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REDUCTION OF THE MODIFIED POINCARÉ DIFFERENTIAL EQUATION TO BIRKHOFF MATRIX FORM

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In this paper is given the reduction of the modified Poincaré linear differential equation with one n-tiple regular singularity to the Birkhoff canonical matrix form.

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1. Introduction

In the previous paper [1] is shown that the general Poincaré linear differential equation reduces to the following Cauchy matrix form $(xI - D)\frac{dY}{dx} = QY$, where I is $n \times n$ unit matrix, Q is $n \times n$ matrix of transformation, $D = diag(d_1, d_2, \cdots, d_n)$ and d_i, $(1 \leq i \leq n)$ a distinct regular singularities. If we substitute $d_i = 0$, $(1 \leq i \leq n)$ into the last equality, then very easy we find that the modified Poincaré linear differential equation with one n-tiple regular singularity reduces to the following Birkhoff matrix form $xI\frac{dY}{dx} = QY$, which means that Birkhoff canonical matrix system is a special case of the Cauchy matrix system. Just, this type of the reduction will be an object of investigation in the present paper.

2. Main Results

Now we will prove the following result.

Theorem. *The modified Poincaré differential equation*

$$x^n y^{(n)} = \sum_{i=1}^{n}\Big(\sum_{j=0}^{ri} a_{ij}x^j\Big)x^{n-i}y^{(n-i)}, \tag{1}$$

reduces to the Birkhoff matrix form

$$xI\frac{dU}{dx} = Q(x)U. \tag{2}$$

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by the linear transformation with polynomials in x^{-1} as its coefficients

$$Q(x)=\begin{bmatrix} q_{11}(x) & x^r & 0 & \cdots & 0 & 0 \\ q_{21}(x) & q_{22}(x) & x^r & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ q_{n-1,1}(x) & q_{n-1,2}(x) & q_{n-1,3}(x) & \cdots & q_{n-1,n-1}(x) & x^r \\ q_{n1}(x) & q_{n2}(x) & q_{n3}(x) & \cdots & q_{n,n-1}(x) & q_{nn}(x) \end{bmatrix}, \tag{3}$$

where

$$\begin{aligned} q_{ii}(x) &= \rho_i-(i-1)r+q_{ii}^1x+\cdots+q_{ii}^{r-1}x^{r-1}, \quad (i\neq n) \\ q_{ij}(x) &= q_{ij}^0+q_{ij}^1x+\cdots+q_{ij}^{r-1}x^{r-1}, \quad (i>j; i\neq n) \\ q_{nj}(x) &= q_{nj}^0+q_{nj}^1x+\cdots+q_{nj}^{r-1}x^{r-1}+a_{n+1-j,r(n+1-j)}x^r, \\ q_{nn}(x) &= \rho_n-(n-1)r+q_{nn}^1x+\cdots+q_{nn}^{r-1}x^{r-1}+a_{1r}x^r. \end{aligned} \tag{4}$$

Proof. For the equation (1) the characteristic constants ρ_i, $(1\le i\le n)$ of the regular singularity at the coordinate origin $x=0$, are given as roots of the equation

$$I(\rho)\equiv[\rho]_n-\sum_{i=1}^n a_{i0}[\rho]_{n-i}=0, \quad [\rho]_k=\rho(\rho-1)\cdots(\rho-k+1), \quad [\rho]_0=1$$

such that $\rho_i\neq\rho_j(mod 1)$, $(i\neq j; 1\le i,j\le n)$ and the principal characteristic constants λ_i, $(1\le i\le n)$ of the irregular singularity of rank 1 at infinity $x=\infty$, are given as roots of the equation

$$J(\lambda)\equiv\lambda^n-\sum_{i=1}^n a_{i,ri}\lambda^{n-i}=0.$$

¿From the equality

$$y_p(x)=x^{-(r-1)p}y^{(p)}, \tag{5}$$

after differentiation yields to

$$xy_p'+(r-1)py_p=x^ry_{p+1}, \quad (0\le p\le n-1). \tag{6}$$

If we multiply the equation (1) by x^{-rn}, then it obtains the form

$$y_n=\sum_{i=1}^n F_i(x)y_{n-i}, \tag{7}$$

where

$$F_i(x)=\sum_{j=0}^{ri} a_{ij}x^{j-ri}, \quad (1\le i\le n). \tag{8}$$

Now according to (6) and (7), we obtain

$$DY = D \begin{bmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_{n-2} \\ y_{n-1} \end{bmatrix} =$$

$$= \begin{bmatrix} s_1 & x^r & 0 & \cdots & 0 & 0 \\ 0 & s_2 & x^r & \cdots & 0 & 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 & 0 & 0 & \cdots & s_{n-1} & x^r \\ x^r F_n(x) & x^r F_{n-1}(x) & x^r F_{n-2}(x) & \cdots & x^r F_2(x) & s_n + x^r F_1(x) \end{bmatrix} \cdot \begin{bmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_{n-2} \\ y_{n-1} \end{bmatrix} =$$

$$= L(x)Y, \qquad (D \equiv x\frac{d}{dx}; \quad s_i = -(i-1)(r-1); \quad 1 \le i \le n). \tag{9}$$

By the substitution of Turrittin [2],

$$U = C(x)Y, \qquad C(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ c_{21}(x) & 1 & 0 & \cdots & 0 & 0 \\ c_{31}(x) & c_{32}(x) & 1 & \cdots & 0 & 0 \\ \cdot \\ \cdot \\ \cdot \\ c_{n1}(x) & c_{n2}(x) & c_{n3}(x) & \cdots & c_{n,n-1}(x) & 1 \end{bmatrix} \tag{10}$$

we obtain the equality

$$DU = [DC(x) + C(x)L(x)]C^{-1}(x)U = Q(x)U, \tag{11}$$

and hence it follows that

$$DC(x) + C(x)L(x) = Q(x)C(x). \tag{12}$$

From (12) can be determined only $c_{ij}(x)$, $(i > j)$ such that all elements of the matrix $Q(x)$ are polynomials of the required form (4).

If we substitute

$$C(x)L(x) = \Big[p_{ij}(x)\Big]_{n\times n} \quad \text{and} \quad Q(x)C(x) = \Big[t_{ij}(x)\Big]_{n\times n},$$

then we obtain

$$p_{ij}(x) = x^r c_{i,j-1}(x) + s_j c_{ij}, \quad (1 \le i \le n-1) \tag{13}$$

$$p_{nj}(x) = x^r c_{n,j-1}(x) + s_j c_{nj}(x) + x^r F_{n-j+1}(x)$$

and

$$t_{ij}(x) = q_{ij}(x) + \sum_{\nu=j+1}^{i} q_{i\nu}(x) c_{\nu j}(x) + x^r c_{i+1,j}(x), \tag{14}$$

where it is

$$c_{ii}(x) = 1; \quad c_{ij}(x) = 0, \quad (j > i); \quad c_{i0}(x) = 0.$$

¿From (12) it also follows that

$$Dc_{ij}(x) + p_{ij}(x) = t_{ij}(x), \quad (j < i+2). \tag{15}$$

These relations are identically satisfied for the elements over the diagonal, because it is $p_{i,i+1}(x) = x^r$ and $t_{i,i+1} = x^r$.

For the diagonal elements

$$p_{ii}(x) = t_{ii}(x),$$

it follows that

$$x^r c_{i,i-1}(x) + s_i = q_{ii}(x) + x^r c_{i+1,i}(x), \quad (1 \le i \le n-1) \tag{16}$$

$$x^r c_{n,n-1}(x) + s_n + x^r F_1(x) = q_{nn}(x), \tag{17}$$

and for the j-th subdiagonal elements we obtain

$$Dc_{i,i-j}(x) + x^r c_{i,i-j-1}(x) + s_{i-j} c_{i,i-j}(x) =$$

$$= q_{i,i-j}(x) + \sum_{\nu=0}^{j-1} q_{i,i-\nu}(x) c_{i-\nu,i-j}(x) + x^r c_{i+1,i-j}(x), \quad (2 \le i \le n-1) \tag{18}$$

$$Dc_{n,n-j}(x) + x^r c_{n,n-j-1}(x) + s_{n-j} c_{n,n-j}(x) + x^r F_{j+1}(x) =$$

$$= q_{n,n-j}(x) + \sum_{\nu=0}^{j-1} q_{n,n-\nu}(x) c_{n-\nu,n-j}(x), \quad (1 \le j \le n-1). \tag{19}$$

¿From the equalities (16) - (19) can be considered the subdiagonal elements of $C(x)$ which can be determined as polynomials of the same degree. Now we will substitute

$$c_{i,i-j}(x) = x^{-r} c^1_{i,i-j}(x) + x^{-2r} c^2_{i,i-j}(x) + \cdots + x^{-jr} c^j_{i,i-j}(x) \tag{20}$$

$$(2 \le i \le n; \quad 1 \le j \le i-1)$$

where $c^m_{i,i-j}(x)$, $(1 \le m \le j)$ are polynomials of x of degree in the best case $r-1$. The coefficients $F_i(x)$, $(1 \le i \le n)$ can be represented in the form

$$F_i(x) = a_{i,ri} + x^{-r} F^1_i(x) + x^{-2r} F^2_i(x) + \cdots + x^{-ir} F^i_i(x), \quad (1 \le i \le n) \tag{21}$$

where $F^m_i(x)$, $(1 \le m \le i)$ are polynomials of degree $r-1$, i.e.

$$F^m_i(x) = F^m_i(0) + F^m_i(1)x + \cdots + F^m_i(r-1)x^{r-1}. \tag{22}$$

Particularly, it is $F_i^i(0) = a_{i0}$, $(1 \le i \le n)$. Now we will introduce the following remarks

$$D\Big[x^{-mr}c^m(x)\Big] = D\Big(x^{-mr}\sum_{j=0}^{r-1}\mu_j x^j\Big) =$$

$$= x^{-mr}\Big[\sum_{j=0}^{r-1}\mu_j(-mr+j)x^j\Big] \equiv x^{-mr}D_{mr}c^m(x),$$

$$q(x)c(x) = \Big(\sum_{j=0}^{r} t_j x^j\Big)\Big(\sum_{j=0}^{r-1}\mu_j x^j\Big) =$$

$$= x^r\Big[(t_r\mu_0 + t_{r-1}\mu_1 + \cdots + t_1\mu_{r-1}) + (t_r\mu_1 + t_{r-1}\mu_2 + \cdots + t_2\mu_{r-1})x + \cdots + (t_r\mu_{r-1})x^{r-1}\Big] +$$

$$+\Big[(t_0\mu_0) + (t_0\mu_1 + t_1\mu_0)x + \cdots + (t_0\mu_{r-1} + t_1\mu_{r-2} + \cdots + t_{r-1}\mu_0)x^{r-1}\Big] \equiv$$

$$\equiv x^r[q(x)c(x)]^0 + [q(x)c(x)]^1.$$

Substituting (20) and (21) into (18) and (19), we obtain

$$D_{mr}c^m_{i,i-j} + s_{i-j}c^m_{i,i-j} + c^{m+1}_{i,i-j-1} =$$

$$= \sum_{\nu=0}^{r-1}\Big\{\Big[q_{i,i-\nu}c^m_{i-\nu,i-j}\Big]^1 + \big[q_{i,i-\nu}c^{m+1}_{i-\nu,i-j}\Big]^0\Big\} + c^{m+1}_{i+1,i-j}, \tag{23}$$

$$q_{i,i-j} = c^1_{i,i-j-1} - \sum_{\nu=0}^{j-1}\Big[q_{i,i-\nu}c^1_{i-\nu,i-j}\Big]^0 - c^1_{i+1,i-j}, \tag{24}$$

$$D_{mr}c^m_{n,n-j} + s_{n-j}c^m_{n,n-j} + c^{m-1}_{n,n-j-1} + F^{m+1}_{j+1} =$$

$$= \sum_{\nu=0}^{r-1}\Big\{\Big[q_{n,n-\nu}c^m_{n-\nu,n-j}\Big]^1 + \Big[q_{n,n-\nu}c^{m+1}_{n-\nu,n-j}\Big]^0\Big\}, \tag{25}$$

$$q_{n,n-j} = c^1_{n,n-j-1} + F^1_{j+1} + a_{j+1,r(j+1)}x^r - \sum_{\nu=0}^{j-1}\Big[q_{n,n-\nu}c^1_{n-\nu,n-j}\Big]^0, \quad (1 \le m \le j) \tag{26}$$

where

$$c^m_{i-\nu,i-j} = 0, \qquad (m > j-\nu).$$

¿From (26) it just follows that

$$q_{n,n-j} = a_{j+1,r(j+1)}x^r + \cdots, \qquad (0 \le j \le n-1). \tag{27}$$

Now we are able to determine all coefficients of the polynomials $c_{ij}^m(x)$ and $q_{ij}(x)$. So, suppose that $\xi_0^1, \xi_0^2, \cdots, \xi_0^n$ are constants, such that holds the following equality

$$[\rho]_n + \xi_0^1[\rho]_{n-1} + \cdots + \xi_0^n = \prod_{\nu=1}^{n}(\rho - \rho_\nu). \tag{28}$$

Now we choose an unknown variable ζ which satisfies the following equalities

$$\begin{gathered} \xi_0^1 + [\zeta - (n-1)] = \mu_0^1 \\ \xi_0^j + [\zeta - (n-j)]\mu_0^{j-1} = \mu_0^j, \quad (2 \le j \le n-1) \\ \zeta\mu_0^{n-1} + \xi_0^n = 0. \end{gathered} \tag{29}$$

Let ζ be one of the values of ρ_ν, for example let $\rho = \rho_n$, such that it holds

$$\begin{vmatrix} \mu_0^1 + \rho - (n-2) & -1 & 0 & \cdots & 0 \\ \mu_0^2 & \rho - (n-3) & -1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \mu_0^{n-2} & 0 & 0 & \cdots & -1 \\ \mu_0^{n-1} & 0 & 0 & \cdots & \rho \end{vmatrix} =$$

$$= [\rho]_{n-1} + \mu_0^1[\rho]_{n-2} + \cdots + \mu_0^{n-1} = \prod_{\nu=1}^{n}(\rho - \rho_\nu). \tag{30}$$

It is easy to verify the validity of (28). Since

$$\begin{vmatrix} \mu_0^1 + y_1 & -1 & 0 & \cdots & 0 \\ \mu_0^2 & y_2 & -1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \mu_0^{n-2} & 0 & 0 & \cdots & -1 \\ \mu_0^{n-1} & 0 & 0 & \cdots & y_{n-1} \end{vmatrix} =$$

$$= y_1 y_2 \cdots y_{n-1} + \mu_0^1 y_2 y_3 \cdots y_{n-1} + \mu_0^2 y_3 y_4 \cdots y_{n-1} + \cdots + \mu_0^{n-1},$$

the determinant can be denoted by $f(\rho)$,

$$f(\rho) = [\rho]_{n-1} + \mu_0^1[\rho]_{n-2} + \cdots + \mu_0^{n-1}$$

and

$$(\rho - \rho_n)f(\rho) = [\rho]_n - (\rho_n - n + 1)[\rho]_{n-1} + \mu_0^1[\rho]_{n-1} - (\rho_n - n + 2)\mu_0^1[\rho]_{n-2} + \cdots + \mu_0^{n-1}[\rho]_1 - \rho_n\mu_0^{n-1} =$$

$$= [\rho]_n + \xi_0^1[\rho]_{n-1} + \cdots + \xi_0^n = \prod_{\nu=1}^{n}(\rho - \rho_\nu) = (\rho - \rho_n)\prod_{\nu=1}^{n-1}(\rho - \rho_\nu).$$

Now we will start by determination of $c^j_{n,n-j}(x)$, $(1 \le j \le n-1)$ and $q_{nn}(x)$. From (17) and (25) we obtain

$$c^1_{n,n-1} + s_n + F^1_1 = q_{nn} - a_{1r}x^r,$$

$$D_{jr}c^j_{n,n-j} + s_{n-j}c^j_{n,n-j} + c^{j+1}_{n,n-j-1} + F^{j+1}_{j+1} = [q_{nn}c^j_{n,n-j}]^1, \qquad (1 \le j \le n-1). \tag{31}$$

If we substitute

$$c^j_{n,n-j} = (\mu^j_0 + \mu^j_1 x + \cdots + \mu^j_{r-1}x^{r-1}), \quad (1 \le j \le n-1)$$

$$q_{nn} - a_{1r}x^r = t_0 + t_1 x + \cdots + t_{r-1}x^{r-1},$$

then from (31) it follows that

$$\mu^1_0 = t_0 - s_n - a_{10} = t_0 + (n-1)r - (n-1) - a_{10}, \tag{32}$$

$$\mu^1_m = t_m - F^1_1(m), \quad (1 \le m \le r-1) \tag{33}$$

$$\begin{aligned}\mu^{j+1}_0 &= (t_0 + jr - s_{n-j})\mu^j_0 - a_{j+1,0} = \\ &= [t_0 + (n-1)r - (n-r-1)]\mu^j_0 - a_{j+1,0},\end{aligned} \tag{34}$$

$$\begin{aligned}\mu^{j+1}_m &= [t_0 + (n-1)r - m - (n-j-1)]\mu^j_m + t_m\mu^j_0 + \\ &+ (t_1\mu^j_{m-1} + \cdots + t_{m-1}\mu^j_1) - F^{j+1}_{j+1}(m) \\ &(1 \le j \le n-1; 1 \le m \le r-1),\end{aligned} \tag{35}$$

where $\mu^n_m = 0$. By application of (28),(29) and (30) into (32) and (34), then it just follows that $t_0 + (n-1)r$ is one of the roots of ρ_i, $(1 \le i \le n)$ of the equation

$$[\rho]_n - \sum_{i=1}^{n} a_{i0}[\rho]_{n-i} = 0.$$

By substitution

$$t_0 + (n-1)r = \rho_n, \tag{36}$$

can be determined μ^j_0, $(1 \le j \le n-1)$. Using (33) and (35) we will determine μ^j_m, $(1 \le j \le n-1)$ and successively t_m. According to the equation (30), the determinant

$$\begin{vmatrix} \mu^1_0 + \rho_n - m - (n-2) & -1 & 0 & \cdots & 0 \\ \mu^2_0 & \rho_n - m - (n-3) & -1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \mu^{n-2}_0 & 0 & 0 & \cdots & -1 \\ \mu^{n-1}_0 & 0 & 0 & \cdots & \rho_n - m \end{vmatrix} =$$

$$= [\rho_n - m]_{n-1} + \mu_0^1[\rho_n - m]_{n-2} + \cdots + \mu_0^{n-1} = \prod_{\nu=1}^{n-1}(\rho_n - m - \rho_\nu)$$

is again different from zero, which means that only the coefficients $c^j_{n,n-j}(x)$ and $q_{nn}(x)$ can be determined.

Further successively will be determined the coefficients $c^j_{i,i-j}(x)$, $(1 \le j \le i-1)$ and $q_{ii}(x)$. ¿From (16) and (23), it follows that

$$c^1_{i,i-1} + s_i = q_{ii} + c^1_{i+1,i}$$

$$D_{jr}c^j_{i,i-j} + s_{i-j}c^j_{i,i-j} + c^{j+1}_{i,i-j-1} = [q_{ii}c^j_{i,i-j}]^1 + c^{j+1}_{i+1,i-j}, \quad (1 \le j \le i-1). \tag{37}$$

Further we apply the mathematical induction. Let

$$c^{j+1}_{i+1,i-j}(x) = \xi_0^{j+1} + \xi_1^{j+1}x + \cdots + \xi_{r-1}^{j+1}x^{r-1}, \qquad (0 \le j \le i-1)$$

be polynomials and let the constants ξ_0^j, $(1 \le j \le i)$ satisfy the equation

$$[\rho]_i + \xi_0^1[\rho]_{i-1} + \cdots + \xi_0^i = \prod_{\nu=1}^{i}(\rho - \rho_\nu). \tag{38}$$

By the changes

$$c^j_{i,i-j} = (\mu_0^j + \mu_1^j x + \cdots + \mu_{r-1}^j x^{r-1}), \qquad (1 \le j \le i-1)$$

$$q_{ii} = (t_0 + t_1 x + \cdots + t_{r-1}x^{r-1}),$$

we obtain

$$\mu_0^1 = t_0 + (i-1)r - (i-1) + \xi_0^1, \tag{39}$$

$$\mu_m^1 = t_m + \xi_m^1, \quad (1 \le m \le r-1) \tag{40}$$

$$\mu_0^{j+1} = [t_0 + (i-1)r - (i-j-1)]\mu_0^j + \xi_0^{j+1}, \tag{41}$$

$$\mu_m^{j+1} = [t_0 + (i-1)r - m - (i-j-1)]\mu_m^j + t_m\mu_0^j + (t_1\mu_{m-1}^j + \cdots + t_{m-1}\mu_1^j) + \xi_m^{j+1} \tag{42}$$

$$(1 \le j \le i-1; \quad 1 \le m \le r-1)$$

where $\mu_m^i = 0$. Applying the equation (28)-(30) to the equations (39)-(40), it follows

$$t_0 + (i-1)r = \rho_i \tag{43}$$

and

$$[\rho]_{i-1} + \mu_0^1[\rho]_{i-2} + \cdots + \mu_0^{i-1} = \prod_{\nu=1}^{i-1}(\rho - \rho_\nu). \tag{44}$$

Again, assuming that (3) and (4) hold, can be determined $c^j_{i,i-j}(x)$, $(1\le j\le i-1)$ and $q_{ii}(x)$. Thus by mathematical induction we proved that for each i, $(1\le i\le n)$, can be determined $c^j_{i,i-j}(x)$, $(1\le j\le i-1)$ and $q_{ii}(x)$. By using of (36), (38) (43) and (44), it obtains also that $q_{ii}(x)$ has the form

$$q_{ii}(x)=\rho_i-(i-1)r+q^1_{ii}x+\cdots+q^{r-1}_{ii}x^{r-1}+q^r_{ii}x^r,\quad (1\le i\le n)$$

where $q^r_{ii}=0$, $(i\ne n)$ and $q^r_{nn}=a_{1r}$.

Finally, we will prove that the sets of polynomials

$$\{q_{n,n-j}(x),c^m_{n-j-m}(x);\quad 1\le m\le n-j-1\}$$

and

$$\{q_{i,i-j}(x),c^m_{i,i-j-m}(x);\quad 1\le m\le i-j-1\},\quad (n-1\ge i\ge 1)$$

can be determined successively for values n and $(1\le j\le n-1)$. It will be proven by induction. Let the sets $\{q_{n,n-\nu},c^m_{n-\nu-m}\}$ and $\{q_{i,i-\nu},c^m_{i-\nu-m}\}$, $(1\le i\le n-1)$ are known for $0\le\nu\le j-1$. Then from (25) we obtain

$$D_{mr}c^m_{n,n-j-m}+s_{n-j-m}c^m_{n,n-j-m}+c^{m+1}_{n,n-j-m-1}=$$

$$=[q_{nn}c^m_{n,n-j-m}]^1+[q_{n,n-j}c^m_{n-j,n-j-m}]^1+$$

$$+\sum_{\nu=1}^{j-1}[q_{n,n-\nu}c^m_{n-\nu,n-j-m}]^1+\sum_{\nu=1}^{j-1}[q_{n,n-\nu}c^{m+1}_{n-\nu,n-j-m}]^0-F^{m+1}_{j+m+1},\qquad (1\le m\le n-j-1).\quad (45)$$

By the change

$$c^j_{n-j,n-j-m}=\xi^m_0+\xi^m_1x+\cdots+\xi^m_{r-1}x^{r-1},\qquad (1\le m\le n-j-1)$$

and using (38) and (44), we obtain

$$[\rho]_{n-j-1}+\xi^1_0[\rho]_{n-j-2}+\cdots+\xi^{n-j-1}_0=\prod_{i=1}^{n-j-1}(\rho-\rho_i).\qquad (46)$$

If we substitute

$$q_{n,n-j}=t_0+t_1x+\cdots+t_{r-1}x^{r-1},$$

$$c^m_{n,n-j-m}=\mu^m_0+\mu^m_1x+\cdots+\mu^m_{r-1}x^{r-1},\qquad (1\le m\le n-j-1)$$

then from (26) follow the equalities

$$t_r=a_{j+1,r(j+1)},\qquad (47)$$

$$t_\nu=\mu^1_\nu+\cdots,\qquad (0\le\nu\le r-1)\qquad (48)$$

and from (45) we obtain

$$(-mr+\nu+s_{n-j-m})\mu^m_\nu+\mu^{m+1}_\nu=$$

$$=(p_0\mu^m_\nu+\cdots+p_\nu\mu^m_0)+(t_0\xi^m_\nu+\cdots+t_\nu\xi^m_0)+\cdots\qquad (49)$$

$$(1 \le m \le n-j-1; \quad 0 \le \nu \le r-1)$$

where we can substitute

$$q_{nn} = p_0 + p_1 x + \cdots + p_{r-1}x^{r-1} + p_r x^r, \qquad (p_0 = \rho_n - (n-1)r).$$

The equation (49) can be represented in the following form

$$\mu_\nu^{m+1} = [\rho_n - jr - \nu - (n-j-m-1)]\mu_\nu^m + t_\nu \xi_0^m +$$
$$+(p_1\mu_{\nu-1}^m + \cdots + p_\nu \mu_0^m) + (t_0\xi_\nu^m + \cdots + t_{\nu-1}\xi_1^m) + \cdots \tag{50}$$

and hence we obtain the determinant

$$\begin{vmatrix} \xi_0^1 + \rho_n - jr - \nu - (n-j-2) & -1 & 0 & \cdots & 0 \\ \xi_0^2 & \rho_n - jr - \nu - (n-j-3) & -1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \xi_0^{n-j-2} & 0 & 0 & \cdots & -1 \\ \xi_0^{n-j-1} & 0 & 0 & \cdots & \rho_n - jr - \nu \end{vmatrix} =$$

$$= [\rho_n - jr - \nu]_{n-j-1} + \xi_0^1[\rho_n - jr - \nu]_{n-j-2} + \cdots + \xi_0^{n-j-1} =$$

$$= \prod_{i=1}^{n-j-1} (\rho_n - rj - \nu - \rho_i) \neq 0.$$

Thus, the set $\{q_{n,n-j}, c_{n,n-j-m}^m\}$ is determined uniquely. In order to prove that the sets $\{q_{i,i-\nu},\ c_{i,i-\nu-m}^m\}$, $(1 \le i \le n-1)$ can uniquely be determined, we apply again mathematical induction. The evaluation of their proof is not need here, since such sets of polynomials are just the same as the previous. ■

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