Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISBN 1083-0464
Issue 1 July 1999, pp. 1 – 12
Submitted: April 4, 1998. Published: July 1, 1999

The Stability of Nash-Cournot Equilibria in Labor Managed Oligopolies

Weiye Li
Ferenc Szidarovszky

Abstract

This paper examines the asymptotical stability of equilibria in discrete dynamic labor-managed oligopolies. First the equivalence of the equilibrium problem of a large class of nonlinear games and the equilibrium problem of a class of discrete dynamic systems is verified. Stability conditions are then derived for a certain class of dynamic models, and these results are finally applied to labor-managed oligopolies. The economic interpretation of the stability conditions are also presented.

1991 A.M.S. Subject Classification Codes. xxxx
Key Words and Phrases: yyyyy zzzz wwww xxxxx

Weiye Li, Department of Mathematics, University of Arizona
Tucson, Arizona, 85721
Ferenc Szidarovszky, Systems and Industrial Engineering Department,
University of Arizona, Tucson, Arizona, 85721
Email: szidar@sie.arizona.edu

1 Introduction

The stability of dynamic economic systems is one of the most frequently discussed problem areas in the modern economic literature. Stability conditions are usually obtained by using the stability theory of linear or nonlinear differential and difference equations.

One of the most important model classes consists of the different variants of the Cournot oligopoly model. A comprehensive summary of such models and stability results in both continuous and discrete cases is presented in Okuguchi [1], and Okuguchi and Szidarovszky [2].

In this paper the stability of labor-managed dynamic oligopolies will be examined. General stability conditions will be first introduced, and will be then applied to special models.

The paper is organized as follows. In Section 2 some general results are presented, and in Section 3, these results are applied to a special class of dynamic models, and in Section 4 the further special case of labor-managed oligopolies will be analyzed. Section 5 concludes the paper.

2 Nash-Cournot Equilibria and Fixed Points of Dynamic Systems

Oligopoly is a state of industry where a small N number of firms produce M homogeneous goods or close substitutes competitively. The multiproduct *Cournot Oligopoly* is an N-person noncooperative game defined as follows. Let $x_k^{(m)}$ denote the output of firm k of product m, then the output of the firm can be characterized by output vector $\mathbf{x_k} = (\mathbf{x_k^{(1)}}, ..., \mathbf{x_k^{(M)}})$. Let $\mathbf{s} = \sum_{\mathbf{k=1}}^{\mathbf{N}} \mathbf{x_k}$ denote the output vector of the industry, and assume that the price vector depends on $\mathbf{s}$: $\mathbf{p} = \mathbf{p}(\mathbf{s})$. If c_k $(k = 1, 2, ..., N)$ denotes the cost function of firm k, then its profit can be expressed as

$$\varphi_k(\mathbf{x_1},\ ...,\ \mathbf{x_N}) = \mathbf{x_k^T}\,\mathbf{p}(\mathbf{s}) - \mathbf{c_k}(\mathbf{x_k}). \tag{2.1}$$

If $X_k \subset \mathbb{R}_+^{\mathbb{M}}$ denotes the set of all feasible outputs for firm k, then the resulting game can be given in strategic form as $\Gamma = (N; X_1, ..., X_N; \varphi_1, ..., \varphi_N)$. In the single product case, $M = 1$, and we may select $X_k = [0, L_k]$, where L_k is the capacity limit of firm k.

A vector $\mathbf{x}^* = (\mathbf{x_1^*}, \mathbf{x_2^*}, ..., \mathbf{x_N^*}) \in \mathbf{X} = \mathbf{X_1} \oplus \mathbf{X_2} \oplus ... \oplus \mathbf{X_N}$ is called a Nash-Cournot equilibrium point of game Γ, if for $k = 1, 2, ..., N$,

1. $\mathbf{x_k^*} \in \mathbf{X_k}$;
2. For arbitrary $\mathbf{x_k} \in \mathbf{X_k}$,

$$\varphi_k(\mathbf{x_1^*}, ..., \mathbf{x_{k-1}^*}, \mathbf{x_k^*}, \mathbf{x_{k+1}^*}, ..., \mathbf{x_N^*}) \quad \geq \quad \varphi_k(\mathbf{x_1^*}, ..., \mathbf{x_{k-1}^*}, \mathbf{x_k}, \mathbf{x_{k+1}^*}, ..., \mathbf{x_N^*}).$$

In other words, the Nash-Cournot equilibrium is an N-tuple of strategies at which each player maximizes his own payoff with respect to his own strategy selection, given the strategy choices of all other players. If a Nash-Cournot equilibrium $\mathbf{x}^*$ exists and is an interior point of X, and all φ_k are continuously differentiable in X_k, then for $k = 1, 2, ..., N$ and $m = 1, 2, ..., M$,

$$\frac{\partial \varphi_k}{\partial x_k^{(m)}} \quad = \quad 0, \tag{2.2}$$

where the partial derivatives are taken with respect to each component.

In the continuous case, it is natural to assume that the rate of change of the k-th player's strategy selection with respect to time is positively proportional to his marginal payoff with respect to his strategy, namely,

$$\dot{\mathbf{x}}_k \quad = \quad \mathbf{C_k} \nabla_\mathbf{k} \varphi_\mathbf{k}(\mathbf{x}), \qquad\qquad \mathbf{k = 1,\ 2,\ ...,\ N}, \tag{2.3}$$

where $\nabla_k \varphi_k$ denotes the gradient of φ_k with respect to $\mathbf{x_k}$, and $\mathbf{C_k}$ is a diagonal constant matrix with positive diagonal elements. It is obvious that the interior Nash-Cournot equilibrium is a fixed point of the dynamical system (2.3).

In the discrete case, each player maximizes his payoff in every time period based on his knowledge on the strategies expected from the other players. Such expectations rely in general on the actual and the expected strategies taken in the last time periods.

Assume that equation (2.2) has a unique soloution $\mathbf{x_k}$ in terms of variables $\mathbf{x_1}, ..., \mathbf{x_{k-1}}, \mathbf{x_{k+1}}, ..., \mathbf{x_N}$ as

$$\mathbf{x_k} \quad = \quad \mathbf{f_k}(\mathbf{x_1},\ ...,\ \mathbf{x_{k-1}},\ \mathbf{x_{k+1}},\ ...,\ \mathbf{x_N}).$$

Besides, let $\mathbf{x_k(t)}$ be the actual strategy of player k and $\tilde{\mathbf{x}}_i^k(t)$ his expectation on the strategy of player i at time t $(i \neq k)$. Assume $\tilde{\mathbf{x}}_i^k(t+1)$ is a function of only $\mathbf{x_i(t)}$ and $\tilde{\mathbf{x}}_i^k(t)$, that is,

$$\tilde{\mathbf{x}}_i^k(t+1) \quad = \quad \mathbf{g_i^k}(\mathbf{x_i(t)},\ \tilde{\mathbf{x}}_\mathbf{i}^\mathbf{k}(\mathbf{t})). \tag{2.4}$$

In addition, we assume that

$$\mathbf{g_i^k}(\mathbf{a}, \mathbf{a}) \quad = \quad \mathbf{a} \tag{2.5}$$

for any vector $\mathbf{a}$ and $k \neq i$, for which the reason is obvious.

As a special case, if $\mathbf{g_i^k}$ is a convex linear combination, i.e.,

$$\mathbf{g_i^k}(\mathbf{a}, \mathbf{b}) \quad = \quad \mathbf{D_i^k}\,\mathbf{a} + (\mathbf{I} - \mathbf{D_i^k})\,\mathbf{b}, \tag{2.6}$$

where $\mathbf{D_i^k}$ is a diagonal matrix with entries in $(0, 1]$. This scheme is called the *adaptive expectation*. Obviously, any adaptive expectation satisfies the natural assumption (2.5).

Due to the fact that each player maximizes his payoff based on the expected strategies of the others, and assuming interior optimum throughout, it is easy to see that for all k,

$$\mathbf{x_k(t)} \quad = \quad \mathbf{f_k}(\tilde{\mathbf{x}}_\mathbf{1}^\mathbf{k}(\mathbf{t}),\ ...,\ \tilde{\mathbf{x}}_\mathbf{k-1}^\mathbf{k}(\mathbf{t}),\ \tilde{\mathbf{x}}_\mathbf{k+1}^\mathbf{k}(\mathbf{t}),\ ...,\ \tilde{\mathbf{x}}_\mathbf{N}^\mathbf{k}(\mathbf{t})). \tag{2.7}$$

The dynamical system with state variables

$$\{\mathbf{x_k(t)},\ \tilde{\mathbf{x}}_\mathbf{i}^\mathbf{k}(\mathbf{t}) \ \Big|\ k \neq i;\ k, i = 1, ..., N\} \tag{2.8}$$

is well determined by relations (2.4), (2.5), (2.7) and the choice of the initial expectations $\{\tilde{\mathbf{x}}_i^k(0) \,\big|\, k \neq i;\ k, i = 1, ..., N\}$. If we start the process at the Nash-Cournot equilibrium by setting the initial expectations $\tilde{\mathbf{x}}_i^k(0) = \mathbf{x_k^*}$ for all $k \neq i$, then $\mathbf{x_k(0)} = \mathbf{x_k^*}$ by (2.7) and (2.2), $\tilde{\mathbf{x}}_i^k(1) = \mathbf{x_k^*}$ by the assumption of $\mathbf{g_k}$, etc. Inductively, we have $\tilde{\mathbf{x}}_i^k(t) = \mathbf{x_k(t)} = \mathbf{x_k^*}$ for all t. Thus the N copies of the Nash-Cournot equilibrium is the fixed point of the dynamical system (2.8) when the order of variables in the system is arraged properly.

The above results can be summarized in the following theorem.

Theorem 2.1 *Assume that for all k, φ_k is continuously differentiable as an MN variable function. Then the interior Nash-Cournot equilibrium of the above N-person game corresponds to some fixed point of the dynamical system governed by (2.3) (when the system is continuous), or (2.2), (2.4), (2.5) and (2.7) (when the system is discrete).*

It is also easy to see that any fixed point is an interior equilibrium if φ_k is concave in $\mathbf{x_k}$ for all k. Therefore a corollary is derived.

Corollary 2.2 *In addition to the assumption of Theorem 2.1, assume that φ_k is concave in $\mathbf{x_k}$ for all k. Then the interior Nash-Cournot equilibrium problem of the above N-person game is equivalent to the fixed point problem of the dynamical system governed by (2.3) (when the system is continuous), or (2.2), (2.4), (2.5) and (2.7) (when the system is discrete).*

The stability or asymptotical stability of the Nash-Cournot equilibrium is of great interest in economic theory. It is well known that the eigenvalues of the Jacobian matrix of the transition function of a dynamical system at the fixed point can determine the stability and asymptotical stability in most cases. See for example, Bellman [3], and Li and Szidarovszky [4]. For instance, for a continuous system, if all eigenvalues have negative real parts, then the fixed point is asymptotically stable; for a discrete system, if all eigenvalues are inside the unit circle, then this fixed point is asymptotically stable. It is also known from the theory of differential and difference equations that if at least one eigenvalue of the Jacobian matrix at the equilibrium has positive real parts then the equilibrium of the continuous system is unstable; and if at least one eigenvalue is outside the unit circle, then the equilibrium of the discret system is unstable. These sufficient conditions are easy to be applied to the continuous system (2.3) by considering the eigenvalues of the Jacobian matrix $\frac{\partial(\mathbf{C_1}\frac{\partial \varphi_\mathbf{1}}{\partial \mathbf{x_1}}\,...,\mathbf{C_N}\frac{\partial \varphi_\mathbf{N}}{\partial \mathbf{x_N}})}{\partial(\mathbf{x_1},...,\mathbf{x_N})}$.

In the rest of this paper the discrete case will be analysed. Introduced first the notations

$$\mathbf{x} \;=\; (\mathbf{x_1},\, \mathbf{x_2},\, ...\,,\, \mathbf{x_N})^\mathbf{T} \in \mathbb{R}^{\mathbb{N}\mathbb{M}},$$

$$\tilde{\mathbf{x}} \;=\; (\tilde{\mathbf{x}}_2^1,\, \tilde{\mathbf{x}}_3^1,\, ...,\, \tilde{\mathbf{x}}_N^1; \tilde{\mathbf{x}}_1^2,\, \tilde{\mathbf{x}}_3^2,\, ...,\, \tilde{\mathbf{x}}_N^2,\; ...;\, \tilde{\mathbf{x}}_1^N,\, \tilde{\mathbf{x}}_2^N,\, ...,\, \tilde{\mathbf{x}}_{N-1}^N)^T \in \mathbb{R}^{(\mathbb{N}-\mathbb{1})\mathbb{N}\mathbb{M}}.$$

Accordingly, denote $\mathbf{g}$, a mapping from $(X_1 \oplus X_2 \oplus \cdots \oplus X_N) \oplus (X_2 \oplus \cdots \oplus X_N) \oplus (X_1 \oplus X_3 \oplus \cdots \oplus X_N) \oplus \cdots \oplus (X_1 \oplus \cdots \oplus X_{N-1})$ into $(X_2 \oplus \cdots \oplus X_N) \oplus (X_1 \oplus X_3 \oplus \cdots \oplus X_N) \oplus \cdots \oplus (X_1 \oplus \cdots \oplus X_{N-1})$, by

$$\mathbf{g}(\mathbf{x}, \tilde{\mathbf{x}}) \;=\; (\mathbf{g_i^k}(\mathbf{x_i},\, \tilde{\mathbf{x}}_\mathbf{i}^\mathbf{k}) \,\big|\, \mathbf{i} \neq \mathbf{k};\, \mathbf{k},\, \mathbf{i} = \mathbf{1}, \mathbf{2}, ..., \mathbf{N}).$$

Denote $\mathbf{f}$, a mapping from $(X_2 \oplus \cdots \oplus X_N) \oplus (X_1 \oplus X_3 \oplus \cdots \oplus X_N) \oplus \cdots \oplus (X_1 \oplus \cdots \oplus X_{N-1})$ into $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)$, by

$$\mathbf{f}(\tilde{\mathbf{x}}) \;=\; (\mathbf{f_k}(\tilde{\mathbf{x}}_\mathbf{i}^\mathbf{k}, ..., \tilde{\mathbf{x}}_\mathbf{k-1}^\mathbf{k}, \tilde{\mathbf{x}}_\mathbf{k+1}^\mathbf{k}, ..., \tilde{\mathbf{x}}_\mathbf{N}^\mathbf{k}) \,\big|\, \mathbf{k} = \mathbf{1}, \mathbf{2}, ..., \mathbf{N}).$$

Then system (2.8) is described by the following simple equalities

$$\begin{aligned} \mathbf{x}(\mathbf{t}) &= \mathbf{f}(\tilde{\mathbf{x}}(\mathbf{t})) \\ \tilde{\mathbf{x}}(t+1) &= \mathbf{g}(\mathbf{x}(\mathbf{t}),\, \tilde{\mathbf{x}}(\mathbf{t})), \end{aligned} \tag{2.9}$$

which can be rewritten as

$$\begin{aligned} \mathbf{x}(\mathbf{t}+\mathbf{1}) &= \mathbf{f}(\mathbf{g}(\mathbf{x}(\mathbf{t}),\, \tilde{\mathbf{x}}(\mathbf{t})) \\ \tilde{\mathbf{x}}(t+1) &= \mathbf{g}(\mathbf{x}(\mathbf{t}),\, \tilde{\mathbf{x}}(\mathbf{t})). \end{aligned} \tag{2.10}$$

The N-direct sum of the Nash-Cournot equilibrium up to some permutation of the components is a fixed point of system (2.9) or (2.10). The Jacobian matrix of the transition function is

$$\frac{\partial(\mathbf{x(t+1)},\, \tilde{\mathbf{x}}\mathbf{(t+1)})}{\partial(\mathbf{x(t)},\, \tilde{\mathbf{x}}\mathbf{(t)})} = \begin{pmatrix} D\mathbf{f}(\mathbf{g}(\mathbf{x(t)},\, \tilde{\mathbf{x}}\mathbf{(t)})) \cdot \mathbf{D_1 g}(\mathbf{x(t)},\, \tilde{\mathbf{x}}\mathbf{(t)}) & D\mathbf{f}(\mathbf{g}(\mathbf{x(t)},\, \tilde{\mathbf{x}}\mathbf{(t)})) \cdot \mathbf{D_2 g}(\mathbf{x(t)},\, \tilde{\mathbf{x}}\mathbf{(t)}) \\ D_1\mathbf{g}(\mathbf{x(t)},\, \tilde{\mathbf{x}}\mathbf{(t)}) & D_2\mathbf{g}(\mathbf{x(t)},\, \tilde{\mathbf{x}}\mathbf{(t)}) \end{pmatrix},$$

where D, D_1, and D_2 are differential operations with respect to $\mathbf{g}$, $\mathbf{x}$, $\tilde{\mathbf{x}}$, respectively. That is, $D\mathbf{f}$, $D_1\mathbf{g}$, and $D_2\mathbf{g}$ are the Jacobian matrices of $\mathbf{f}$ and $\mathbf{g}$ with respect to these variables. For simplicity, let us denote the Jacobian matrix of (2.10) at the fixed point as

$$\mathbf{J} = \begin{pmatrix} D\mathbf{f} \cdot \mathbf{D_1 g} & D\mathbf{f} \cdot \mathbf{D_2 g} \\ D_1\mathbf{g} & D_2\mathbf{g} \end{pmatrix}.$$

Let $(\mathbf{u}, \mathbf{v})^{\mathbf{T}}$ be an eigenvector associated to an eigenvalue λ of $\mathbf{J}$. Then

$$\begin{aligned} D\mathbf{f}(\mathbf{D_1 g}\,\mathbf{u} + \mathbf{D_2 g}\,\mathbf{v}) &= \lambda\mathbf{u} \\ D_1\mathbf{g}\,\mathbf{u} + \mathbf{D_2 g}\,\mathbf{v} &= \lambda\mathbf{v}. \end{aligned}$$

Simple substitution derives either $\lambda = 0$ or $\mathbf{u} = \mathbf{Df}\,\mathbf{v}$. A zero eigenvalue does not distroy stability and asymptotical stability. If $\lambda \neq 0$, then

$$(D_1\mathbf{g}\,\mathbf{Df} + \mathbf{D_2 g})\,\mathbf{v} = \lambda\mathbf{v}.$$

That is, any nonzero eigenvalue of $\mathbf{J}$ must be an eigenvalue of matrix

$$D_1\mathbf{g}\,\mathbf{Df} + \mathbf{D_2 g}, \tag{2.11}$$

whose size is only $(N-1)NM \times (N-1)NM$. If this matrix has all eigenvalues inside the unit circle, then the Nash-Cournot equilibrium is asymptotically stable.

3 Stability in Single-product Oligopolies with Payoff Fuctions $\varphi_k(x_k,\, \Sigma_{i\neq k}\, x_i)$ and Adapted Expectations

The general approach outlined above requires a tedious computation in manipulating with matrix (2.11). In practice, this approach can be handled tackfully. The main idea is to reduce the number of variables in the dynamical system. This idea can be realized when the firms form expectations on the output of the rest of the industry, since the $N(M-1)$-dimensional expectation variable becomes only M-dimensional. Another case, when reduction in the dimension is possible, especially when the M goods are mutaually independent, i.e., the payoff function of each player is the sum of M functions where each of them is a payoff corresponding to a single good. Thus system (2.9) can be decomposed into M equivalent smaller dimensional subsystems.

Consider next a special single product case $\Gamma = (N; X_1, ..., X_N; \varphi_\mathbf{1}, ..., \varphi_\mathbf{N})$, where $X_k = [0, L_k]$, $\varphi_k(x_1,\, x_2,\, ...,\, x_N) = \phi(x_k, \sum_{i\neq k} x_i)$. For example, in the case of single product oligopolies without product differentiation, $\varphi_k(x_k, \sum_{i\neq k} x_i) = x_k\, p(x_k + \sum_{i\neq k} x_i) - c_k(x_k)$.

If adaptive expectations are assumed, it coincides with $g_i^k(a,b) = d_k a + (1-d_k)b$ for all $i \neq k$, and thus

$$\tilde{s}_k(t+1) \;=\; d_k s_k(t) + (1-d_k)\tilde{s}_k(t)$$

with $s_k(t) = \sum_{i\neq k} x_i(t)$ and $\tilde{s}_k(t) = \sum_{i\neq k} \tilde{x}_i(t)$. This gives us

$$x_k(t) \;=\; f_k(\tilde{s}_k(t)), \qquad \text{and} \qquad \tilde{s}_k(t+1) \;=\; g_k(s_k(t), \tilde{s}_k(t)).$$

Thus system (2.9) can be reduced to

$$\begin{aligned} \mathbf{x(t)} &= \mathbf{f}(\tilde{\mathbf{s}}(\mathbf{t})) \\ \tilde{\mathbf{s}}(t+1) &= \mathbf{g}(\mathbf{s}(\mathbf{t}), \tilde{\mathbf{s}}(\mathbf{t})) \end{aligned} \tag{3.1}$$

In this case, matrix (2.11) becomes

$$\begin{aligned} & D_1\mathbf{g}\,\mathbf{Df} + \mathbf{D_2g} \\ = & \begin{pmatrix} 0 & d_1 & \cdots & d_1 \\ d_2 & 0 & \cdots & d_2 \\ \vdots & \vdots & & \vdots \\ d_N & d_N & \cdots & d_N \end{pmatrix} \begin{pmatrix} Df_1 & & & \\ & Df_2 & & \\ & & \ddots & \\ & & & Df_N \end{pmatrix} + \begin{pmatrix} 1-d_1 & & & \\ & 1-d_2 & & \\ & & \ddots & \\ & & & 1-d_N \end{pmatrix} \\ = & \Lambda + \mathbf{a}\,\mathbf{b^T}, \end{aligned}$$

with

$$\Lambda \;=\; \begin{pmatrix} 1-d_1-d_1Df_1 & & & \\ & 1-d_2-d_2Df_2 & & \\ & & \ddots & \\ & & & 1-d_N-d_NDf_N \end{pmatrix},$$

$$\mathbf{a} \;=\; \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{pmatrix}, \qquad \text{and} \qquad \mathbf{b} \;=\; \begin{pmatrix} Df_1 \\ Df_2 \\ \vdots \\ Df_N \end{pmatrix}.$$

The characteristic polynomial of this matrix is given as

$$\begin{aligned} \phi(\lambda) &= \det(\Lambda - \lambda\mathbf{I} + \mathbf{ab}^T) \\ &= \det(\Lambda - \lambda\mathbf{I}) \cdot \det(\mathbf{I} + (\Lambda - \lambda\mathbf{I})^{-1}\mathbf{ab}^T)\,. \end{aligned}$$

It is well known that if $\mathbf{I}$ is the $n \times n$ identity matrix, and $\mathbf{u}$ and $\mathbf{v}$ are n-dimensional real (or complex) vectors, then

$$\det(\mathbf{I} + \mathbf{uv}^T) \;=\; 1 + \mathbf{v}^T\mathbf{u},$$

which can be proved easily by using finite induction with respect to n. Therefore

$$\begin{aligned}\phi(\lambda) &= \det(\Lambda - \lambda \mathbf{I}) \cdot [1 + \mathbf{b}^T(\Lambda - \lambda \mathbf{I})^{-1}\mathbf{a}] \\ &= \prod_{k=1}^{N}(1 - d_k - d_k\, Df_k - \lambda)\left(1 + \sum_{k=1}^{N} \frac{d_k\, Df_k}{1 - d_k - d_k\, Df_k - \lambda}\right).\end{aligned} \tag{3.2}$$

Introduce the notation $\gamma_k = 1 - d_k - d_k\, Df_k$, $\xi_k = d_k\, Df_k$, and denote by $\eta_1 < \eta_2 < \cdots < \eta_s$ the distinct γ_k values with multiplicities $r_1, r_2, ..., r_s$. Let $I_j = \{i \,\big|\, \gamma_i = \eta_j\}$, and $\theta_j = \sum_{i \in I_j} \xi_i$ for $j = 1, 2, ..., s$. Then solving for the roots of (3.2) becomes finding the solution to equation

$$\prod_{j=1}^{s}(\lambda - \eta_j)^{r_j}\left(\sum_{j=1}^{s} \frac{\theta_j}{\lambda - \eta_j} - 1\right) = 0. \tag{3.3}$$

The left hand side is an Nth-degree polynomial with roots η_j with multiplicities $r_j - 1$ when $\theta_j \neq 0$, or with multiplicities r_j when $\theta_j = 0$. All other roots of (3.3) are the roots of function

$$G(\lambda) = \sum_{j=1}^{s} \frac{\theta_j}{\lambda - \eta_j} - 1. \tag{3.4}$$

Finding the roots of this function is equivalent to finding the solution of a polynomial equation of degree which equals the number of nonzero θ_j. It is easy to see that if all roots of $G(\cdot)$ are real, then (3.3) has N real roots.

The following result gives a characterization of the roots of $G(\cdot)$.

Lemma 3.1 *Assume that all $\theta_j \neq 0$, and the number of sign changes of sequence $\{\theta_1, \theta_2, ..., \theta_s\}$ is at most 1 in such a way that when the sign change occurs, it is from "$-$" to "$+$". Then (3.4) has exactly s real roots. If there is one sign change, then $s - 2$ roots are in (η_1, η_s), one is in $(-\infty, \eta_1)$ and one is in (η_s, ∞). If there is no sign change, then $s - 1$ roots are in (η_1, η_s), and the last root is in $(-\infty, \eta_1)$ when all $\theta_j < 0$, or in (η_s, ∞) when all $\theta_j > 0$.*

Proof: Under the assumption of the lemma, $G(\cdot)$ has the following properties.

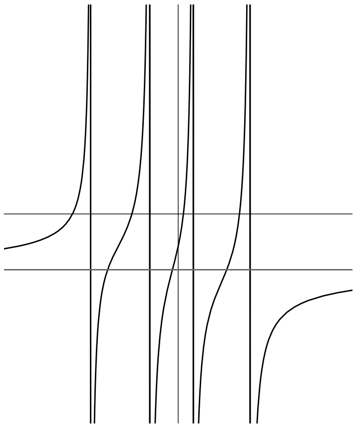
(a) $G(-\infty) = G(\infty) = -1$.

(b) If all $\theta_j < 0$, then $G(\lambda) > -1$ for any $\lambda < \eta_1$; $G(\lambda) < -1$ for any $\lambda > \eta_s$. Furthermore, $\lim_{\lambda \to \eta_j - 0} G(\lambda) = +\infty$, $\lim_{\lambda \to \eta_j + 0} G(\lambda) = -\infty$.

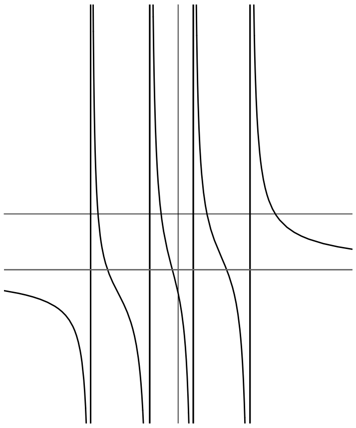
(c) If all $\theta_j > 0$, then $G(\lambda) < -1$ for any $\lambda < \eta_1$; $G(\lambda) > -1$ for any $\lambda > \eta_s$. Furthermore, $\lim_{\lambda \to \eta_j - 0} G(\lambda) = -\infty$, $\lim_{\lambda \to \eta_j + 0} G(\lambda) = +\infty$.

(d) If the only sign change is at l and $\theta_l < 0$ and $\theta_{l+1} > 0$, then $G(\lambda) < -1$ for any $\lambda \in (\eta_l, \eta_{l+1})$. Furthermore, $\lim_{\lambda \to \eta_j - 0} G(\lambda) = +\infty$, $\lim_{\lambda \to \eta_j + 0} G(\lambda) = -\infty$ for $j \leq l$; $\lim_{\lambda \to \eta_j - 0} G(\lambda) = -\infty$, $\lim_{\lambda \to \eta_j + 0} G(\lambda) = +\infty$ for $j > l$.

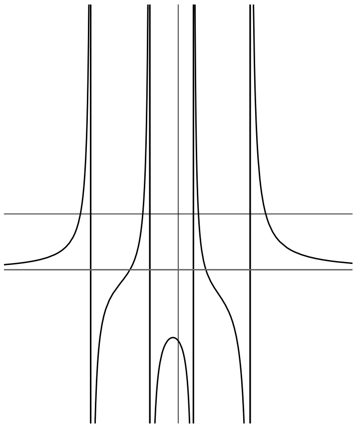
Using the Intermediate Value Theorem and the above properties of $G(\cdot)$, it is easy to see that the graph of $G(\cdot)$ has only three patterns which correspond to the three cases as given before, and the graph intersects the horizontal axis exactly s times in all cases.



All $\theta_j < 0$



All $\theta_j > 0$



One sign change

□

From the above discussion, we know that the characteristic equation of matrix (2.11) has at most $N-2$ roots in the closed interval $[\eta_1,\, \eta_s]$. To ensure that all the eigenvalues have absolute values less than 1, we need that all $|\eta_j| < 1$ plus either $G(-1) < 0$ or $G(1) < 0$, or both $G(-1) < 0$ and $G(1) < 0$, fully depending whether there are either all $\theta_j \le 0$ or all $\theta_j \ge 0$, or there is one sign change occuring in the way as stated in Lemma 3.1.

Theorem 3.2 *Assume that system (3.1) has an interior Nash-Cournot equilibrium and all conditions of Lemma 3.1 are met. Then any one of the following three cases ensures that the Nash-Cournot equilibrium is asymptotically stable:*

1. All $|\eta_j| < 1$, all $\theta_j \le 0$, and $G(-1) < 0$.

2. All $|\eta_j| < 1$, all $\theta_j \ge 0$, and $G(1) < 0$.

3. All $|\eta_j| < 1$, $\theta_j \le 0$ for $j < l$, $\theta_l < 0$, $\theta_{l+1} > 0$, $\theta_j \ge 0$ for $j > l+1$ and $G(-1) < 0$, $G(1) < 0$.

As a special case, assume that all $Df_k \in (-1,\, 0]$. Then all $\gamma_k = 1 - m_k - m_k\, Df_k \in [0,\, 1)$, and all $\xi_k = m_k\, Df_k \le 0$. So all $|\eta_j| < 1$, and all $\theta_j = \sum_{i \in I_j} \xi_i \le 0$. Therefore, we have

Corollary 3.3 *Assume that system (3.1) has an interior Nash-Cournot equilibrium. If for all k, $-1 < Df_k \le 0$, and $G(-1) < 0$, then the equilibrium is asymptotically stable.*

Similarly to Theorem 3.2, one can prove the following necessary stability conditions.

Theorem 3.4 *Assume that system (3.1) has an interior stable Nash-Cournot equilibrium, all conditions of Lemma 3.1 are met, and $s \ge 2$.*

1. If all $\theta_j \le 0$, then $|\eta_j| < 1$ for $j = 1, 2, ..., s-1$. Furthermore, either $\eta_s \le 1$ and $G(-1) \le 0$ or $\eta_s > 1$ and $G(-1) \le 0$ and $G(1) \ge 0$;

2. If all $\theta_j \ge 0$, then $|\eta_j| < 1$ for $j = 2, ..., s$. Furthermore, either $\eta_1 \ge -1$ and $G(1) \le 0$ or $\eta_1 < -1$ and $G(-1) \ge 0$ and $G(1) \le 0$;

3. If $\theta_j \ge 0$ for $j < l$, $\theta_l < 0$, $\theta_{l+1} > 0$, $\theta_j \ge 0$ for $j > l+1$, then $|\eta_j| < 1$ for $j = 1, 2, ..., s$. Furthermore,$G(-1) \le 0$ and $G(1) \le 0$.

4 Stability in Labor Managed Oligopolies

As a particular example, we will discuss the asymptotical stability of the Nash-Cournot equilibrium of the single product case of labor managed oligopolies. Assume now that N firms produce a single good and the payoff function of each firm is the surplus per labor of the firm. Let p be the price (or inverse demand) function which is assumed to be a function of the total production of the industry. Let w be the competitive wage rate, and c_k the fixed cost of firm k. Let h_k be the number of labors in firm k as a function of the production level of firm k. Then the payoff function of firm k is given as

$$\varphi_k(x_1,\, ...,\, x_N) = \frac{x_k\, p(s) - w\, h_k(x_k) - c_k}{h_k(x_k)} \tag{4.1}$$

with $s = \sum_{i=1}^{N} x_i$. Assuming interior equilibrium, we have

$$\frac{\partial \varphi_k}{\partial x_k} = \frac{(p(s) + x_k\, p'(s))\, h_k(x_k) - (x_k\, p(s) - c_k)\, h_k'(x_k)}{h_k^2(x_k)} = 0. \tag{4.2}$$

The existence of the Nash-Cournot equilibrium has been proved under realistic conditions in Okuguchi [5]. From (4.2), we get

$$(p(s) + x_k\, p'(s))\, h_k(x_k) - (x_k\, p(s) - c_k)\, h_k'(x_k) = 0. \tag{4.3}$$

Assume that for each $s_k = \sum_{i \neq k} x_i$, there is a unique solution for x_k, $x_k = f_k(s_k)$. Using the chain rule to differentiate both sides of equality (4.3) with respect to s_k, we get

$$\begin{aligned}
&[p'(s)\,(1 + Df_k) + Df_k\, p'(s) + x_k\, p''(s)\,(1 + Df_k)]\, h_k(x_k) + [p(s) + x_k\, p'(s)]\, h_k'(x_k)\, Df_k \\
&\qquad - [Df_k\, p(s) + x_k p'(s)(1 + Df_k)]\, h_k'(x_k) - [x_k\, p(s) - c_k]\, h_k''\, Df_k \;=\; 0.
\end{aligned}$$

Solve for Df_k to get

$$\begin{aligned}
Df_k &= \frac{x_k\, p'(s)\, h_k'(x_k) - [p'(s) + x_k\, p''(s)]\, h_k(x_k)}{[2p'(s) + x_k\, p''(s)]\, h_k(x_k) - [x_k\, p(s) - c_k]\, h_k''(x_k)} \\
&= \frac{p'(s)\, [x_k\, h_k(x_k)]' - [x_k\, p(x_k)]''\, h_k(x_k)}{[x_k\, p(x_k)]''\, h_k(x_k) - [x_k\, p(s) - c_k]\, h_k''(x_k)}.
\end{aligned} \tag{4.4}$$

We can study the magnitudes of Df_k to draw conclusions on the stability or asymptotical stability of the labor-managed oligopoly by using the results in Section 3. As a special case, consider the linear case, when $p(s) = As + b$ with $A < 0$, $b > 0$, and $h_k(x_k) = a_k x_k + b_k$, with $a_k > 0$, $b_k \geq 0$ for all k. From (4.4), $Df_k = -\frac{b_k}{2h_k} \in (-1, 0]$. Thus Corollary 3.3 implies the following result:

Corollary 4.1 *Assume that a linear labor managed oligopoly has an interior Nash-Cournot equilibrium, and assume that* $G(-1) = \sum_{k=1}^{N} \frac{\frac{d_k b_k}{2h_k}}{2 - d_k\left(1 - \frac{b_k}{2h_k}\right)} - 1 < 0$ *at the equilibrium. Then the equilibrium is asymptotically stable. If* $G(-1) > 0$*, then the equilibrium is unstable.*

5 Economic Interpretation

Notice that the stability condition can be rewritten as

$$\sum_{k=1}^{N} \frac{\frac{d_k b_k}{2h_k}}{2 - d_k\left(1 - \frac{b_k}{2h_k}\right)} \quad < \quad 1. \tag{5.1}$$

Here $d_k \in (0, 1]$, $h_k > 0$, $b_k \geq 0$. This condition holds if all the numbers $\frac{d_k b_k}{2h_k}$ are sufficiently small. It happens if for all k, either d_k or b_k is small enough, or h_k is sufficiently large. That is, either the speed of adjustment is small requiring slow speed in following the actual data in the adaptive expectation, or the labor requirement for producing zero output is small, or the labor requirement for producing equilibrium output is large enough. As a further special case assume symmetry, that is, assume that $d_k \equiv d$, $b_k \equiv b$, and $h_k \equiv h$. Then (5.1) reduces to the following:

$$\frac{\frac{dbN}{2h}}{2 - d\left(1 - \frac{b}{2h}\right)} \quad < \quad 1,$$

which can be further simplified as follows:

$$\frac{db}{2h}(N-1) + d \quad < \quad 2. \tag{5.2}$$

If b and h are given, then (5.2) is equivalent to the relation

$$d \quad < \quad \frac{2}{1 + \frac{b(N-1)}{2h}} \; . \tag{5.3}$$

If d and h are fixed, then (5.2) can be rewritten as

$$b \quad < \quad \frac{2-d}{\frac{d(N-1)}{2h}} \; , \tag{5.4}$$

and if b and d are given, then

$$h \quad > \quad \frac{db(N-1)}{2(2-d)} \; , \tag{5.5}$$

given actual upper bounds for the different variables. In the literature of labor managed oligopolies several authors assume that $h_k(0) = 0$ for all k. In this case $b_k = 0$, and therefore condition (5.1) is always satisfied showing that the interior equilibrium is always asymptotically stable.

6 Conclusions

In this paper dynamic labor managed oligopolies were examined with discrete time scales. In addition to deriving general stability conditions for the nonlinear case, the special case of linear price and labor functions was investigated in details. We have shown that the interior equilibrium is asymptotically stable if for all firms, either the speed of adjustment, or the labor requirement for zero output is sufficiently small, or the labor needed to produce the equilibrium output is large enough. Particular bounds have been derived for the relevant parameters.

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