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ON THE TRANSLATED EXPONENTIAL MODEL WITH CENSORING

A. A. Bartolucci[1], J. M. Dickey[2], K. P. Singh[1], M. A. Tabatabai[3]

Abstract. Two distinct Bayesian methodologies are given for "exact"inference on translated exponential survival models in one and two population problems with censoring. The first approach, based on the use of natural conjugate families of prior distributions, differs from the traditional frequentist analysis in that the derived posterior distributions are exact under the model. The second method is based on noncontinuous-type extensions of natural conjugate priors, and involves the use of Bayes factors for sharply defined hypotheses. The two methods are intended for different states of prior information and would yield different inferences for similar data. Inferences from the two methods are presented comparing data in two breeds of mice to determine if there are genetic differences with respect to surviving the bacterial strain salmonella typhimurium.

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1. INTRODUCTION.

An exponential survivor function is often indicated when sample data is plotted by the Kaplan-Meier (1958) technique. At the cellular immunobiology laboratory of the University of Alabama at Birmingham (U.A.B.) Cancer Center recent survival analyses of mice exposed to a bacterial strain have suggested exponential fits with a possible non zero location parameter. We thus consider a time to failure T measured from the time of exposure, as having the probability density

$$\mu^{-1}\exp[-(t-\delta)/\mu], \quad t>\delta, \tag{1.1}$$

that is,

$$T|(\mu,\delta) \sim \text{Exponential}(\mu,\delta), \quad \mu>0, \quad \delta>0. \tag{1.2}$$

Zelen (1966) discussed the role of δ in clinical trial experience as the "guarantee time,"or minimal time until the first possible failure. The parameter μ is then the mean time to failure from time δ. We address here the important issue of inference on the mean total time to failure

$$\tau = E(t) = \mu + \delta$$

[1]Department of Biostatistics, University of Alabama at Birmingham, E-mail address: abartolucci@crl.soph.uab.deu
[2]School of Statistics, University of Minnesota
[3]Department of Mathematical Sciences, Cameron University, E-mail address: mohammadt@cameron.edu

for μ and δ both unknown. In the two-sample case, i.e., comparing samples from (1.2) and the distribution $T^*|(\mu^*, \delta^*) \sim$ Exponential (μ^*, δ^*), we also extend our inferential procedures to the quotient of means, $\tau/\tau^* = (\mu + \delta)/(\mu^* + \delta^*)$.

Our first approach to the problem of inference on τ and τ/τ^*, given in Sections 2 and 3, respectively, is a Bayesian prior-to-posterior analysis from smooth prior densities in a new conjugate prior family. The posterior distributions are employed in straightforward fashion to yield probability regions for the parameters of interest.

A second approach, Section 4, offers a genuine alternative to these methods for inference on point or sharp hypotheses. This is intended for situations where there is appreciable prior density mounted up over a narrow neighborhood surrounding a simplified sampling model, that is , situations where the experimenter seriously suspects that the simplified model may be a good approximation to the unknown actual sampling model. This will involve the use, as approximations, of positive probabilities assigned to the lower dimensional hypothesis sets in the parameter space, and bayes factors expressed in terms of density ordinates rather than tail areas (Dickey and Lientz 1970). This is a further refinement and extension of the approach discussed in the stochastic curtailing procedure of Halpern et al. (1982), Lan et al. (1983), and Lachin and Lan (1992).

In Section 5, we give numerical illustrations of the methods using the bacterial infection example.

2. A SINGLE POPULATION.

Under the model (1.1) those observations which are censored have likelihood contributions in the form of right tail area probabilities. Namely, a clinical trial patient known only to survive beyond time t contributes a factor to the likelihood,

$$\exp[-(t-\delta)/\mu \tag{2.1}$$

A sample of n failure times and $m - n \geq 0$ censored times for a single treatment regimen,

$$\text{Data}: \; t_i, \; i = 1, \ldots, n, \ldots, m \tag{2.2}$$

yields the likelihood function of μ and δ,

$$\mu^{-n}\exp\{-n[\bar{t} - (m/n)\delta]/\mu\}, \quad 0 \leq \delta \leq d, \, \mu \geq 0, \tag{2.3}$$

where

$$d = \min(t_1, \ldots, t_n), \bar{t} = \sum_{i=1}^{m} t_1/n. \tag{2.4}$$

For notational convenience, we have assigned the n initial indices to the observed failures. Note that the total number of patients is m. Under type-II censoring for the case $\delta = 0$, the reciprocal observed failure rate $\bar{t}$ is distributed as $\mu\chi^2_{2n}/2n$, and hence it would estimate μ unbiasedly (Bain 1978).

The family of distributions which is natural-conjugate to the family of likelihoods (2.3) will have the form with density proportional to (2.3.),

$$p(\mu, \delta) = k^{'}\mu^{-n^{'}-1}\exp\{-n^{'}[\bar{t}^{'} - (m^{'}/n^{'})\delta]/\mu\} \tag{2.5}$$

$$0 < \delta < d^{'} \quad \mu > o, \quad d^{'} > 0,$$

where the normalizing constant can be seen to be

$$\begin{aligned} k^{'} &= \{m^{'}(n^{'}-1)n^{n^{'}-1}/\Gamma(n^{'})\} \\ &/\{[\bar{t}^{'} - (m^{'}/n^{'})d^{'}]^{-(n^{'}-1)} - [\bar{t}^{'}]^{-(n^{'}-1)}\}. \end{aligned}$$

This family is parameterized (or "hyper parameterized") by $n^{'}, \bar{t}^{'}, m^{'}, d^{'} \geq 0$. Assuming $n^{'}\bar{t}^{'} - m^{'}d^{'} \geq 0$ and $n^{'}, m^{'} > 0$, we have that with prior probability one, $n^{'}t^{'} - m^{'}\delta > 0$.

Defining

$$S^{'}(\delta) = \bar{t}^{'} - (m^{'}/n^{'})\delta,$$

for which $S'(\delta) > 0$, we see that the conditional density of μ given δ takes the recognizable chi-square form,

$$S^{'}(\delta)/\mu|\delta \sim \chi^2_{2n'}/2n^{'}, \tag{2.6}$$

whereas under the reverse conditioning

$$p(\delta|\mu) = (m^{'}/\mu)e^{m^{'}\delta/\mu}/(e^{m^{'}d^{'}/\mu} - 1), \tag{2.7}$$

In both cases, (2.6) and (2.7), note that the ranges of the conditional distributions do not depend on the conditioning variables.

The marginal prior density for μ is

$$p(\mu) = k^{'}m^{-1}\mu^{-n^{'}}e^{-n^{'}\bar{t}^{'}/\mu}(e^{m^{'}d^{'}/\mu} - 1), \quad \mu > 0. \tag{2.8}$$

i.e., a linear combination (with one coefficient negative) of two reciprocal chi-square densities each with $2(n^{'} - 1)$ degrees of freedom (variables $2(n^{'}\bar{t}^{'} - m^{'}d^{'})/\mu$ and $2n^{'}\bar{t}^{'}/\mu$). The marginal density of δ takes the simple form,

$$p(\delta) = k^{'}\Gamma(n^{'})\{n^{'}S^{'}(\delta)\}^{-n^{'}}, \quad 0 \leq \delta \leq d. \tag{2.9}$$

We have found this family to be large enough in practice to allow realistic modeling of prior opinions. This model, like many models (Dickey 1979), can bring order into expert-opinion processes. The elicitation of a distribution within the family, to be described here in Section 5, involves a statement of choice of survival quartiles by the expert investigator (e.g. clinician). For further general theory on elicitation through prior percentiles, see Kadane et al. (1980) and Dickey (1979).

In this family, the prior-to-posterior transformation of hyper parameters for updating to observational data by Bayes Theorem takes a pooled-sample form, as is usual with conjugate families for general-exponential-form sampling models,

$$\begin{aligned} n^{''} &= n^{'} + n \\ \bar{t}^{''} &= (n^{'}\bar{t}^{'} + n\bar{t})/(n^{'} + n) \\ m^{''} &= m^{'} + m \\ d^{''} &= \min(d^{'}, d). \end{aligned} \tag{2.10}$$

If the prior density is parameterized by $n^{'}, \bar{t}^{'}, m^{'}, d^{'},$, then the corresponding posterior density will have parameters $n^{''}, \bar{t}^{''}, m^{''}, d^{''}$. (Note that there are two sample-size type hyper parameters, n and m.) Again, $\bar{t}^{''} > (m^{''}/n^{''})/d^{''}$. The marginal posterior distributions of μ and of δ, respectively, are then given by (2.8) and (2.9) with $n^{'}, t^{'}, m^{'}, d^{'}$, replaced by $n^{''}, \bar{t}^{''}, m^{''}, d^{''}$. These are not recognized as traditional distributions.

Note that for such linear functions of μ and δ as the overall mean survival,

$$\tau = \mu + \delta, \tag{2.11}$$

an estimate can be obtained as a linear function of the two posterior means, as $E(\mu) + E(\delta)$. The actual prior density of this parameter τ has the form,

$$p(\tau) \propto \begin{cases} \Gamma \quad (n', a/\tau) \quad \text{if max} \quad (0, \tau - d') = 0 \\ \Gamma \quad (n', a/\tau) - \Gamma(n', a/(\tau - d')) \quad \text{if max} \quad (0, \tau - d') = \tau - d' \end{cases} \tag{2.12}$$

where $a = n'\bar{t}' - m'\tau$ and in a standard notation (Abramowitz and Stegun 1964),

$$\Gamma(x, y) = \int_y^\infty z^{x-1} e^{-z} dz. \tag{2.13}$$

The corresponding posterior density is

$$p(\tau|\text{Data}) \propto \begin{cases} \Gamma \quad (n'', a''/\tau) \quad \text{if max} \quad (0, \tau - d'') = 0 \\ \Gamma \quad (n'', a''/\tau) - \Gamma(n'', a''/\tau - d'')) \quad \text{if max} \quad (0, \tau - d'') = \tau - d'' \end{cases} \tag{2.14}$$

and $a'' = n''\bar{t}'' - m''\tau$. Numerical computation of posterior credible regions for τ are feasible by use of equation (2.14).

3. COMPARING TWO POPULATIONS.

Consider also an independent second sample of patient survival and censored times,

$$\text{Data}^* : t_i^*, i = 1, \ldots, n^*, \ldots, m^*, \tag{3.1}$$

of which n^* are actual failures ($n^* \leq m^*$). This data may represent the observed survival experience for a second therapy with underlying mean, $\tau^* = \mu^* = \delta^*$. We establish in this section inference procedures on the relation of τ to τ^* in either situation of the unknown locations δ, δ^* being assumed equal or unequal.

In the second of these two cases, that is $\delta \neq \delta^*$, assuming the natural conjugate prior distribution for (μ^*, δ^*) independent of (μ, δ), with parameters $n^*, \bar{t}^{*'}, m^{*'}$ and $d^{\star'}$, one obtains a posterior distribution for $(\mu^\star, \delta^\star)$ independent of (μ, δ) and identical in form to that given in Section 2. (All the prior and posterior derived distributions there are taken to apply again to the second population parameters $\mu^\star, \delta^\star, \tau^\star$, but with appropriately starred hyper parameters.)

To facilitate the comparison of the population means $\tau = \mu + \delta$ and $\tau^\star = \mu^\star + \delta^\star$, we perform the change of variables to (η, τ), where

$$\eta = \tau/\tau^*. \tag{3.2}$$

Inference can now be focused on suitable null values, $\eta = \eta_0$, where the natural choice of $\eta_0 = 1$ implies equality of the two population means. From (3.2), the marginal posterior density of η, in proportionality form, is

$$p(\eta|\text{Data, Data}^*) \propto I_1 + I_2 + I_3, \tag{3.3}$$

where, assuming $d'' \leq d^{\star''}$,

$$
\begin{aligned}
I_1 =& \frac{1}{\eta^2}\int_0^{d''}\left\{\tau^2(b-c\tau)^{1-g}\left[\frac{b^*\eta-c^*\tau}{\eta}\right]^{i-g^*}\right.\\
&\left.\cdot\,\Gamma\left(g-1,\frac{b-c\tau}{\tau}\right)\Gamma\left(g^*-1,\frac{b^*\eta-c^*\tau}{\tau}\right)\right\}d\tau\\
I_1 =& \frac{1}{\eta^2}\int_0^{d''}\left\{\tau^2(b-c\tau)^{1-g}\left[\frac{b^*\eta-c^*\tau}{\eta}\right]^{1-g^*}\right.\\
&\cdot\left[\Gamma\left(g-1,\frac{b-c\tau}{\tau}\right)-\Gamma\left(g-1,\frac{b-c\tau}{\tau-d''}\right)\right]\\
&\left.\cdot\,\Gamma\left(g^*-1,\frac{b^*\eta-c^*\tau}{\tau}\right)\right\}.d\tau\\
I_3 =& \frac{1}{\eta^2}\int_{d^{*''}}^{\infty}\left\{\tau^2[b-c\tau]^{1-g}\left[\frac{b^*\eta-c^*\tau}{\eta}\right]^{1-g^*}\right.\\
&\cdot\left[\Gamma\left(g-1,\frac{b-c\tau}{\tau}\right)-\Gamma\left(g-1,\frac{b-c\tau}{\tau-d''}\right)\right]\\
&\left.\cdot\left[\Gamma\left(g^*-1,\frac{b^*\eta-c^*\tau}{\tau}\right)-\Gamma\left(g^*-1,\frac{b^*\eta-c^*\tau}{\eta(\eta-d^{*''})}\right)\right]\right\}d\tau
\end{aligned}
$$

and where

$$
\begin{aligned}
&P(\eta\leq\eta_0|Data,Data^*).\\
&b=n''\bar{t}'', && b^*=n^{*''}\bar{t}^{*''}\\
&c=m'', && c^*=m^{*''}\\
&g=n'', && g^*=n^*.
\end{aligned}
\tag{3.4}
$$

For given fixed values, η_0, expression (3.3) can be computed numerically to yield posterior credible regions.

For the remaining case of equal but unknown location, $\delta=\delta^\star$, our results are much simpler. A consideration of $\tau\leq\tau^\star$ reduces to $\mu\leq\mu^\star$. Thus the mean survivals τ and $\tau^\star$ can be compared in terms of μ and $\mu^\star$. It is well known that the conjugate prior family for a simplified sampling model can be obtained by conditioning in the conjugate prior family of a more complicated sampling model, provided one uses a conditioning variable which is a linear combination of the joint variables (Gunel and Dickey 1974). We condition on the variable and value $\delta-\delta^\star=0$. The preceding natural conjugate prior assumption on the parameters (μ,δ) and $(\mu^\star,\delta^\star)$ taken independently yields the joint posterior density $p(\mu,\delta|Data)\cdot p(\mu^\star,\delta^\star|Data^\star)$, each factor in the form (2.5). Then taking $\delta=\delta^\star$ and integrating with respect to δ yields the density $p(\mu,\mu^\star|\delta-\delta^\star=0,Data,Data^\star)$. Finally, by changing variable to (ρ,μ),

$$\rho=\mu/\mu^*,$$

and integrating with respect to μ, we obtain

$$
\begin{aligned}
&p(\rho|\delta-\delta^*=0,Data,Data^*)\\
&=k''k^{*''}\Gamma(n''+n^{*''}-1)\left\{\rho^{n^{*''}-1}(m''+m^{*''}\rho)\right\}\\
&\cdot\left\{[n''\bar{t}''+n^{*''}\bar{t}^{*''}\rho-min(d'',d^{*''})(m''+m^{*''}\rho)]^{-(n''+^{*''}-1}\right.\\
&\left.-[n''\bar{t}+n^{*''}\bar{t}^{*''}\rho]^{-(n''+n^{*''}-1)}\right\}.
\end{aligned}
\tag{3.5}
$$

Expression (3.5) can be used to compute posterior credible regions of the form

$$p\{\rho \leq \rho_0 | \delta - \delta^* = 0, Data, Data^*\} \tag{3.6}$$

for fixed values of ρ_0.

The marginal posterior density of the common value δ in the case of $\delta = \delta^\star$ has the form,

$$\begin{aligned} p(\delta|\delta - \delta^* = 0, Data, Data^*) &\propto p(\delta|Data) \cdot p(\delta^* = \delta|Data^*) \\ &\propto S''(\delta)^{-n''} S^{*\prime\prime}(\delta)^{-n^{*\prime\prime}} \\ &0 < \delta < min(d'', d^{*\prime\prime}) \end{aligned} \tag{3.7}$$

where S'' and $S^{\star\prime\prime}$ are the posterior version of S' and $S^{\star\prime}$ defined in Section 2.

Sometimes, it may even be of interest to infer whether $\mu \leq \mu^\star$ in the case of $\delta \neq \delta^\star$. Given our preceding prior distributional assumptions on (μ, δ) and $(\mu^\star, \delta^\star)$, the marginal posterior density of $\rho = \mu/\mu^\star$ is

$$\begin{aligned} p(\rho|Data, Data^*) = &k'' k^{*\prime\prime} \Gamma(Q)(m'' m^{*\prime\prime})^1 \rho^{n^{*\prime\prime}-2} \\ \cdot\{&[n''\bar{t}'' - m''d'' + (n^{*\prime\prime}\bar{t}^{*\prime\prime} - m^{*\prime\prime}d^{*\prime\prime})\rho]^{-Q} \\ &- [(n''\bar{t}'' - m''d'') + n^{*\prime\prime}\bar{t}^{*\prime\prime}\rho]^{-Q} \\ &- [n''\bar{t}'' + (n^{*\prime\prime}\bar{t}^{*\prime\prime} - m^{*\prime\prime}d^{*\prime\prime})\rho]^{-Q} \\ &+ [n''\bar{t}'' + n^{*\prime\prime}\bar{t}^{*\prime\prime}\rho]^{-Q}\} \\ &for Q = n'' + n^{*\prime\prime} - 2. \end{aligned} \tag{3.8}$$

4 THE BAYES FACTOR.

A lack of realism in parameterized families of ordinary continuous-type joint prior distributions appears during attempts to assign appreciable prior probability to small neighborhood hypotheses. Such an attempt is obviously desirable, to model situations where the experimenter's prior belief includes appreciable suspicion (but not near certainty) that the unknown true sampling model is contained in such a specified neighborhood. If prior distributions express prior uncertainty, they should be able to express such a nontrivial suspicion. If the experimenter has prior uncertainty of this kind, he should use the following kind of inference method. If he does not, he should use other kinds of methods, as in the preceding sections. Consider an interval or "strip "hypothesis of the form

$$H_\varepsilon : |\eta - \eta_0| < \varepsilon. \tag{4.1}$$

The preceding joint prior densities give nearly zero prior probability to H_ε for small ε. Dickey (1976) corrected for this deficiency by considering a mixture of ordinary continuous distributions, which then has a mound of local high density over the region H_ε. However, working with a mixture is mathematically cumbersome. We approximate such a distribution by using a positive prior probability for the sharp hypothesis H,

$$H : \eta = \eta_0 vs. H^c : \eta \neq \eta_0. \tag{4.2}$$

The approximation of appreciable probability for H_ε (4.1) by positive probability for H (4.2) leads to inferences in terms of the Bayes factor or weighted likelihood ratio, as discussed by I. J. Good (1950), Savage et al. (1963) and Dickey and Lientz (1970). This ratio has the basic form,

$$\begin{aligned} B(H) = &\int lik(\tau, \eta = \eta_0|Data) p(\tau|H) d\tau \\ &/ \int lik(\eta, \tau|Data) p(\eta, \tau|H^c) d\eta d\tau \end{aligned} \tag{4.3}$$

and was described by I. J. Good (1950) as the "factor in favor of the hypothesis, H, in virtue of the result of the experiment." Dickey and Lientz (1970) showed that the Bayes factor can be expressed as Savage's Density ratio:

$$B(H) = \lim_{\eta \to \eta_0} [p(\eta|H^c, Data)/p(\eta|H^c] \tag{4.4}$$

For our present sampling model, the Bayes factor can be calculated using our continuous-case prior densities in the form,

$$p(\eta_0|Data)/p(\eta_0). \tag{4.5}$$

Bartolucci and Dickey (1977) developed $B(H)$ for survival data problems in which the guarantee time is known.

One observes that if the prior degrees of freedom for $p(\eta)$ are taken to approach zero, i.e., $n^{'} = n^{\star'} \to 0$, then the posterior density corresponding to the non-integrable, "ignorance" prior density $p(\eta)$ proportional to $1/\eta$ is obtained immediately from our conjugate posterior density (3.3). But then the data becomes less and less likely under H^c, and $B(H) \to \infty$. However, if the prior density ordinate is not sent to zero but is rather assumed to be non-trivial and to locally approximately satisfy the same proportionality as for $n^{'} = n^{\star'} = 0$, namely $p(\eta|H^c) = constant/\eta$ (implying log η has a locally uniform prior density), the $B(H)$ takes the simple form,

$$B(H) = p(\eta_0|Data)/(constant/\eta_0). \tag{4.6}$$

In addition to the formal decision-theory criterion based on a loss function (Dickey and Leinz 1970), inference from the Bayes factor can also be carried out in an informal way by noting its rough order of magnitude. An extreme value will tend to indicate a strong inference common to a large range of opinions.

Of course, the Bayes factor gives the ratio of the posterior odds to the prior odds for $H(orH_e)$,

$$B(H) = [P(H|Data)/P(H^c|Data)]/[P(H)/P(H^c)],$$

and this ratio makes good inferential sense even without actually assigning a specific value to the prior probability $P(H)$.

The explanations and results of this section also apply verbatim to the hypothesis $H : \rho = \rho_0$ for our parameterization $\rho = \mu/\mu^\star$. Examples demonstrating the methods are given in the following section.

5 EXAMPLE.

Scientists at UAB initiated a study to compare the survival of two different breeds of mice exposed to salmonella typhimurium bacteria. This bacterium is most often the strain found in humans suffering from "food poisoning." Table 1 is a summary example of survival results at the time of analysis. Notice that the data follow a Type II censoring pattern.

To demonstrate the methods developed here, we have listed in Tables 3 and 4 Bayesian inference for the hypotheses $\eta_0 = 1(\eta = \tau/\tau^*$ and $\rho_0 = 1(\rho = \mu/\mu^*)$, respectively. We take sampling models, $T_i|(\mu, \delta) \sim$ Exponential $(\mu, \delta)(i = l, \ldots, 29)$ in Breed A and $T_j^*|(\mu^*, \delta^*) \sim Exponential(\mu^*, \delta^*)(j = l, \ldots, 35)$ for Breed B.

It is not uncommon in planning clinical trials for investigators to quote survival percentiles based on their previous experience in treating the particular disease type. The medical literature contains numerous examples of quotations of median survival and other percentiles, especially in articles on investigative therapies. (See, for example, Presant, C.A., et al. 1978 and Smalley, R.V. et al. 1976.)

One method of incorporating such information in prior probability assessments is seen in Birch and Bartolucci (1983). Expert opinions can be characterized by the predictive distribution of time to failure conditional on the hyper parameters. That is, the distribution after averaging out the sampling parameters according to their prior distribution. In the case of the exponential distribution (1.2) and the prior density (2.5) the predictive density takes the form

$$p(t|\bar{t}^{'}, n^{'}, m^{'}, d^{'}) \propto (n^{'}\bar{t}^{'} + t - (m^{'} + 1)d^{'})^{-n^{'}} - (n^{'}\bar{t}^{'} + t)^{-n^{'}} 0 < t < \infty \tag{5.1}$$

One can simply elicit percentiles of this predictive distribution from the experimenter and solve for the prior parameters. This basically involves eliciting four percentiles and solving interactively four equations with four unknown parameters. To do this we use a Newton-Raphson iterative technique. Let the f_i^{th} percentile of (5.1) be $t_i i = 1, \ldots, 4$. Define

$$\int_0^{t_i} p(t|\bar{t}', n', m', d')dt - f_i = \phi_i(\bar{t}', n', m', d')i = 1, \ldots, 4. \tag{5.2}$$

The ususal methodology would be to expand (5.2) in Taylor Series of first order terms, i.e.

$$\begin{aligned}\phi_i(\bar{t}', n', m', d') &= \phi_i(\bar{t}_0, n_0, m_0, d_0)\\ &= \frac{\partial \phi_i}{\partial \bar{t}'}\partial_{\bar{t}'} + \frac{\partial \phi_i}{\partial n'}\partial_{n'} + \frac{\partial \phi_i}{\partial m'}\partial_{m'} + \frac{\partial \phi_i}{\partial d'}\partial_{d'} = 0\\ i &=, \ldots, 4\end{aligned}$$

where $\bar{t}_0, n_0, m_0$ and d_0' are initial values of t', n', m', and d', respectively and $\delta_\chi = \chi - \chi_0$. In our case we choose to condition on d' and m' to keep the number of equations and elicited percentiles to a manageable number. To expect to elicit more than two or three percentiles from an investigator may be unrealistic. In our example, we condition on d' and m' since they are most easily determined for our problem. That is to say, the choice $d' = 5$ or $d' = 8$ is most consistent with the minimum expected survival for the salmonella experiment and m' is chosen initially to be 30 since the value of m' will usually vary from 20 to 50. Thus, we can solve iteratively for n' and $\bar{t}'$ given d' and m'. We have an interative program that will perform this task. Kadane et al. (1980) investigated the assessment of the parameters of a matrix t distribution by a similar elicitation procedure.

In Table 2 we have the elicitation results for two sets of percentiles, (12, 62) and (10, 50), respectively. Primarily to demonstrate our analysis technique, in Tables 3 and 4 we held n' fixed for each set of percentiles and give results varying m'. the lower and upper percentiles are given in parentheses in the n' column. The resulting prior densities form models of reasonable opinions for the experimental situation we are considering.

The assumption in table 3 is that of unequal and unknown locations $\delta \neq \delta^\star$, with independent prior distributions for (μ, δ) and $(\mu^\star, \delta^\star)$,namely (2.5) and the starred version of (2.5), respectively. The hypothesis of interest is $H : \eta = 1$. Expression (3.3) is the density kernel used for calculating the posterior tail areas (column 3 for $\eta_0 = 1$). These tail probabilities and Bayes factors cohere with the various types of prior input as listed and with $d' = 5, \quad d'' = min \quad (5, 8) = 5 \quad \text{and} \quad d^{\star'} = 8, \quad d^{\star''} = min(8, 8) = 8$.

For illustration we have taken equality of the prior parameters, $m' = m^{\star'}, n = n^{\star'}$, and $\quad \bar{t}' = \bar{t}^{\star'}$ although such an assumption is not necessary.

The prior probabilities for H are zero in the calculation of the posterior tail area, and arbitrary in the calculation of the Bayes factor, where they do not enter. The posterior tail probabilities in Table 3 range from 0.000006 to 0.048608. The Bayes factors, $B(H)$, were computed as in expression (4.4) and are roughly consistent with the interpretation of the tail areas, in that we have moderate evidence against $\eta_0 = 1$ as $B(H)$ ranges from 0.00010 to 0.36098.

In table 4, equations (3.5) and (4.4), with $\rho = \mu/\mu^\star$ in place of η, were used in the computations of columns 3 and 4 respectively. The assumption here is that of equal but unknown locations, $\delta = \delta^\star$, and the hypothesis of interest is $H : \rho = 1$. The realistic choice of $d' = d^{\star'} = 8, d'' = 8$ leads to results fairly consistent with Table 3 in that there is moderate evidence against $\rho_0 = 1$. The results of both tables are fairly robust when other values in the neighborhood of $\eta_0 = 1 \quad \text{or} \quad \rho_0 = 1$ considered.

Upon further examination of the Bayes factor in each table for the case $m' = m^{\star'} = 25$, one notes that as n' and $n^{\star'}$ decreases, $B(H)$ increases. For the case $n' = n^{\star'} = 0$, our effective prior uncertainty range in either case of $\eta \quad or \quad \rho$ is chosen to be the closed interval [1/5, 5], implying $p(log_{10}\eta|H^c) = 0.716$ or equivalently, $p(\eta|H^c) = 0.716/\eta$ or $p(\rho|H^c) = 0.716/\rho$.

For traditional inference for testing $\rho_0 = 1.0$, we have available the "approximate"F distribution of Gehan (1965) or Cox (1953) which, based on our data, yields the statistic

$$(33/)[n\bar{t} - 29t_{(1)}]/[n^*\bar{t}^* - 35t^*_{(1)}] = 1.62 \tag{5.3}$$

for $t(1) = 10$ and $t^\star_{(1)} = 8$. The tabled 5 percent critical value on 54 and 66 degrees of freedom is about 1.56. Thus, classically, we also have rejection of $\rho_0 = 1$ at the 0.05 level, which harmonizes with our "exact "Bayesian results.

The inference problem for which the traditional expression (5.3) may be considered is one of the "dividing "type, $H : \rho \leq \rho_0$ (Cox 1977). The posterior tail area could be considered as a Bayesian analogue for this situation (Box and Tiao 1973). However, it does not involve positive prior probability on H as does the Bayes factor. Different prior distributions for different stages of prior uncertainty are involved in the two Bayesian methods, as explained at the beginning of Section 4. Examining Table 4, one also sees an instance of the general tendency for traditional tests to reject more readily than do reasonable thresholds on Bayes factors (Lindley 1957; Dickey 1976). Figures 1 to 4 give the posterior densities for the cases in Tables 3 and 4 for which $m^{'} = m^{\star'} = 30.0$. For the sake of standardization each density point was divided by the maximum value. The overall conclusion from the analysis is that the Breed A mice appear to survive the salmonella exposure for longer periods of time than do the Breed B mice.

We have given here a methodology for Bayesian inference on the survival mean parameters $\tau = \mu + \delta$ and $\tau^\star = \mu^\star + \delta^\star$ for $\delta \neq \delta^\star$, whereas most traditional comparisons of exponential samples either impose the restriction $\delta = \delta^\star$ or impose appropriate first-order-statistic estimates on δ and $\delta^\star$ and then proceed with approximate tests of $H : \rho_0 = 1$, especially in the cases involving censoring. In this paper we have overcome these constraints and provided "exact "Bayesian inference procedures for the parameters of interest. Our methods help answer the need for a cautious coherent approach to analyzing data with proper consideration of the influence of relevant predata uncertainty for reasonable differences in prior opinions.

TABLE 1.

Survival Time in Days.

Breed A			Breed B		
10	16	29	8	12	15
10	16	34	9	12	15
11	16	34	9	12	16
11	18	37+	9	12	18
11	18	50+	9	13	18
11	19		10	13	19
11	19		10	13	19
14	23		11	13	23
15	23		11	14	23
15	23		11	14	23+
16	23		12	14	27+
16	23		12	14	

$$m = 29 \qquad \bar{t} = 21.185 \qquad m^\star = 35 \qquad t^\star = 14.939$$
$$n = 27 \qquad t_{(1)} = 10.0 \qquad n^\star = 33 \qquad t^\star_{(1)} = 8.0$$

+ indicates a censored observation

TABLE 2.

Percentiles		Prior Shape	Prior Mean
0.25	0.75	$n^{'}$	$\bar{t}^{'}$
12	62	8.018	19.687
10	50	15.023	11.763

TABLE 3.

Unknown Unequal Locations.

$$d^{''} = 8.0 \qquad d^{\star''} = 5.0 \qquad \eta = \tau/\tau^\star$$

$n^{'} = n^{\star'}$	$m^{'} = m^{\star'}$	$\bar{t}^{'} = \bar{t}^{*'}$	$P[\eta < 1 \vert Data, Data^\star]$	$B(H)$ for $H : \eta = 1$
0	25		0.048608	0.36098
8.018	25	19.687	0.02555	0.22527
$(12, 62)$				
	30	22.805	0.01593	0.15085
	40	29.041	0.00508	0.05541
15.023	25	11.763	0.000069	0.00105
$(10, 50)$				
	30	13.4276	0.000032	0.00052
	40	16.7558	0.000006	0.00010

TABLE 4.

Unknown Equal Locations.

$$d^{''} = d^{\star ''} = 8.0 \qquad \rho = \mu/\mu^{\star}$$

$n^{'} = n^{\star '}$	$m^{'} = m^{\star '}$	$\bar{t}^{'} = \bar{t}^{*'}$	$P[\rho < 1 \vert \delta = \delta^{\star}, Data, Data^{\star}]$	$B(H)$ for $H : \rho = 1$
0	14		.005324	0.77740
8.018 (12, 62)	25	29.0411	0.04196	0.49886
	30	34.0299	0.04095	0.48878
	40	44.0075	0.03923	0.47168
15.023 (10, 50)	25	16.7558	0.01088	0.20527
	30	19.41784	0.01066	0.20168
	40	24.7435	0.00997	0.18995

FIGURE 1
POSTERIOR DENSITY

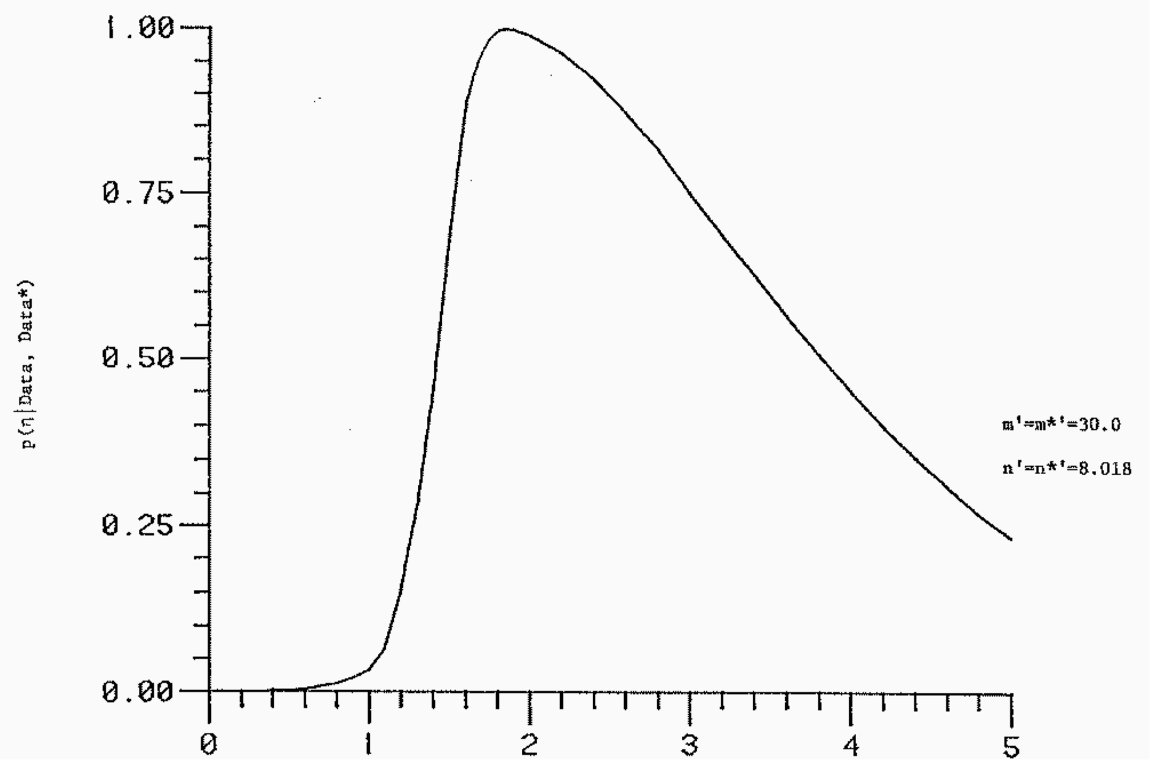


FIGURE 2
POSTERIOR DENSITY

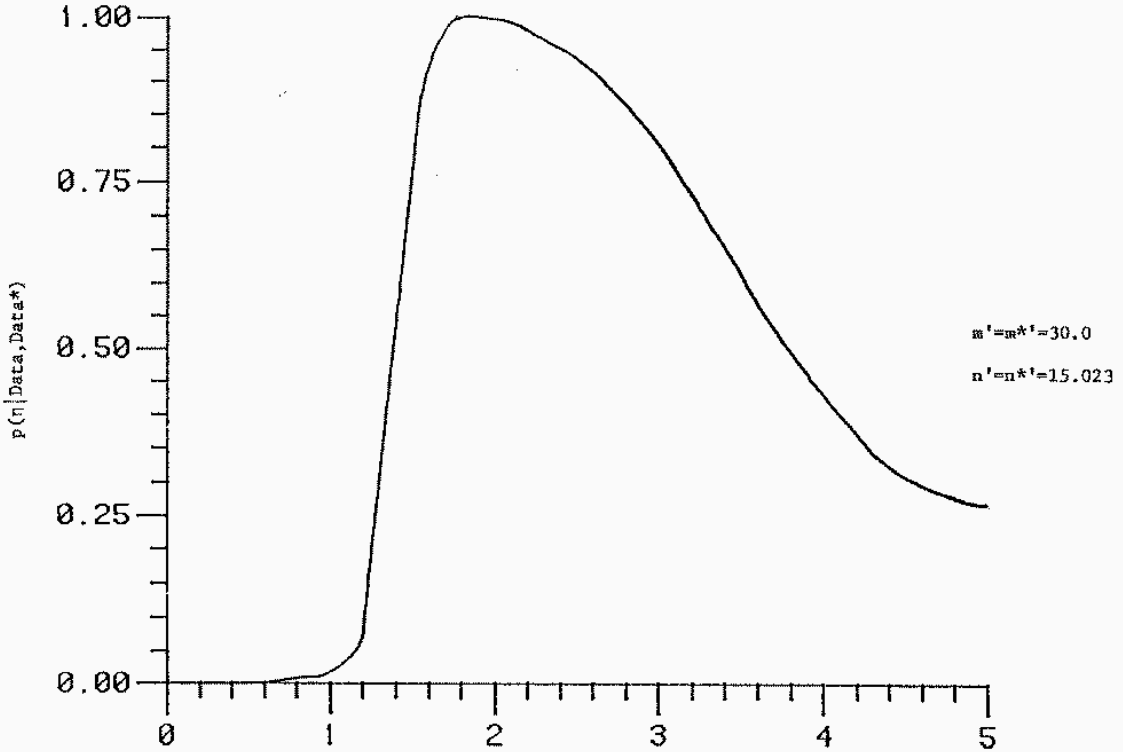
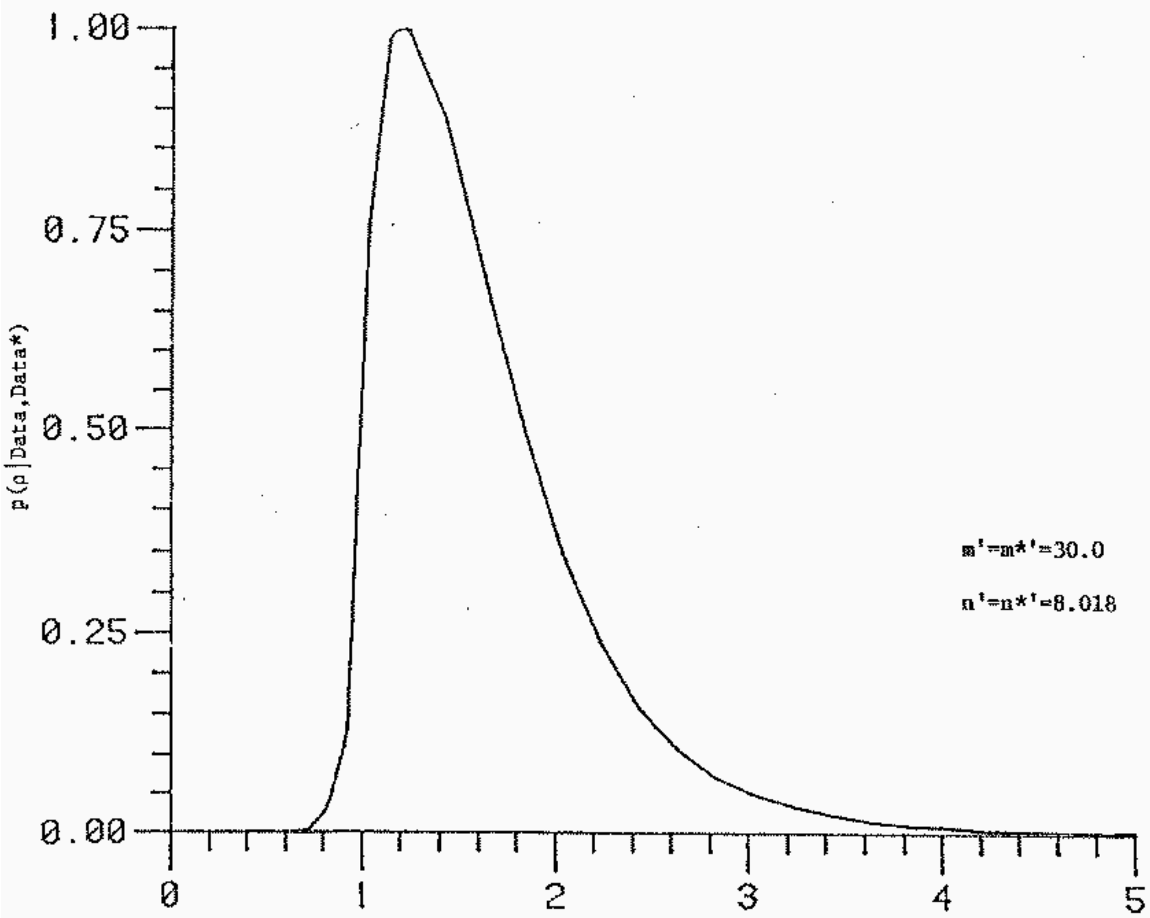
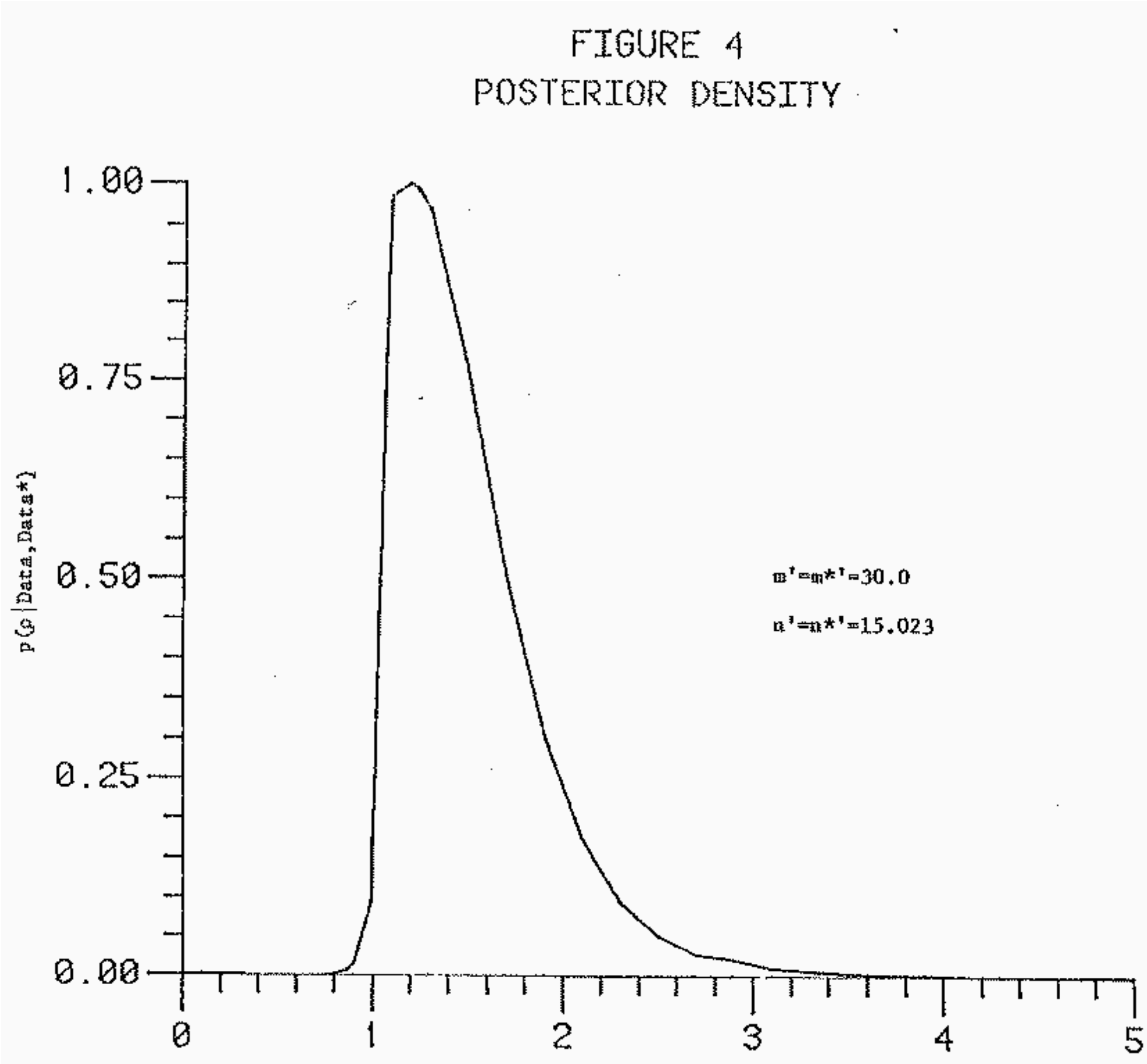


FIGURE 3
POSTERIOR DENSITY





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