'$ *for sufficiently large* n. *Then, we deduce* $F(x^*) = F(y^*) = 0$.

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