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Space-time distributions

Mihaela D. Iftime

Abstract

The space-time foliation Σ compatible with the gravitational field g on a 4-manifold M determines a fibration π of M, $\pi : M \longrightarrow \mathcal{N}$ is surjective submersion over the 1-dimensional leaves space $\mathcal{N}$. M is then written as a disjoint union of the leaves of Σ, which are 3-dimensional spacelike surfaces on M.

The decomposition, $TM = \Sigma \oplus T^0M$, also implies that we can define a lift of the curves on $\mathcal{N}$ to curves (non-spacelike) on M.

The stable causality condition holds on M [2] means that Σ is a causal space-time distribution, generated by an exact timelike 1-form $\omega^0 = \mathrm{d}t$ where t is some real function on M. In this case M is written as a disjoint union of a family of spacelike 3-surfaces of constant t, which cover $D^+(S)$ of a initial 3-surface S of M.

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Suceava, 5800 Romania
E-mail: mtime@warpnet.ro

1 Introduction

Note: We will denote by small Latin indices $a, b, \ldots$ numbers ranging from 0 to 3 and the small Greek indices $\alpha, \beta, \ldots$ numbers ranging from 1 to 3.

The space-time structure on a 4-dimentional manifold M is a uniquely defined by a triplet (g^R, g, Σ), where g and g^R represent a Riemannian, respectively a Lorentzian metric on M and $\Sigma \subset \tau_M$, the space-time distribution on M compatible with the gravitational field g, is a 3-dimentional spatial tangent subbundle on M orthocomplement (with respect both g and g^R) of the 1-dimentional timelike subbundle T^0M generated by the timelike globally defined vector field $h^0 \in \mathcal{X}(M)$, $g^R(h^0\,, h^0) = -g(h^0\,, h^0) = 1$ which indicates the time orientation (locally) at every point of M.

If g^R is a Riemannian metric on M (which exists on every paracompact space) and a time-orientation defined by $h^0 \in \mathcal{X}(M)$ then we have a Lorentz metric on M determined by the expression :

$$(1) \quad g(X,\, Y) = g^R(X,\, Y) - 2g^R(X,\, h^0)g^R(Y,\, h^0) \quad \forall X,\, Y \in \mathcal{X}(M)$$

Locally, relating to an orthonormal frame $\{e_a\} = \{h^\alpha,\, h^0\}$ we find:

$$(2) \quad g_{ab} = g^R_{ab} - 2\omega^0_a\omega^0_b,$$

where $\omega^0 = g^R(h^0,\, \cdot)$ is the unitary timelike 1-form.

The Pfaff equation $\omega^0 = 0$ generates the 3-dimensional spatial spaces Σ_p in each fibre T_pM of the tangent bundle τ_M and these subspaces are orthogonal to the timelike field h^0 in $p \in M$.

The tensor

$$(3) \quad h_{ab} = g_{ab} + \omega^0_a\omega^0_b$$

defines a Riemannian metric on the tangent subbundle Σ.

We have the following decomposition of tangent bundle:

$$(4) \quad TM = \Sigma \oplus T^0M$$

So, given a Riemannian metric g^R_{ab} on M, each 3-dimensional distribution Σ on M generated by a nonzero 1-form σ on M represents a space-time distribution compatible with a gravitational field g defined by the expression (2), where we take $\omega^0 = -\frac{\sigma}{g^R(\sigma,\sigma)}$.

Inverse, given a gravitational field g on M we find a spacetime distribution Σ compatible with g and a Riemannian metric g^R which satisfy the expression (2).

Two such different triplets $(g_i,\, g_i^R,\, \Sigma_i),\;\; i = 1,\, 2$ will describe the same space-time structure on M if the space-time distributions coincide, $\Sigma_1 = \Sigma_2$.

The geometric equivalence principle — which states that we may introduce an orthonormal tetrad field $\{h^\alpha,\, h^0\}$ at each point of M — is also sufficient for defining the space-time structure on a 4-manifold M.

The metric g_{ab} can be transformed to

(5) $\quad g_{ab} = \eta_{ab} = diag(1,\, 1,\, 1,\, -1)$

at any point of M.

Also for any point $p \in M$ and its normal convex neighbourhood $U_p \subset M,\;\; U_p = \exp_p W_0$, where $W_0 \subset T_pM$ is an open neighbourhood of the origin of T_pM, we can have a causal structure; all events q in U_p are divided into three groups according whether the causal relation to p is timelike, null or spacelike and these relations are independent of the choice of coordinate system. But we don't know how to split an arbitrary space-time in space and time globally.

An integrable distribution

$$\Sigma = \bigcup_{p\in M} \Sigma_p$$

is a spacetime foliation of spatial hypersurfaces S of M; $\;\Sigma_p = T_pS,\;\; \forall p \in S$.

In this case, at every point $p \in M$ we can define a local coordinate neighbourhood $U_i \subset M$ with local adapted coordinates (x^a) so that the spacelike vector fields $\{\frac{\partial}{\partial x^\alpha}\}$ generate Σ locally.

The Pfaff equation $x^0 =$ const. locally defines the spatial hypersurfaces S of M (i.e. the maximal integral manifolds or the leaves of Σ).

We have,

$$\mathcal{U} = \{U_i,\;\; t_i : \tau_M^{-1}(U_i) \longrightarrow U_i \times \mathrm{R}^4\}$$

a bundle atlas on TM which is compatible with the subbundle $\Sigma \subset \tau_M$, so that

$$\mathcal{U}^\Sigma = \{U_i,\;\; t_i^{-1} : \tau_M^{-1}(U_i) \cap \Sigma \longrightarrow U_i \times \mathrm{R}^3\}$$

is a bundle atlas on Σ, $t_i(\tau_M^{-1}(U_i) \cap \Sigma) = U_i \times \mathrm{R}^3$.

We can have directly: for any adapted local coordinate system $(U_i,\, (x^a))$, then $(x^a,\, \dot{x}^a)$ are local bundle coordinates on TM and $\{\frac{\partial}{\partial x^a}\}$ are the dual sections. Then, from the local trivialisation of τ_M

(6) $\quad t_i = (\tau_{M_{|\tau_M^{-1}(U_i)}},\, \dot{x}^a) : \tau_M^{-1}(U_i) \longrightarrow U_i \times \mathrm{R}^4$

we obtain a local trivialisation on subbundle Σ

(7) $\quad t_i^\Sigma = (\tau_{M_{|\tau_M^{-1}(U_i)\cap\Sigma}},\, \dot{x}^\alpha) : \tau_M^{-1}(U_i) \cap \Sigma \longrightarrow U_i \times \mathrm{R}^3.$

2 Family of spatial hypersurfaces of spacetime

A space-time foliation Σ on M generates a fibration π as follows: we define an equivalence relation on M, $x, y \in M$, $x \sim y$ if x, y are in the same leaf of the foliation.

We denote by $\mathcal{N} = M_{|\sim}$ the space of leaves which has the natural quotient structure of 1-dimensional differentiable manifold, so that $\pi : M \longrightarrow \mathcal{N}$ is a surjective submersion. The leaves of the foliation Σ coincide with the fibres of π, $\pi^{-1}(\nu) = S_\nu$ and $\Sigma = \mathrm{Ker}\pi_*$. We also may suppose that the fibres of π are simply connected even if it is not a necessary condition.

We can define an vertical lift of the tangent vectors on $\mathcal{N}$ to tangent vectors on M which belong to Σ.

For $\bar{X} \in T_\nu \mathcal{N}$ its vertical lift is $X^v \in T_p M$, $X^{va} = h^a_b(p) X^b$, $\forall X \in T_p M$, where $\pi_* X = \bar{X}$.

Here

$$(8) \quad h = (h^a_b) : TM \longrightarrow TM$$
$$h^a_b = g^{ac} h_{cb}$$

is an endomorphism on τ_M, a projection operator, i.e. $h^a_b h^b_c = h^a_c$, which projects any vector $X \in T_p M$ into its part lying in the subspace Σ_p,

$$(9) \quad X^a = h^a_b X^b - h^{0a} \omega^0_b X^b$$

and any $X \in \mathcal{X}(\mathcal{N})$ then $X^v \in \mathcal{V}(\Sigma)$ defined by $X^v = h(X)$ for any $X \in \mathcal{X}(M)$ so that $\pi_* X = \bar{X}$.

Here $\mathcal{V}(\Sigma) = \{X \in \mathcal{X}_{loc}(M) \ / X_p \in \Sigma_p, \forall p\}$.

The Lorentzian (metric) connection ∇ on M induces by restriction a connection ∇^S on each leaf of Σ ,

$$(10) \quad \nabla_{X^v} Y^v = \nabla^S_{X^v} h(Y) = h(\nabla_{X^v} Y) \in \Sigma.$$

We used

$$(11) \quad 0 = (\nabla_{X^v} h) Y = \nabla_{X^v} h(Y) - h(\nabla_{X^v} Y)$$

and

$$(12) \quad h_{ab;c} = (g_{ab} + \omega^0_a \omega^0_b)_{;c} = 0$$

It is worth notice that the space-time is a disjoint union of the leaves of the foliation Σ, i.e.

$$M = \bigcup_{\nu \in \mathcal{N}} S_\nu$$

where $S_\nu = \pi^{-1}(\nu)$ represents a family of spatial hypersurfaces of M. From decomposition (4) we can define a transport of the those fibres S_ν along the curves $\gamma(\tau)$ on $\mathcal{N}$. That is to say, a curve $\gamma(\tau) : [0, 1] \longrightarrow \mathcal{N}$, $\gamma(0) = \nu_0$, $\gamma(1) = \nu_1$ could be lifted on M from each point of the fibre $x_i \in \pi^{-1}(\nu_0) = S$ to a nonspatial future directed curve on M of end-point $y_i \in \pi^{-1}(\nu_1) = S_{\nu_1}$.

This way we obtain an 1:1 correspondence between S and S_{ν_1}. Then we have a family of maps (homeomorphisms)

(13) $\alpha_\gamma : S_{\gamma(0)} \longmapsto S_{\gamma(1)}, \quad \alpha_\gamma : x_i \longrightarrow y_i$

satisfying natural conditions: $\alpha_{\gamma_1\gamma_2} = \alpha_{\gamma_1}\alpha_{\gamma_2}$, $\alpha_{\gamma^{-1}} = (\alpha_\gamma)^{-1}$, $\alpha_\gamma = \mathrm{id}$ for $\gamma = \text{const.}$, α_γ is independent of the choice of the parameter τ on γ.

On the other hand $\mathcal{N}$ is 1-dimensional (connected) manifold. This implies that $\mathcal{N}$ must be diffeomorphic with an interval $[0, 1]$ or with circle S^1.

Supposing $\mathcal{N}$ has a fixed orientation given by the unique unitary $\bar{X} \in \mathcal{X}(\mathcal{N})$, then $\mathcal{N} = \gamma_{\bar{X}}([0, 1])$, where $\gamma_{\bar{X}} : [0, 1] \longrightarrow \mathcal{N}$ is the integral curve of $\bar{X}$. We will have the two cases: when $\gamma_{\bar{X}}$ is an injective integral curve, $\mathcal{N} = [0, 1]$ and when $\gamma_{\bar{X}}$ admits cross-points (i. e. $\gamma_{\bar{X}}$ may intersect itself), $\mathcal{N} = S^1$.

Lifting $\gamma_{\bar{X}}$ from each point $x_i \in \pi^{-1}(\nu_0) = S$ we obtain a family of timelike future directed curves $\tilde{\gamma}_{\bar{X}} : [0, 1] \longrightarrow M$, $\dot{\tilde{\gamma}}_{\bar{X}}(\tau) \in T^0_p M$, $\pi(p) = \nu$ which start on S and intersect each leaf S_ν orthogonally.

The stable causality condition holds on M [1] means that the space-time foliation Σ is causal, being generated by a globally defined timelike 1-form $\omega^0 \in \wedge^1 M$ which is exact, i.e. $\omega^0 = \mathrm{d}t$ where t is some real function on M.

In this case, the leaves of Σ are spatial hypersurfaces of M of constant t, and

$$M = \bigcup_{\nu \in \mathcal{N}} S_\nu$$

is the disjoint union of a family of spatial hypersurfaces S_ν of constant t on M.

If we suppose that the initial surface S is a partial Cauchy surface of M we obtain M as a disjoint union of such homeomorphic partial Cauchy surfaces S_ν, which in addition satisfy the following conditions: for $\nu_i < \nu_j$ then S_{ν_j} lies to the future of S_{ν_i} and they cover $D^+(S)$.

[1] This condition of stability is equivalent to the existence of a function f on M, whose ∇f is everywhere timelike.

In the case when $\mathcal{N}$ is diffeomorphic with S^1, provided that S is a partial Cauchy surface on M, a curve $\gamma_{\bar{X}} : [0, 1] \longrightarrow \mathcal{N}$, $\gamma_{\bar{X}}(0) = \gamma_{\bar{X}}(1) \in S^1$ is lifted from a point x_i of S to a closed timelike curve $\tilde{\gamma}$ on M with $\tilde{\gamma}(0) = x_i = \tilde{\gamma}(1)$.

3 Some discussion

Physically, it seems more convenient to choose the surfaces S_ν so that they intersect $\mathcal{I}^+$ [2]. This means that every S_ν tend asymptotically to null surfaces.

Also by assumption of strong asymptotic predictability (i.e. there exists a partial Cauchy surface S of M such that (a) $\mathcal{I}^+$ lies in the boundary of $D^+(S)$ and (b) $J^+(S) \cap \partial J^-(\mathcal{I}^+)$ lies in $D^+(S)$) for some ν, the surfaces S_{ν_i} for $\nu_i > \nu$ will intersect the event horizon $\partial J^-(\mathcal{I}^+)$. Then a connected component of the nonempty set $S_{\nu_i} \backslash J^-(\mathcal{I}^+)$ will represent a black hole on the surface S_{ν_i}.

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