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On Multivariate Approximation by Mejer-Konig-Zeller Type Polynomials

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Abstract

Multivariate positive linear operators have many good properties, such as convergence, convexity preserving, simultaneous approximation, and so on. They can been used in the field of computer graphics. The typical example is multivariate Bernstein polynomial. Recently, much effort was devoted to study the properties of multivariate positive linear operators. In this paper, we give a multivariate version of Mejer-Konig-Zeller operator and find out the approximation properties of this operator. Our results are theoretical, but they will provide the benchmark for the use of such operators in practice.

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1 Introduction

Multivariate positive linear operators have many good properties, such as convergence, convexity preserving, simultaneous approximation, and so on. They can been used in the field of computer graphics. The typical example is multivariate Bernstein polynomial. Recently, much effort was devoted to study the properties of multivariate positive linear operators. We only metion references [1-6]. In this paper, we give a multivariate version of Mejer-Konig-Zeller type operator and find out the approximation properties of this operator. Our results are theoretical, but they will provide the benchmark for the use of such operators in practice.

Let $\Delta = \{(x, y) : x, y \geq 0, x + y \leq 1\}$. For positive integer r, we define $C(\Delta)$ and $C^r(\Delta)$ as follows

$$C(\Delta) = \{f : f \quad \text{is continuous on} \quad \Delta \quad \text{and with real value}\}.$$

$$C^r(\Delta) = \left\{ f \in C(\Delta) : \frac{\partial^k f}{\partial^i \partial^{k-i}} \in C(\Delta), 0 \leq i \leq k, 0 \leq k \leq r \right\}.$$

For $f \in C(\Delta)$ and $\delta > 0$, we define a various modules of continuity of f as follows.

$$\begin{aligned} \omega(f;\delta) = \sup\{f(x', y') - f(x, y)| : (x', y'), (x, y) \in \Delta \\ |x' - x| \leq \delta, |y' - y| \leq \delta\}. \end{aligned}$$

$$\omega_1(f;\delta) = \sup\{|f(x_1, y) - f(x_2, y)| : (x_1, y), (x_2, y) \in \Delta |x_1 - x_2| \leq \delta\}.$$

$$\begin{aligned} \omega_2(f;\delta) = \sup\{f(x, y_1) - f(x, y_2)| : (x, y_1), (x, y_2) \in \Delta \\ |y_1 - y_2| \leq \delta\}. \end{aligned}$$

For $f \in C^1(\Delta)$, we define $\omega_{10}(f;\delta), \omega_{01}(f;\delta)$ as

$$\omega_{10}(f;\delta) = \omega(f'_x;\delta); \omega_{01}(f;\delta) = \omega(f'_y;\delta),$$

where f'_x and f'_y are the first order partial derivatives of f with respect to variables x and y, respectively. Now we define $L(\alpha, \delta)$ as

$$L(\alpha, \delta) = \{f \in C(\Delta) : \omega(f, \delta) = O(\delta^\alpha), \text{as} \quad \delta \to 0+\}.$$

Let $f \in L(\Delta)$. We define the multivariate Mejer-Konig-Zeller type operator $L_{mn}(f; x, y)$ as

$$L_{mn}(f; x, y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} f(\frac{k}{m+k}, \frac{l}{n+l}) M_{mnkl}(x, y),$$

where

$$M_{mnkl} = \left(\begin{array}{c} m+k \\ k \end{array} \right) x^k (1-x)^{m+1} \left(\begin{array}{c} n+l \\ l \end{array} \right) y^l (1-y)^{n+1}.$$

The rest of this is organized as follows. Section 2 provides some lemmas which will be used in the course of poving the main results. Section 3 includes all approximate properties of the multivariate Mejer-Konig-Zeller type and their proofs.

2 Lemmas

In order to prove our main results we need the following lemmas.

Lemma 2.1

$$L_{mn}(1; x, y) = 1 \tag{1}$$

$$L_{mn}(s; x, y) = x \tag{2}$$

$$L_{mn}(t; x, y) = y. \tag{3}$$

$$L_{mn}(s^2; x, y) = x^2 + \varepsilon_m(x), \tag{4}$$

where

$$\varepsilon_m(x) \leq \frac{x(1-x)}{m}. \tag{5}$$

$$L_{mn}(t^2; x, y) = x^2 + \varepsilon_n(y), \tag{6}$$

where

$$\varepsilon_n(y) \leq \frac{y(1-y)}{n}. \tag{7}$$

Proof: (1)-(3) are obvious.
Now we prove (4) and (5).

$$L_{mn}(s^2; x, y) = \sum_{k=1}^{\infty} \frac{(m+k-1)!}{(k-1)!m!} \frac{k}{m+k} x^k (1-x)^{m+1}$$
$$= \sum_{k=2}^{\infty} \frac{(m+k-2)!}{(k-2)!m!} x^k (1-x)^{m+1}$$

$$
\begin{aligned}
&+\sum_{k=1}^{\infty}\frac{(m+k-1)!}{(k-1)!m!}\left(\frac{k}{m+k}-\frac{k-1}{m+k-1}\right)x^k(1-x)^{m+1}\\
&=x^2+\sum_{k=1}^{\infty}\frac{(m+k-1)!}{(k-1)!m!}\frac{m}{(m+k)(m+k-1)}x^k(1-x)^{m+1}\\
&=x^2+x(1-x)\sum_{k=1}^{\infty}\frac{(m-1+k-1)!}{(k-1)!(m-1)!}\frac{1}{m+k}x^{k-1}(1-x)^m\\
&=x^2+x(1-x)\sum_{k=0}^{\infty}\frac{(m-1+k)!}{(k)!(m-1)!}\frac{1}{m+k+1}x^k(1-x)^m\\
&:=x^2+\varepsilon_m(x).
\end{aligned}
$$

Since

$$\frac{1}{m+k+1}\le\frac{1}{m}$$

we have

$$\varepsilon_m(x)\le\frac{x(1-x)}{m}.$$

Similarly, we can prove (6) and (7). This completes the proof of Lemma 2.1.

Lemma 2.2

$$L_{mn}((s-x)^2;x,y)\le\frac{x(1-x)}{m} \tag{8}$$

$$L_{mn}((t-y)^2;x,y)\le\frac{y(1-y)}{n}, \tag{9}$$

$$L_{mn}(|s-x|;x,y)\le\sqrt{\frac{x(1-x)}{m}}, \tag{10}$$

$$L_{mn}(|t-y|;x,y)\le\sqrt{\frac{y(1-y)}{n}}, \tag{11}$$

Proof: By Lemma 2.1 we have

$$
\begin{aligned}
L_{mn}((s-x)^2;x,y)&=L_{mn}(x^2;x,y)-2xL_{mn}(s;x,y)+L_{mn}(s^2;x,y)\\
&=\varepsilon_m(x)\le\frac{x(1-x)}{m}.
\end{aligned}
$$

Similarly, we can prove (9). By Holder inequality, we have

$$L_{mn}(|s-x|;x,y)\le\left(\sqrt{L_{mn}((s-x)^2;x,y)}\right)^{1/2}.$$

From this and (5) we can obtain (10). Similarly we can obtain (11). This completes the proof of Lemma 2.2.

3 Main Results and Their Proofs

Theorem 3.1 *If function $f(s,t)$ is continuous at $(x,y) \in \Delta$, then*

$$\lim_{m,n\to\infty} L_{mn}(f;x,y) = f(x,y). \tag{12}$$

Moreover, if $f \in C(\Delta)$, $L_{mn}(f;x,y)$ is uniformly continuous on Δ.

Proof: From Lemma 2.1, as $m,n \to \infty$ we have

$$L_{mn}(1;x,y) \rightrightarrows 1,$$

$$L_{mn}(s;x,y) \rightrightarrows x,$$

$$L_{mn}(t;x,y) \rightrightarrows y,$$

$$L_{mn}(s^2+t^2;x,y) \rightrightarrows x^2+y^2,$$

By Korovkin's theorem [7], this yields Theorem 3.1.

Theorem 3.2 *For any $f \in C(\Delta)$, as $m,n \to \infty$, we have*

$$\|L_{mn}(f) - f\|_{C(\Delta)} \le 2\omega(f, 1/\sqrt{\frac{1}{m}+\frac{1}{n}}). \tag{13}$$

Proof: $\forall \delta \in (0,1)$, by Holder's inequality and Lemma 2.2 we have

$$\begin{aligned}
|L_{mn}(f;x,y) - f(x,y)| &\le \omega(f,\delta)\left(1 + \frac{1}{\delta}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} M_{mnkl}(x,y)\|(\frac{k}{m+k},\frac{l}{n+l}) - (x,y)\|\right) \\
&\le \omega(f,\delta)\left(1 + \frac{1}{\delta}\{\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} M_{mnkl}(x,y)[(\frac{k}{m+k}-x)^2 + (\frac{l}{n+l}-y)^2]\}^{1/2}\right) \\
&\le \omega(f,\delta)\left(1 + \frac{1}{\delta}\{L_{mn}((s-x)^2+(t-y)^2;x,y)\}^{1/2}\right) \\
&\le \omega(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{\frac{x(1-x)}{m} + \frac{y(1-y)}{n}}\right) \\
&\le \omega(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{\frac{1}{m}+\frac{1}{n}}\right).
\end{aligned}$$

Let $\delta = 1/\sqrt{\frac{1}{m}+\frac{1}{n}}$, we can obtain the desired result. This completes the proof of Theorem 3.2.

Corollary 3.1 *If $f \in Lip(\alpha; \Delta)(0 < \alpha \leq 1)$, then*

$$\|L_{mn}f - f\| \leq K_f(\frac{1}{m} + \frac{1}{n})^{\alpha/2},$$

where K_f is a constant only depending on f.

Theorem 3.3 *For any $f \in C^1(\Delta)$, as $m, n \to \infty$, we have*

$$\|L_{mn}(f; x, y) - f(x, y)\|_{C(\Delta)} \leq 2\omega_{10}\left(f, 1/\sqrt{x(1-x)/m + y(1-y)/n}\right)$$
$$+2\omega_{01}(f, 1/\sqrt{x(1-x)/m + y(1-y)/n}).$$

Proof: Let $f \in C^1(\Delta), \forall (x, y) \in \Delta$, by Taylor's formula [8, pp.129-139] we have

$$\begin{aligned} f(s,t) - f(x,y) &= f'_x(x,y)(s-x) + f'_y(x,y)(t-y) \\ &+ \left[f'_x(x+\theta_1(s-x), y+)(s-x), y+\theta_2(t-y)) - f'_x(x,y)\right](s-x) \\ &+ \left[f'_y(x+\theta_1(s-x), y+)(s-x), y+\theta_2(t-y)) - f'_y(x,y)\right](t-y), \end{aligned}$$

where $0 < \theta_1, \theta_2 < 1$.

Let $\delta > 0$, by Lemma 2.2 we have

$$\begin{aligned} &|L_{mn}(f; x, y) - f(x, y)| \\ &\leq \omega_{10}(f, \delta)\left\{C_{mn}(x,y) + \frac{1}{\delta}[C^2_{mn}(x,y) + C_{mn}(x,y)d_{mn}(x,y)]\right\} \\ &+\omega_{01}(f, \delta)\left\{C_{mn}(x,y) + \frac{1}{\delta}[C_{mn}(x,y)d_{mn}(x,y) + C^2_{mn}(x,y)]\right\} \end{aligned}$$

where

$$C_{mn}(x,y) = \sqrt{L_{mn}((s-x)^2; x, y)} \leq \sqrt{\frac{x(1-x)}{m}}.$$

$$d_{mn}(x,y) = \sqrt{L_{mn}((t-y)^2; x, y)} \leq \sqrt{\frac{y(1-y)}{m}}.$$

It is clear that

$$C^2_{mn}(x,y) + C_{mn}(x,y)d_{mn}(x,y) \leq \sqrt{\frac{x(1-x)}{m} + \frac{y(1-y)}{n}}.$$

$$C_{mn}(x,y)d_{mn}(x,y) + d^2_{mn}(x,y) \leq \sqrt{\frac{x(1-x)}{m} + \frac{y(1-y)}{n}}.$$

Let $\delta = \sqrt{x(1-x)/m + y(1-y)/n}$, we can obtain the requred result. This completes the proof of Theorem 3.3.

Corollary 3.2 *For any $f \in C^1(\Delta)$, if $f'_x, f'_y \in Lip(1,\Delta)$, as $m,n \to \infty$, we have*

$$\|L_{mn}(f) - f\|_{C(\Delta)} = O(\frac{1}{m} + \frac{1}{n}).$$

Theorem 3.4 *Denote $L_n(f;x,y) = L_{nn}(f;x,,y)$. For any $f \in C^2(\Delta)$, we have*

$$\begin{aligned}
&\lim_{m,n\to\infty} n\left\{L_n(f;x,y) - f(x,y)\right\} \\
&= (1-3x)f'_x(x,y) + (1-3y)f'_y(x,y) \\
&+x(1-x)f''_{xx}(x,y) + (x+y-xy)f''_{xy}(x,y) + y(1-y)f''_{yy}(x,y),
\end{aligned}$$

where f''_{xx} and f''_{yy} are the second order partial derivatives of f with respect to variables x and y, respectively, and f''_{xy} is the second order mixed partial derivative of f with respect to variables x and y.

Proof: Let $f \in C^2(\Delta), \forall(x,y) \in \Delta$, by Taylor's formula [8, pp.129-139] we have

$$\begin{aligned}
f(s,t) &= f(x,y) + f'_x(x,y)(s-x) + f'_y(x,y)(t-y) \\
&+\frac{1}{2}\Big\{f''_{xx}(x,y)(s-x)^2 + 2f''_{xy}(x,y)(s-x)(t-y) \\
&+f''_{yy}(x,y)(t-y)^2\Big\} + R_2(s,t;x,y), \qquad (14)
\end{aligned}$$

where

$$R_2(s,t;x,y) = \sum_{i=0}^{2} \alpha_i(s,t)(s-t)^{2-i}(t-y)^i.$$

Here $\alpha_i(s,t)$ is bounded on Δ and

$$\lim_{(s,t)\to(x,y)} \alpha_i(s,t) = 0. \qquad (15)$$

From the properties of $\alpha_i(s,t)$ and Lemma 2.1 we have

$$L_n(R_2;x,y) = o(\frac{1}{n}), \quad (n\to\infty). \qquad (16)$$

By (14) and Lemma 2.1 we have

$$\begin{aligned}
L_n(f;x,y) &= f(x,y) + f'_x(x,y)L_n(s-x;x,y) + f'_y(x,y)L_n(t-y;x,y) \\
&+\frac{1}{2}\Big\{f''_{xx}(x,y)L_n((s-x)^2;x,y) + 2f''_{xy}(x,y)L_n((s-x)(t-y);x,y) \\
&+f''_{yy}(x,y)L_n((t-y)^2;x,y)\Big\} + L_n(R_2;x,y). \qquad (17)
\end{aligned}$$

From Lemma 2.1 and (15)-(17) we can obtain the required result. This completes the proof of Theorem 3.4.

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