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Newton-type Iterations as Approximations of Chebyshev's Method

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Abstract

This paper examines two convergence theorems, under the same assumptions, for a modification of the Chebyshev method which are based upon majorant principle and recurrence relations. It is shown that from the last we obtain better results.

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1 Introduction

Let X and Y be Banach spaces and $F : \Omega \subseteq X \to Y$ be once Fréchet differentiable in an open convex domain Ω. Then the Chebyshev method ([1], [3], [9]), one of the most famous third-order iterations for solving the equation

$$F(x) = 0, \tag{1}$$

which starts from the initial approximation $x_0 \in \Omega$, is defined by

$$x_{n+1} = x_n - \left[I + \frac{1}{2}\Gamma_n F''(x_n)\Gamma_n F(x_n)\right]\Gamma_n F(x_n), \quad n \geq 0,$$

where $\Gamma_n = F'(x_n)^{-1}$, provided that $F'(x_n)$ is invertible at each step.

In [4], the goal of the paper is to reduce operational cost and at the same time to drop the conditions on the second Fréchet-derivative by replacing it with a fixed bilinear operator. So it was introduced an iteration, which was called a modification of Chebyshev's method, for solving (1) and defined by

$$\begin{aligned} x_{n+1} &= x_n - \Gamma_n F(x_n) - \tfrac{1}{2}\Gamma_n A(\Gamma_n F(x_n))^2 \\ &= x_n - \left[I + \tfrac{1}{2}\Gamma_n A \Gamma_n F(x_n)\right]\Gamma_n F(x_n), \quad n \geq 0, \end{aligned} \tag{2}$$

where $A : X \times X \to Y$ is a general bounded bilinear operator such that $\|A\| = \alpha$.

The "penalty" for this approach is the order of convergence drops from three to two. On the other hand, notice that (2) can be written in the form

$$\begin{aligned} F'(x_n)(y_n - x_n) &= -F(x_n), \\ F'(x_n)(x_{n+1} - y_n) &= -A(y_n - x_n)^2/2, \quad n \geq 0. \end{aligned}$$

We have therefore to solve twice the same system appearing in Newton's method with the only modification of the independent terms. As a result, under a similar computational cost, (2) remains faster than Newton's method. Observe that (2) is reduced to the Newton method if $A \equiv 0$.

To develop a convergence theorem for (2), using Kantorovich type assumptions ([5], [6]), two different analyses were done in [4]. The convergence assumptions were relaxed in the second analysis and the following was assumed:

(c$_1$) there exists a continous linear operator $\Gamma_0 = F'(x_0)^{-1}$, $x_0 \in \Omega$,

(c$_2$) $\|\Gamma_0(F'(x) - F'(y))\| \leq k\|x - y\|$, $x, y \in \Omega$, $\quad k \geq 0$,

(c$_3$) $\|A\| = \alpha$, $\quad \|\Gamma_0 A\| \leq \alpha/b$, $\quad \|\Gamma_0 F(x_0)\| = a/b$,

(c$_4$) $b - 2ak \geq 0$.

It was also employed the equation

$$p(t) \equiv \frac{kN}{2}t^2 - bt + a = 0 \tag{3}$$

has two positive roots r_1 and r_2 such that $r_1 \leq r_2$, where N is a parameter satisfying $N \in \left[b + \frac{2\alpha}{k}, \frac{b^2}{2ak}\right]$.

After it was defined the scalar sequence $\{t_n\}$ by

$$t_0 = 0, \quad t_{n+1} = t_n - \frac{p(t_n)}{p'(t_n)}\left(1 + \frac{\alpha}{2}\frac{p(t_n)}{p'(t_n)^2}\right), \quad n \geq 0, \tag{4}$$

and proved that $\{t_n\}$ is well-defined for $n \geq 0$, monotonously increases and converges quadratically to r_1 for all $0 \leq \alpha \leq \frac{b}{4a}(b - 2ak)$.

We now state the convergence theorem for (2) which is based upon majorant principle. We denote $\overline{B(x, r)} = \{y \in X; \|y - x\| \leq r\}$ and $B(x, r) = \{y \in X; \|y - x\| < r\}$.

Theorem 1.1 *Let us assume that conditions* $(\mathbf{c}_1)$–$(\mathbf{c}_4)$ *hold and* $0 \leq \alpha \leq \dfrac{b}{4a}(b - 2ak) = \alpha_1$. *Then the sequence* $\{x_n\}$ *defined by (2) converges to a solution* x^* *of (1) in* $\overline{B(x_0, r_1)} \cap \Omega$ *for* $N \in \left[b + \dfrac{2\alpha}{k}, \dfrac{b^2}{2ak}\right]$. *The limit* x^* *is the unique solution of (1) in* $B(x_0, r) \cap \Omega$ *where* $r = r_2 + \dfrac{2(N - b)}{kN}$. *Moreover* $\|x^* - x_n\| \leq r_1 - t_n$, $n \geq 0$.

Following that and taking into account "test" function (3), we observe, by analysing the asymptotic error constant $C_\alpha = \left|\dfrac{p''(r_1) - \alpha}{2p'(r_1)}\right|$, sequence (4) converges faster to r_1 for increasing values of the parameter $\alpha \in [0, kN]$.

Notice that the speed of convergence of sequence (2) is majorized by scalar sequence (4). Then this fact leads to extend the values of the parameter α so that the number of the general bounded bilinear operator A considered in (2) is increased. To reach that, we establish another result on the convergence of method (2) under the same general assumpptions $(\mathbf{c}_1)$–$(\mathbf{c}_4)$. A new technique consisting of a system of recurrence relations is used and we can also obtain better error estimates in practice as we can see in example 2.

In section 2, we derive several basic results on a scalar sequence and present the recurrence relations. Next, in section 3, we apply the results to establish a convergence theorem for process (2) and see the upper bound for α appearing in theorem 1.1 is improved, since the interval of valid α is extended. Finally, in section 4, we yield two examples where theses improvements are shown.

2 Preliminaires

Let us assume $(\mathbf{c}_1)$–$(\mathbf{c}_4)$ and define a sequence of positive real numbers depending on two parameters by the following recurrence relation. Denote $\delta = ak/b$ and $a_0 = a\alpha/b^2$. Set

$$a_{n+1} = a_n f(a_n)^2 g(a_n), \quad n \geq 0, \tag{5}$$

where

$$f(x) = \frac{2}{2(1-\delta) - \delta x} \tag{6}$$

and

$$g(x) = \frac{1}{2}\left[x + \delta\left(1 + \frac{x}{2}\right)^2\right]. \tag{7}$$

We first need a lemma whose proof is trivial.

Lemma 2.1 *Let f and g the two real functions defined in (6) and (7) respectively. Then*

(i) *f and g are increasing for $\delta \geq 0$,*

(ii) *$f(\gamma x) \leq f(x)$ and $g(\gamma x) \leq \gamma g(x)$ for $\gamma \in (0, 1)$.*

After that some properties for the sequence $\{a_n\}$ given by (5) are provided.

Theorem 2.2 *Let f and g the two real functions given by (6) and (7) respectively. Let $0 \leq \alpha < \dfrac{b^2}{a} h\left(\dfrac{ak}{b}\right) = \alpha_2$ where*

$$h(x) = \frac{2(2x^2 - 3x - 1 + \sqrt{1 + 8x - 4x^2})}{x(1 - 2x)}. \tag{8}$$

Then

(i) *$f(a_0)^2 g(a_0) < 1$,*

$(\mathbf{ii}_n)$ *$a_{n+1} \leq a_n$ for all $n \geq 0$,*

$(\mathbf{iii}_n)$ *$\delta(2 + a_n) < 2$ for all $n \geq 0$.*

Proof. ¿From the hypotheses it is easy to check **(i)** and $(\mathbf{iii}_0)$. Consequently, $a_1 \leq a_0$. We now suppose that $(\mathbf{ii}_j)$–$(\mathbf{iii}_j)$ are true for $j = 1, 2, \ldots, n-1$. Taking into account lemma 2.1 (i), we deduce $(\mathbf{ii}_n)$ and $(\mathbf{iii}_n)$. ∎

Theorem 2.3 *Assume the hypotheses of theorem 2.2 are satisfied. Let us define* $\gamma = a_1/a_0$. *Then*

(i) $a_n \leq \gamma^{2^n-1} a_0$ *for all* $n \geq 2$,

(ii) $f(a_n)g(a_n) \leq \dfrac{\gamma^{2^n}}{f(a_0)}$ *for all* $n \geq 1$.

Proof. Notice that $\gamma = a_1/a_0 = f(a_0)^2 g(a_0) < 1$ by theorem 2.2 (i), and $a_2 = a_1 f(a_1)^2 g(a_1) \leq \gamma^2 a_0 f(a_0)^2 g(a_0) = \gamma^2 a_1$ by lemma 2.1 (ii). If $a_j \leq \gamma^{2^{j-1}} a_{j-1}$ is supposed for $j = 2, 3, \ldots, n-1$, then

$$a_n = a_{n-1} f(a_{n-1})^2 g(a_{n-1}) \leq \gamma^{2^{n-1}} a_{n-2} f(a_{n-2})^2 g(a_{n-2}) = \gamma^{2^{n-1}} a_{n-1}$$

and, as a result, $a_n \leq \gamma^{2^n-1} a_0$.

Finally, for all $n \geq 1$, we have

$$f(a_n)g(a_n) \leq f(\gamma^{2^n-1} a_0) g(\gamma^{2^n-1} a_0) \leq \gamma^{2^n-1} f(a_0) g(a_0) = \frac{\gamma^{2^n}}{f(a_0)}$$

and **(ii)** also holds. ∎

Firstly, we observe that taking into account initial hypotheses $(\mathbf{c}_1)$–$(\mathbf{c}_4)$ and assuming that $y_0 \in \Omega$ we have

$$\|\Gamma_0 A \Gamma_0 F(x_0)\| \leq \|\Gamma_0 A\| \|\Gamma_0 F(x_0)\| \leq a_0.$$

Then the iterate x_1 is well defined and

$$\|x_1 - x_0\| \leq \left\| \left[I + \frac{1}{2} \Gamma_0 A \Gamma_0 F(x_0) \right] \right\| \|\Gamma_0 F(x_0)\| \leq \left(1 + \frac{a_0}{2}\right) \|\Gamma_0 F(x_0)\|.$$

Next we prove the following items are true for all $n \geq 1$ by mathematical induction:

$[\mathbf{I}_n]$ $\|\Gamma_n\| = \|F'(x_n)^{-1}\| \leq f(a_{n-1}) \|\Gamma_{n-1}\|$,

$[\mathbf{II}_n]$ $\|\Gamma_n F(x_n)\| \leq f(a_{n-1}) g(a_{n-1}) \|\Gamma_{n-1} F(x_{n-1})\|$,

$[\mathbf{III}_n]$ $\|\Gamma_n A\| \|\Gamma_n F(x_n)\| \leq a_n$,

$[\mathbf{IV}_n]$ $\|x_{n+1} - x_n\| \leq \left(1 + \dfrac{a_n}{2}\right) \|\Gamma_n F(x_n)\|$.

We now assume that $[\mathbf{I}_n]$–$[\mathbf{IV}_n]$ are true for some n and $x_n \in \Omega$. ¿From lemma 2.1 (i) and theorem 2.2 if $x_{n+1} \in \Omega$, it is follows that the operator $\Gamma_n F'(x_{n+1})$ is invertible, since

$$\|I - \Gamma_n F'(x_{n+1})\| \leq \prod_{j=0}^{n-1} \|\Gamma_{j+1} F'(x_j)\| \|\Gamma_0 (F'(x_n) - F'(x_{n+1}))\|$$

$$\leq k\|x_{n+1} - x_n\| \prod_{j=0}^{n-1} f(a_j) \leq k\left(1 + \frac{a_n}{2}\right) \|\Gamma_0 F(x_0)\| \prod_{j=0}^{n-1} f(a_j)^2 g(a_j) \leq \delta\left(1 + \frac{a_n}{2}\right) < 1,$$

and, by the Banach lemma, Γ_{n+1} is defined and

$$\|\Gamma_{n+1}\| = \|\Gamma_{n+1} F'(x_n)\Gamma_n\| \leq \frac{\|\Gamma_n\|}{1 - \|I - \Gamma_n F'(x_{n+1})\|} \leq \frac{\|\Gamma_n\|}{1 - \delta\left(1 + \frac{a_n}{2}\right)} = f(a_n)\|\Gamma_n\|.$$

Therefore $[\mathbf{I}_{n+1}]$ holds.

Using from (2) the expression for $x_{n+1} - x_n$, we have

$$F(x_n) + F'(x_n)(x_{n+1} - x_n) + \int_0^1 [F'(x_n + t(x_{n+1} - x_n)) - F'(x_n)](x_{n+1} - x_n)\, dt$$

$$= -\frac{A}{2}[\Gamma_n F(x_n)]^2 + \int_0^1 \prod_{j=0}^{n-1} [\Gamma_{j+1} F'(x_j)] \Gamma_0 [F'(x_n + t(x_{n+1} - x_n)) - F'(x_n)](x_{n+1} - x_n)\, dt,$$

so that

$$\|\Gamma_n F(x_{n+1})\| \leq \frac{1}{2}\|\Gamma_n A\| \|\Gamma_n F(x_n)\|^2 + \frac{k}{2}\|x_{n+1} - x_n\|^2 \prod_{j=0}^{n-1} \|\Gamma_{j+1} F'(x_j)\|$$

$$\leq \frac{1}{2}\left[\|\Gamma_n A\| \|\Gamma_n F(x_n)\| + k\left(1 + \frac{a_n}{2}\right)^2 \|\Gamma_n F(x_n)\| \prod_{j=0}^{n-1} f(a_j)\right] \|\Gamma_n F(x_n)\|$$

$$\leq \frac{1}{2}\left[a_n + \delta\left(1 + \frac{a_n}{2}\right)^2 \prod_{j=0}^{n-1} f(a_j)^2 g(a_j)\right] \|\Gamma_n F(x_n)\|$$

$$\leq \frac{1}{2}\left[a_n + \delta\left(1 + \frac{a_n}{2}\right)^2\right] \|\Gamma_n F(x_n)\| = g(a_n)\|\Gamma_n F(x_n)\|.$$

follows as a consequence of theorem 2.2. Hence $[\mathbf{II}_{n+1}]$ is true, since

$$\|\Gamma_{n+1} F(x_{n+1})\| \leq \|\Gamma_{n+1} F'(x_n)\| \|\Gamma_n F(x_{n+1})\| \leq f(a_n) g(a_n) \|\Gamma_n F(x_n)\|.$$

Now it is easy to see $[\mathbf{III}_{n+1}]$ and $[\mathbf{IV}_{n+1}]$:

$$\|\Gamma_{n+1} A\| \|\Gamma_{n+1} F(x_{n+1})\| \leq a_n f(a_n)^2 g(a_n) = a_{n+1}$$

and

$$\|x_{n+2}-x_{n+1}\| \leq \left(1+\frac{1}{2}\|\Gamma_{n+1}A\|\|\Gamma_{n+1}F(x_{n+1})\|\right)\|\Gamma_{n+1}F(x_{n+1})\|$$
$$\leq \left(1+\frac{a_{n+1}}{2}\right)\|\Gamma_{n+1}F(x_{n+1})\|.$$

Then the induction is complete.

3 Main convergence theorem

After last section we are ready to prove the following convergence theorem.

Theorem 3.1 *In addition to assumptions* $(\mathbf{c}_1)$–$(\mathbf{c}_4)$ *and definition (5), assume that* $B(x_0, 1/k) \subset \Omega$ *and* $0 \leq \alpha < \frac{b^2}{a}h\left(\frac{ak}{b}\right) = \alpha_2$ *where* h *is given by (8). Then iterates (2) are well-defined for all* $n \geq 0$*, lie in* $B(x_0, 1/k)$ *for* $n \geq 0$ *and converge R-quadratically at least to a unique solution* $x^* \in B(x_0, 1/k)$ *of equation (1). Furthermore the error bounds*

$$\|x^*-x_n\| \leq \left(1+\frac{a_0}{2}\gamma^{2^n-1}\right)\gamma^{2^n-1}\frac{\Delta^n}{1-\Delta}\frac{a}{b}, \quad n \geq 0, \tag{9}$$

hold, where $\gamma = a_1/a_0 < 1$ *and* $\Delta = 1/f(a_0) < 1$.

Proof. Firstly, we note

$$\left(1+\frac{a_n}{2}\right)\|\Gamma_n F(x_n)\| \leq \left(1+\frac{a_n}{2}\right) f(a_{n-1})g(a_{n-1})\|\Gamma_{n-1}F(x_{n-1})\|$$
$$\leq \cdots \leq \left(1+\frac{a_n}{2}\right)\|\Gamma_0 F(x_0)\| \prod_{j=0}^{n-1} f(a_j)g(a_j)$$

as a consequence of estimate $[\mathbf{II}_n]$. Next, from theorem 2.3, we have

$$\prod_{j=0}^{n-1} f(a_j)g(a_j) \leq \prod_{j=0}^{n-1}\left(\gamma^{2^j}\Delta\right) = \gamma^{2^n-1}\Delta^n$$

where $\gamma = a_1/a_0 < 1$ and $\Delta = 1/f(a_0) < 1$. Thus

$$\|x_{n+m}-x_n\| \leq \|x_{n+m}-x_{n+m-1}\| + \|x_{n+m-1}-x_{n+m-2}\| + \cdots + \|x_{n+1}-x_n\|$$
$$\leq \left(1+\frac{a_n}{2}\right)\sum_{i=n-1}^{n+m-2}\left[\prod_{j=0}^{i} f(a_j)g(a_j)\right]\|\Gamma_0 F(x_0)\|$$

$$\leq \left(1+\frac{a_0}{2}\gamma^{2^n-1}\right)\gamma^{2^n-1}\frac{1-\Delta^m}{1-\Delta}\Delta^n\|\Gamma_0 F(x_0)\|. \tag{10}$$

As a result, we have for $n=0$

$$\|x_m - x_0\| < \left(1+\frac{a_0}{2}\right)\frac{1-\Delta^m}{1-\Delta}\|\Gamma_0 F(x_0)\| = 1/k$$

and $x_m \in B(x_0, 1/k)$ for all $m \geq 0$. Moreover, by letting $m \to \infty$ in (10), we get

$$\|x^* - x_n\| \leq \left(1+\frac{a_0}{2}\gamma^{2^n-1}\right)\gamma^{2^n-1}\frac{\Delta^n}{1-\Delta}\frac{a}{b}.$$

This proves that $\{x_n\}$ converges to a limit $x^* \in B(x_0, 1/k)$ and bounds (9) hold.

To see that x^* is a solution of (1), we have $\|\Gamma_n F(x_n)\| \to 0$ as $n \to \infty$. Taking into account that $\|F(x_n)\| \leq \|F'(x_n)\|\|\Gamma_n F(x_n)\|$ and the sequence $\{\|F'(x_n)\|\}$ is bounded, we infer that $\|F(x_n)\| \to 0$ as $n \to \infty$. Consequently, we obtain $F(x^*) = 0$ by the continuity of F.

To prove the uniqueness, assume a some other solution y^* of (1) in $B(x_0, 1/k)$. According to Argyros and Chen ([1], [2]), from the approximation,

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))\,dt\,(y^* - x^*),$$

we have to prove that the operator $\int_0^1 F'(x^* + t(y^* - x^*))\,dt$ is invertible and then $y^* = x^*$. Indeed, from

$$\int_0^1 \|\Gamma_0[F'(x^* + t(y^* - x^*)) - F'(x_0)]\|\,dt \leq k\int_0^1 \|x^* + t(y^* - x^*) - x_0\|\,dt$$

$$\leq k\int_0^1 \left[(1-t)\|x^* - x_0\| + t\|y^* - x_0\|\right]\,dt < 1,$$

it follows that $\left[\int_0^1 F'(x^* + t(y^* - x^*))\,dt\right]^{-1}$ exists.

It remains to be shown that sequence (2) has R-order of convergence two at least. We deduce from (9) that

$$\|x^* - x_n\| \leq \left(1+\frac{a_0}{2}\right)\frac{a/b}{\gamma(1-\Delta)}\gamma^{2^n}, \quad n \geq 0.$$

Hence the convergence of (2) is R-quadratical at least (see [8]). ■

Remark 1. Observe that the interval of valid α has been improved under the same general assumptions $(\mathbf{c}_1)$–$(\mathbf{c}_4)$. Note that $\alpha_1 = \frac{b}{4a}(b - 2ak)$ and $\alpha_2 = \frac{b^2}{a}h(\frac{ak}{b})$ where h is defined in (8) are the upper bounds for α appearing in theorem 1.1 and theorem 3.1 respectively. It is easy to prove that $\alpha_2 = \max\{\alpha_1, \alpha_2\}$, and the values of α are therefore extended.

4 Examples

In this section we provide two examples where we compare the results obtained in this paper with those from theorem 1.1.

Example 1. We first apply iterative method (2) to the cubic function, introduced by Ostrowski [7], $F : [-4, 4] \to \mathbb{R}$ where $F(x) = x^3 - 2x - 5$. We choose the initial value $x_0 = 2.4$. Then we can easily find the parameters appearing in theorem 3.1.:

$$a = 4.024, \quad b = 15.28 \quad \text{and} \quad k = 2.63916.$$

In consequence $\alpha < \alpha_2 = 8.36148$. Notice the obtained upper bound α_2 for α is much bigger than the upper bound $\alpha_1 = 2.50537$ appearing in theorem 1.1.

As the hypotheses of theorem 3.1 are satisfied, sequence (2), where $A = \alpha$, converges to $x^* = 2.09455148154$ with 12 significant digits for any $\alpha \in [0, \alpha_2)$. Moreover this solution is unique in $B(x_0, 0.636667)$.

On the other hand, observe that the higher the value of $\alpha \in [0, 8.36148)$ is, the smaller the asymptotic error constant $C_{e,\alpha} = \dfrac{12.5673 - \alpha}{22.3229}$ is. As a result, sequence (2) converges faster to x^* for increasing values of parameter $\alpha \in [0, 8.36148)$ (see table I).

n	x_n ($\alpha = 5$)	x_n ($\alpha = 8$)
0	2.40000000000	2.40000000000
1	2.11395496200	2.10033841040
2	2.09459866332	2.09454655415
3	2.09455148180	2.09455148154

Table I

Example 2. A more interesting application is given by the following example. We take the differential equation quoted in [4]

$$\begin{cases} y'' + y^2 = 0, \\ y(0) = 0 = y(1). \end{cases} \tag{11}$$

Divide the interval $[0, 1]$ into n subintervals and we set $h = 1/n$. We denote the points of subdivision by $0 = w_0 < w_1 < \ldots < w_n = 1$ with the corresponding values of the function $y_0 = y(w_0), y_1 = y(w_1), \ldots, y_n = y(w_n)$. A standard approximation for the second derivative at these points is

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}, \quad i = 1, 2, \ldots, n-1.$$

Noting that $y_0 = 0 = y_n$, define the operator $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$F(y) = Gy + h^2 g(y)$$

where

$$G = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix}, \quad g(y) = \begin{pmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_{n-1}^2 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix}.$$

Then

$$F'(y) = G - 2h^2 \begin{pmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_{n-1} \end{pmatrix}.$$

Let $y \in \mathbb{R}^{n-1}$, then the norm will be $\|y\| = \max\limits_{1 \le i \le n-1} |y_i|$. The corresponding norm on $G \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ is

$$\|G\| = \max_{1 \le i \le n-1} \sum_{k=1}^{n-1} |g_{ik}| = 4.$$

We note that for all $x, y \in \mathbb{R}^{n-1}$:

$$\|F'(x) - F'(y)\| = \|\mathrm{diag}\{2h^2(x_p - y_i)\}\| = 2h^2 \max_{1 \le i \le n-1} |x_i - y_i| \le 2h^2 \|x - y\|.$$

To show the convergence of (2) to a solution y^*, set $n = 10$ and as the solution would vanish at the endpoints and be positive in the interior, a reasonable choice of initial approximation seems to be $\frac{1}{4} \sin \pi x$. This gives us the following vector:

$$x_0 = \begin{pmatrix} 0.0772542 \\ 0.1469460 \\ 0.2022540 \\ 0.2377640 \\ 0.2500000 \\ 0.2377640 \\ 0.2022540 \\ 0.1469460 \\ 0.0772542 \end{pmatrix}.$$

For this initial approximation we obtain the following parameters:

$$a = 0.0195386, \quad b = 0.0763837 \quad \text{and} \quad k = 0.261836.$$

Then, under the same hypotheses, we obtain $\alpha < \alpha_2 = 0.412466$ (theorem 3.1) versus $\alpha \leq \alpha_1 = 0.0646531$ (theorem 1.1). In consequence the valid interval of α has been quite increased.

By theorem 3.1, iteration (2) converges to a unique solution $y^* = (y_1^*, y_2^*, \ldots, y_9^*)$ in $B(x_0, 3.81919)$.

On the other hand, the error estimates obtained by theorem 1.1 and theorem 3.1, under the same value of $\alpha = 0.06$ (see [4]), are compared in table II. It is shown that better error bounds are obtained from the last one.

n	error estimates from th. 1.1	error estimates from th. 3.1
1	0.1535885817753795	0.0466942344593571
2	0.0406992460289781	0.0032037407239804
3	0.0046814846994008	0.0002411379471532
4	0.0000777752985228	0.0000275164672344

Table II

Remark 2. We indicate the choice of the matrix A is an open study for future papers. In this paper, we only choose those with a little operational cost taking into account the influence of $\|A\|$ in the speed of convergence of iteration (2).

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