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Exact Internal Controllability of the Elasticity System

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Abstract

We study the exact internal controllability of the system:

$$\varphi'' - \mu\Delta\varphi - (\lambda + \mu)\nabla(\alpha(x) div\, \varphi) = h \quad \text{in } \Omega\times]0, T[,$$

where Ω is a bounded domain in $\mathbf{R}^n$, λ and μ are the Lamè's constants, and α is a positive real function defined in $\overline{\Omega}$. The result is obtained by applying HUM, Hilbert Uniqueness Method, due to J. L. Lions.

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1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^n$ with a C^2-boundary Γ and let ω be a nonempty open subset of Ω. For $T > 0$, we denote by Q the finite cylinder $\Omega\times]0, T[$ and by Σ its lateral boundary $\Gamma\times]0, T[$.

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In this paper we are concerned with the exact internal controllability of the elasticity system:

$$
(1.1) \qquad \left| \begin{array}{l} \varphi'' - \mu\Delta\varphi - (\lambda+\mu)\nabla(\alpha(x) div\varphi) = h\chi_\omega \quad \text{in} \quad Q \\ \varphi = 0 \quad \text{on} \quad \Sigma \\ \varphi(0) = \varphi^0, \quad \varphi'(0) = \varphi^1 \quad \text{in} \quad \Omega, \end{array} \right.
$$

where χ_ω denotes the characteristic function of ω, $\lambda > 0$ and $\mu > 0$ are the Lamè's constants, and α is a positive real function defined in $\overline{\Omega}$.

The exact controllability problem for the above system is formulated as follows:

" Given $T > 0$, find a Hilbert space $\mathcal{H}$ such that for each pair $\{\varphi_0, \varphi_1\} \in \mathcal{H}$ there exists a control function $h \in L^2(0,T;(L^2(\omega))^n)$ such that the solution φ of the system (1.1) satisfies:

$$\varphi(T) = \varphi'(T) = 0."$$

Exact controllability of the elasticity system was firstly studied by J. L. Lions [10]. Later, R. Fuentes Apolaya [5] considered the problem in noncylindrical domains and E. Zuazua [16] attacked the question of the approximate controllability. However, all of them assume $\alpha(x) \equiv$ constant. The main difficulty found in the resolution of the problem lies in the fact that we cannot apply the Holmgren's Theorem to differential operators with non constant coefficients. To bypass this difficulty, we establish a new argument based on regularization techniques. We refer the reader to [2], [3], [4], [6], [7] and [9] for other works on exact controllability.

In Section 2 we establish our main result Theorem 2.1. By assuming an "inverse inequality", we prove Theorem 2.1 with application of HUM. The arguments are standard, yet nontrivial. The proof of the "inverse inequality" is given in Section 3 by means of an argument which is in part related to the one used in Prates Filho-Soriano [13].

2. The Main Result and Application of HUM

For a fixed $x^0 \in \mathbf{R}^n$ we define

$$m(x) = x - x^0, \quad \forall\, x \in \overline{\Omega}$$

and

$$R(x^0) = \sup_{x\in\overline{\Omega}} \|m(x)\| = \sup_{x\in\overline{\Omega}} (\sum_{i=1}^{n} |x_i - x_i^0|^2)^{\frac{1}{2}}.$$

We also define

$$\Gamma_0 = \Gamma(x^0) = \{x \in \Gamma;\, m(x)\cdot\nu(x) > 0\}$$

and

$$\Gamma_*(x^0) = \Gamma \backslash \Gamma_0,$$

where $\nu(x)$ denotes the unit exterior normal at the point x of Γ.

We assume that $\alpha \in C^1(\overline{\Omega})$ and there exists some $\alpha_0 > 0$ such that

$$\alpha(x) \geq \alpha_0 \quad \forall\, x \in \overline{\Omega} \quad \text{and} \quad \|\nabla \alpha\|_{L^\infty(\Omega)} < \frac{\alpha_0}{R(x^0)}. \tag{2.1}$$

The following functional spaces will be considered:

$$H = [L^2(\Omega)]^n, \quad (u, v)_H = \sum_{i=1}^{n} (u_i, v_i) \quad \forall\, u, v \in H,$$

$$V = [H_0^1(\Omega)]^n, \quad ((u, v))_V = \sum_{i=1}^{n} (\nabla u_i, \nabla v_i) \quad \forall\, u, v \in V,$$

where $(\cdot, \cdot)$ denotes the standard inner-product of $L^2(\Omega)$. If we denote by V' the topological dual of V, we then have

$$V' = [H^{-1}(\Omega)]^n.$$

Now we are ready to state our main result.

Theorem 2.1 *Assume that condition* (2.1) *holds for a fixed* $x_0 \in \mathbf{R}^n$ *and let* ω *be a neighbourhood of* $\overline{\Gamma}_0$*, that is, there exists an open subset* $\mathcal{O}$ *of* $\mathbf{R}^n$ *such that* $\overline{\Gamma}_0 \subset \mathcal{O}$ *and* $\omega = \Omega \cap \mathcal{O}$*.*

Then given $T > T_0$ *where*

$$T_0 = \frac{2R(x^0)}{\sqrt{\mu}(1 - ||\nabla \alpha||_{L^\infty(\Omega)} R(x^0) \alpha_0^{-1})}, \tag{2.2}$$

there exists, for each pair $\{\varphi^0, \varphi^1\} \in H \times V$*, a control* $h \in L^2(0, T; (L^2(\omega))^n)$ *such that the solution* φ *of the problem* (1.1) *satisfies*

$$\varphi(T) = \varphi'(T) = 0.$$

Proof. First we consider a pair $\{\phi^0, \phi^1\} \in [D(\Omega)]^n \times [D(\Omega)]^n$. Then from the smoothness of the inital data, the homogeneous system

$$\left| \begin{array}{l} \phi'' - \mu \Delta \phi - (\lambda + \mu) \nabla (\alpha(x) div \phi) = 0 \quad \text{in } \; Q \\ \phi = 0 \quad \text{on } \; \Sigma \\ \phi(0) = \phi^0, \quad \phi'(0) = \phi^1 \quad \text{in } \; \Omega, \end{array} \right. \tag{2.3}$$

has a unique regular solution. Next we consider the backward problem

$$(2.4) \qquad \left| \begin{array}{l} \psi'' - \mu\Delta\psi - (\lambda+\mu)\nabla(\alpha(x) div\psi) = -\phi\chi_\omega \quad \text{in } Q \\ \psi = 0 \quad \text{on } \Sigma \\ \psi(0) = \psi^0, \quad \psi'(0) = \psi^1 \quad \text{in } \Omega, \end{array} \right.$$

where ϕ is the unique solution of (2.3).

Choosing $\{\psi^0, \psi^1\} \in V \times H$ and taking into account the regularity of ϕ, problem (2.4) has a unique solution ψ satisfying

$$\psi \in C^0([0,T];V) \cap C^1([0,T];H).$$

Therefore the mapping $\Lambda : [D(\Omega)]^n \times [D(\Omega)]^n \to H \times V$ given by

$$(2.5) \qquad \Lambda\{\phi^0, \phi^1\} = \{\psi'(0), -\psi(0)\}$$

is well defined.

On the other hand, if we choose $\{\phi^0, \phi^1\} \in H \times V'$, we get the so called ultra-weak solution ϕ of (2.3) by a transposition argument (cf. Lions [10]). More precisely, $\phi \in L^\infty(0,T;H)$ and satisfies

$$\sum_{i=1}^n \int_Q \phi_i f_i dxdt = \langle \phi^1, \zeta(0)\rangle_{V',V} - (\phi^0, \zeta'(0))_H,$$

for every weak solution of

$$\left| \begin{array}{l} \zeta'' - \mu\Delta\zeta - (\lambda+\mu)\nabla(\alpha(x) div\zeta) = f \quad \text{in } Q \\ \zeta = 0 \quad \text{on } \Sigma \\ \zeta(T) = \zeta'(T) = 0 \quad \text{in } \Omega, \end{array} \right.$$

where $\zeta \in L^1(0,T;H)$. Then, in this case, the ultra-weak solution ϕ of (2.3) satisfies

$$(2.6) \qquad \sum_{i=1}^n \int_0^T \int_\omega |\phi_i(x,t)|^2 dxdt = (\phi^0, \psi'(0))_H + \langle \phi^1, \psi(0)\rangle_{V',V}.$$

This is done by multiplying ψ and the equation in (2.3), and noting that $[D(\Omega)]^n \times [D(\Omega)]^n$ is dense in $H \times V'$.

Next, we define the bilinear form $(\cdot,\cdot)_* : ([D(\Omega)]^n)^2 \times ([D(\Omega)]^n)^2 \to \mathbf{R}$ by

$$(2.7) \qquad (\{\phi^0, \phi^1\}, \{\zeta^0, \zeta^1\})_* = \sum_{i=1}^n \int_0^T \int_\omega \phi_i \zeta_i dxdt$$

where ϕ and ζ are the solutions of (2.3) associated to the initial data $\{\phi^0, \phi^1\}$ and $\{\zeta^0, \zeta^1\}$, respectively. The functional given in (2.7) defines a inner product in $[(D(\Omega))^n]^2 \times [(D(\Omega))^n]^2$ and, consequently, we can consider the following norm in $[D(\Omega)]^n \times [D(\Omega)]^n$:

$$\|\{\phi^0, \phi^1\}\|_* = (\sum_{i=1}^n \int_0^T \int_\omega |\phi_i(x,t)|^2 dxdt)^{\frac{1}{2}}.$$

In order to extend Λ to the whole space $H \times V'$, we consider the Hilbert space $\mathcal{F}$ obtained by taking the closure of $[D(\Omega)]^n \times [D(\Omega)]^n$ with respect to $\| \cdot \|_*$. Then observing that if ϕ is the ultra-weak solution of (2.3), there exists a constant $k_2 > 0$ such that

$$\sum_{i=1}^{n} \int_0^T \int_\omega |\phi_i(x,t)|^2 dxdt \leq k_2(\|\phi^0\|_H^2 + |\phi^1|_{V'}^2).$$

Symmetrically, we claim that if $T > T_0$, then there exists a constant $k_1 > 0$ such that

$$k_1(|\phi^0|_H^2 + |\phi^1|_{V'}^2) \leq \sum_{i=1}^{n} \int_0^T \int_\omega |\phi_i(x,t)|^2 dxdt. \tag{2.8}$$

The proof of this "inverse inequality" is rather technical and it is postponed to Section 3. It follows that

$$\mathcal{F} = H \times V',$$

and therefore Λ has a unique continuous extension to the whole $\mathcal{F}$ which satisfies, for all $\{\phi^0, \phi^1\}, \{\zeta^0, \zeta^1\} \in \mathcal{F}$,

$$\langle \Lambda\{\phi^0, \phi^1\}, \{\zeta^0, \zeta^1\} \rangle_{\mathcal{F},\mathcal{F}'} = (\{\phi^0, \phi^1\}, \{\zeta^0, \zeta^1\})_*.$$

Moreover, since $\mathcal{F}$ is a Hilbert space when equiped with the inner product $(\cdot, \cdot)_*$, from the Lax-Milgran's Lemma for each pair $\{y^0, y^1\} \in \mathcal{F}'$ there exists only one pair $\{\phi^0, \phi^1\} \in \mathcal{F}$ such that

$$\Lambda\{\phi^0, \phi^1\} = \{y^0, y^1\}.$$

Therefore, assuming $\{\varphi^0, \varphi^1\} \in V \times H$, there exists only one pair $\{\phi^0, \phi^1\} \in H \times V'$ such that $\Lambda\{\phi^0, \phi^1\} = \{\varphi^1, -\varphi^0\}$, and from (2.5) we obtain that

$$\{\varphi^1, -\varphi^0\} = \{\psi'(0), -\psi(0)\}, \tag{2.9}$$

where ψ is the solution of (2.4) and ϕ is the solution of (2.3) with initial data $\{\phi^0, \phi^1\}$.

Finally we summarize our arguments. For $T > T_0$ and $\{\varphi^0, \varphi^1\} \in H \times V$ there exists only one pair $\{\phi^0, \phi^1\} \in H \times V'$ such that if ϕ is the solution of (2.3) and ψ is the solution of (2.4), we obtain from (2.9) that φ and ψ are solutions of the original problem (1.1) associated to the initial data $\{\varphi^0, \varphi^1\}$ when we choose

$$h = -\phi \in L^2(0,T;(L^2(\omega))^n).$$

¿From the unicity of solution of the elasticity system (1.1) we have that

$$\varphi(T) = \varphi'(T) = 0.$$

This ends the proof. □

3. Proof of the Inverse Inequality

This section is dedicated to the proof of the inequality (2.8), which is a consequence of the following theorem.

Theorem 3.1 (Inverse Inequality) *Let ω be a neighbourhood of $\overline{\Gamma}_0$. If $T > T_0$, T_0 given by* (2.2)*, there exists a constant $c = c(T) > 0$ such that:*

$$|\phi^0|_H^2 + |\phi^1|_{V'}^2 \leq c \sum_{i=1}^{n} \int_0^T \int_\omega |\phi_i(x,t)|^2 dx dt \tag{3.1}$$

holds for every $\{\phi^0, \phi^1\} \in H \times V'$, where $\phi = (\phi_1, \cdots \phi_n)$ is the corresponding ultra-weak solution of (2.3).

Proof. Assume that if θ is the weak solution of (2.3) associated to the initial data $\{\theta^0, \theta^1\} \in V \times H$, we have for some $c_1 = c_1(T)$ the following estimate:

$$||\theta^0||_V^2 + |\theta^1|_H^2 + |div\, \theta^0|_{L^2(\Omega)}^2 \leq c_1 \sum_{i=1}^{n} \int_0^T \int_\omega |\theta_i^{'}(x,t)|^2 dx dt. \tag{3.2}$$

Consider now $\{\phi^0, \phi^1\} \in H \times V^{'}$ and let ϕ be the ultra-weak solution of (2.3). Defining on $V \times V$ a coercive, continuous and bilinear form $b(u,v)$ given by

$$b(u,v) = \mu \sum_{i,j=1}^{n} \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx + (\lambda + \mu) \int_\Omega \alpha(x)(div\, u)(div\, v) dx, \tag{3.3}$$

we obtain from the Lax-Milgran's Lemma that there exists only one $\eta \in V$ such that

$$b(\eta, v) = \langle -\phi^1, v \rangle_{V',V}, \quad \forall v \in V.$$

We observe that, provided (3.3) holds, it comes from the above identity that:

$$\mu \Delta \eta + (\lambda + \mu) \nabla(\alpha(x) div \eta) = \phi^1 \quad \text{in } V'. \tag{3.4}$$

Now, let us define

$$\Psi(t) = \int_0^t \phi(s) ds + \eta \tag{3.5}$$

Then, Ψ is the weak solution of

$$\left| \begin{array}{l} \Psi'' - \mu \Delta \Psi - (\lambda + \mu) \nabla(\alpha(x) div \Psi) = 0 \text{ in } \quad Q \\ \Psi = 0 \quad \text{on } \Sigma \\ \Psi(0) = \eta, \quad \Psi'(0) = \phi^0 \quad \text{in } \Omega, \end{array} \right.$$

and consequently Ψ satisfies (3.2), that is,

$$\|\eta\|_V^2 + |\phi^0|_H^2 + |div\, \eta|_{L^2(\Omega)}^2 \leq c_1 \sum_{i=1}^{n} \int_0^T \int_\omega |\phi_i(x,t)|^2 dxdt. \tag{3.6}$$

Since $-\Delta : H_o^1(\Omega) \to H^{-1}(\Omega)$ is a isometry, it follows from (3.4) and (3.6) that

$$|\phi^0|_H^2 + |\phi^1|_{V'}^2 \leq c \sum_{i=1}^{n} \int_0^T \int_\omega |\phi_i(x,t)|^2 dxdt,$$

which is the desired inequality. □

Then, from the above explanation, we conclude that to prove Theorem 3.1 we have only to prove that every weak solution of (2.3) satisfies the inequality given in (3.2). This is our next objective.

We recall that the energy of the system (2.3), when weak solutions are concerned, is given by:

$$E(t) = \frac{1}{2}|\phi'(t)|_H^2 + \frac{\mu}{2}\|\phi(t)\|_V^2 + \frac{(\lambda+\mu)}{2}|\alpha^{\frac{1}{2}}(x) div\, \phi(t)|_{L^2(\Omega)}^2. \tag{3.7}$$

So, if we denote by E_0 the initial energy, we have:

$$E_0 = \frac{1}{2}|\phi^1|_H^2 + \frac{\mu}{2}||\phi^0||_V^2 + \frac{(\lambda+\mu)}{2}|\alpha^{\frac{1}{2}}(x) div\, \phi^0|_{L^2(\Omega)}^2. \tag{3.8}$$

Lemma 3.2 *For every $T > T_0$, with T_0 defined in* (2.2)*, and every weak solution θ of (2.3), we have the following inequality:*

$$(T - T_0)E_0 \leq c_3 R(x^0) \int_0^T \int_{\Gamma_0} \{\mu \sum_{i=1}^{n} (\frac{\partial \theta_i}{\partial \nu})^2 + (\lambda+\mu)\alpha(x)(div\theta)^2\} d\Gamma dt \tag{3.9}$$

where E_0 is given in (3.8)*.*

Proof. We are going to use the multiplier method as in J. L. Lions [10]. Consider $m = x - x^0$ and let m_k be its k-component. If θ is the weak solution of (2.3) and we multiply the i component of the equation

$$\theta'' - \mu\Delta\theta - (\lambda+\mu)(\alpha(x) div\theta) = 0$$

by $m_k \frac{\partial \theta_i}{\partial x_k}$, we obtain after using Green and Gauss' Theorems and integration by parts the following expression:

$$\frac{1}{2}\int_\Sigma \{\mu \sum_{i=1}^{n} (\frac{\partial \theta_i}{\partial \nu})^2 + (\lambda+\mu)\alpha(x)(div\theta)^2\} m \cdot \nu d\Sigma =$$

$$\sum_{i,k=1}^{n}(\theta_i^{'}(t),(x_k-x_k^0)\frac{\partial\theta_i(t)}{\partial x_k})|_{t=0}^{t=T}+$$

$$\frac{(n-1)}{2}\int_Q\{\sum_{i=1}^{n}|\theta_i^{'}(t)|^2-\mu\sum_{i=1}^{n}|\nabla\theta_i|^2-(\lambda+\mu)\alpha(x)(div\theta)^2\}dxdt+$$

$$\int_0^T E(t)dt-\frac{(\lambda+\mu)}{2}\int_Q(div\theta)^2\nabla\alpha\cdot mdxdt.$$

¿From the above expression we have that:

$$\frac{R(x^0)}{2}\int_0^T\int_{\Gamma_0}\{\mu\sum_{i=1}^{n}(\frac{\partial\theta_i}{\partial\nu})^2+(\lambda+\mu)\alpha(x)(div\theta)^2\}d\Gamma dt\geq$$

$$\sum_{i=1}^{n}(\theta_i^{'}(t),\sum_{k=1}^{n}(x_k-x_k^0)\frac{\partial\theta_i(t)}{\partial x_k}+\frac{(n-1)}{2}\theta_i(t))|_{t=0}^{t=T}+$$

$$(1-||\nabla\alpha||_{L^\infty(\Omega)}R(x^0)\alpha_0^{-1})\int_0^T E(t)dt\geq$$

$$\frac{-2\mathrm{R}(x^0)}{\sqrt{\mu}}E_0+(1-||\nabla\alpha||_{L^\infty(\Omega)}R(x^0)\alpha_0^{-1})TE_0,$$

which proves the lemma. □

Lemma 3.3 *Let us consider $T>T_0$ and $\varepsilon>0$ such that $T-2\varepsilon>T_0$. Then, for every weak solution θ of (2.3) we have:*

$$||\theta^0||_V^2+|\theta^1|_H^2+|div\theta^0|_{L^2(\Omega)}^2$$

$$\leq c_4(T)\int_\varepsilon^{T-\varepsilon}\int_{\Gamma_0}\{\mu\sum_{i=1}^{n}(\frac{\partial\theta_i}{\partial\nu})^2+(\lambda+\mu)\alpha(x)(div\theta)^2\}d\Sigma \tag{3.10}$$

Proof. Consider $T>T_0$ and $\varepsilon>0$ such that $T-T_0>2\varepsilon$.

Let θ be the weak solution of (2.3) associated to the initial data $\{\theta^0,\theta^1\}\in V\times H$ and let $E(t)$ be the energy relation defined in (3.7).

If we define

$$\zeta(t)=\theta(t+\varepsilon),\quad t\in[0,T-\varepsilon]$$

we have that ζ is the weak solution of

$$\left|\begin{array}{l}\zeta''-\mu\Delta\zeta-(\lambda+\mu)\nabla(\alpha(x)div\zeta)=0\ \text{ in }\ C^0([0,T-2\varepsilon];V')\\ \zeta=0\quad on\ \ \Sigma\\ \zeta(0)=\theta(\varepsilon),\quad\zeta'(0)=\theta'(\varepsilon)\quad\text{in }\ \Omega,\end{array}\right.$$

and consequently from (3.9) it comes that

$$E(\varepsilon) \leq c_5 \int_0^{T-2\varepsilon} \int_{\Gamma_0} \{\mu \sum_{i=1}^n (\frac{\partial \zeta_i}{\partial \nu})^2 + (\lambda+\mu)\alpha(x)(div\zeta)^2\} d\Gamma dt.$$

If we make the change of variables $s = t + \varepsilon$ in the above integral and taking into account that the energy is conservative we obtain the desired result. □

Consider $T > T_0$ and let $\varepsilon > 0$ be such that $T - 2\varepsilon > 0$. Let us define a vector field

$$q = q(x,t) \in [W^{1,\infty}(Q)]^n \tag{3.11}$$

which satisfies the following properties:
(i) $q(x,t) = \nu(x), \quad \forall (x,t) \in \Gamma_0 \times]\varepsilon, T-\varepsilon[.$
(ii) $q(x,t) \cdot \nu(x) \geq 0, \quad \forall (x,t) \in \Gamma \times]0,T[.$
(iii) $q(x,0) = q(x,T) = 0, \quad \forall x \in \Omega.$
(iv) $q(x,t) = 0, \quad \forall (x,t) \in (\Omega \backslash \omega) \times]0,T[.$

Let us consider at a first time the strong solution θ of (2.3). If we multiply the i-component of the equation given in (2.3) by $\nabla\theta_i \cdot q$, if we observe the properties of the field q and if we use Lemma 3.3; we obtain from a density argument the following result:

Lemma 3.4 *Consider $T > T_0$ and $\varepsilon > 0$ such that $T - 2\varepsilon > 0$. If θ is the weak solution of (2.3), there exists a constant $c_6 = c_6(T) > 0$ which satisfies:*

$$||\theta^0||_V^2 + |\theta^1|_H^2 + |div\theta^0|^2_{L^2(\Omega)}$$

$$\leq c_6 \int_\varepsilon^{T-\varepsilon} \int_\omega \{\sum_{i=1}^n |\theta_i'(x,t)|^2 + \sum_{i=1}^n |\nabla\theta_i(x,t)|^2 + |div\theta|^2\} dxdt. \tag{3.12}$$

Remark 3.5 Consider $\hat{\omega}$ a neighbourhood of $\overline{\Gamma}_0$ such that $\Omega \cap \overline{\hat{\omega}} \subset \omega$. So, if we construct to $\hat{\omega}$ a field $\hat{q}$ similar to the field q constructed to ω in (3.11) we obtain:

$$||\theta^0||_V^2 + |\theta^1|_H^2 + |div\theta^0|^2_{L^2(\Omega)} \leq$$

$$c_7 \int_\varepsilon^{T-\varepsilon} \int_{\hat{\omega}} \{\sum_{i=1}^n |\theta_i'(x,t)|^2 + \sum_{i=1}^n |\nabla\theta_i(x,t)|^2 + |div\theta|^2\} dxdt. \tag{3.13}$$

Let us consider now a function

$$p = p(x,t) \in W^{1,\infty}(Q)$$

which satisfies:
(*i*) $p(x,t) \geq 0$, $\forall (x,t) \in \Omega \times]0,T[$
(*ii*) $p(x,t) = 1$, $\forall (x,t) \in \hat{\omega} \times]\varepsilon, T-\varepsilon[$
(*iii*) $p(x,t) = 0$, $\forall (x,t) \in (\Omega \backslash \omega) \times]0,T[$
(*iv*) $0 < p(x,t) < 1$, $\forall (x,t) \in \{\omega \backslash \hat{\omega} \times]0,T[\} \cup \{\hat{\omega} \times]o,\varepsilon] \cup [T-\varepsilon, T[)\}$
(*v*) $p(x,0) = p(x,T) = 0$, $\forall x \in \Omega$
(*vi*) $\frac{|\nabla p|^2}{p} \in L^\infty(\omega \times]0,T[)$.

Lemma 3.6 *Consider $T > T_0$ and $\varepsilon > 0$ such that $T - 2\varepsilon > T_0$. Let ω and $\hat{\omega}$ be the neighbourhoods of $\overline{\Gamma_0}$ mentioned above and let θ be the weak solution of (2.3). Then, we have*

$$\int_\varepsilon^{T-\varepsilon} \int_{\hat{\omega}} \{\sum_{i=1}^n |\nabla \theta_i(x,t)|^2 + |div\theta|^2\} dxdt \leq c_8 \sum_{i=1}^n \int_0^T \int_\omega (|\theta_i(x,t)|^2 + |\theta_i^{'}(x,t)|^2) dxdt.$$

Proof. Suppose that θ is the strong solution of (2.3). If we multiply the equation

$$\theta^{''} - \mu \Delta \theta - (\lambda + \mu) \nabla (\alpha(x) div\theta) = 0$$

by $p\theta$ we obtain, by using Green and Gauss' Theorems and integration by parts, that:

$$\mu \sum_{i=1}^n \int_0^T \int_\omega |\nabla \theta_i|^2 p dxdt = \sum_{i=1}^n \int_0^T \int_\omega p(\theta_i^{'})^2 dxdt + \sum_{i=1}^n \int_0^T \int_\omega p^{'} \theta_i \theta_i^{'} dxdt -$$

$$\mu \sum_{i=1}^n \int_0^T \int_\omega (\nabla \theta_i \cdot \nabla p) \theta_i dxdt - (\lambda+\mu) \int_0^T \int_\omega \alpha(x) div\theta (\nabla p \cdot \theta) dxdt -$$

$$(\lambda+\mu) \int_0^T \int_\omega \alpha(x) (div\theta)^2 p dxdt.$$

¿From the properties of the function p and provided that $0 < \alpha_0 \leq \alpha(x) \leq \alpha_1$ on $\overline{\Omega}$, we obtain the desired result by using a density argument in the above inequality. □

Lemma 3.6 combined with Remark 3.5 allow us to prove that:

$$||\theta^0||_V^2 + |\theta^1|_H^2 + |div\theta^0|_{L^2(\Omega)}^2 \leq c_9 \sum_{i=1}^n \int_0^T \int_\omega (|\theta_i(x,t)|^2 + |\theta_i^{'}(x,t)|^2) dxdt, \tag{3.14}$$

holds for every weak solution of (2.3).

¿From (3.14) we conclude that in order to prove inequality (3.2) for every weak solution of (2.3), we have only to prove that there exists $c_{10} > 0$ which verifies:

$$\sum_{i=1}^n \int_0^T \int_\omega |\theta_i(x,t)|^2 dxdt \leq c_{10} \sum_{i=1}^n \int_0^T \int_\omega |\theta_i^{'}(x,t)|^2 dxdt, \tag{3.15}$$

for every weak solution θ of (2.3). Indeed, if we suppose that (3.15) is not true, we obtain for each $k \in N$ a pair of initial data $\{\theta_k^0, \theta_k^1\} \in V \times H$ such that if θ_k is the weak solution of (2.3) associated to these data, we have that:

$$|\psi_k|_{L^2(0,T;(L^2(\omega))^n)} = 1\,, \quad k \in N \tag{3.16}$$

and

$$|\psi_k^{'}|_{L^2(0,T;(L^2(\omega))^n)} \leq \frac{1}{\sqrt{k}}\,, \quad k \in N \tag{3.17}$$

where

$$\psi_k = \frac{\theta_k}{|\theta_k|_{L^2(0,T;(L^2(\omega))^n)}}.$$

Taking into account that θ_k is the weak solution of (2.3) associated to $\{\theta_k^0,\, \theta_k^1\}$, we obtain, from the linearity of the system, that ψ_k is the weak solution of (2.3) associated to $\{\psi_k^0, \psi_k^1\}$ where

$$\psi_k^0 = \frac{\theta_k^0}{|\theta_k|_{L^2(0,T;(L^2(\omega))^n)}}$$

and

$$\psi_k^1 = \frac{\theta_k^1}{|\theta_k|_{L^2(0,T;(L^2(\omega))^n)}}.$$

Then, we obtain from (3.14), (3.16) and (3.17) that there exists $c_{11} > 0$ such that:

$$||\psi_k^0||_V^2 + |\psi_k^1|_H^2 + |div\psi_k^0|_{L^2(\Omega)}^2 \leq c_{11} \;\;, \quad \forall k \in \mathbf{N}. \tag{3.18}$$

So, we are able to extract subsequences from $\{\psi_k^0\}_{k\in N}$ and $\{\psi_k^1\}_{k\in N}$, which we are going to denote by the same notation, such that:

$$\psi_k^0 \rightharpoonup \psi^0 \quad \text{in } V \quad \text{and} \quad \psi_k^1 \rightharpoonup \psi^1 \quad \text{in } H. \tag{3.19}$$

In addition, considering that the energy is conservative and observing the expression given in (3.18) we have that

$$\{\psi_k\}_{k\in N} \;\text{ is bounded in }\; L^\infty(0,T;V)$$

and

$$\{\psi_k'\}_{k\in N} \;\text{ is bounded in }\; L^\infty(0,T;H). \tag{3.20}$$

Consequently, from Aubin-Lions' Lemma we can conclude that there exists a subsequence which satisfies

$$\psi_k \to \psi \quad in \;\; L^2(0,T;H).$$

We also obtain from (3.20) that there exists a subsequence such that

$$\psi_k' \rightharpoonup \psi' \quad in \ \ L^2(0,T;H)$$

which implies, in combination with (3.17), that

$$\psi'(x,t) = 0 \ \ a.e. \ \ in \ \ \omega\times]0,T[. \tag{3.21}$$

On the other hand, as ψ_k is the weak solution of (2.3) associated to $\{\psi_k^0, \psi_k^1\}$, we obtain that ψ is the weak solution of (2.3) associated to the inital data $\{\psi^0, \psi^1\}$. Consequently we have that $u = \psi'$ is the ultra-weak solution of

$$\left| \begin{array}{l} u'' - \mu\Delta u - (\lambda+\mu)\nabla(\alpha(x) div\, u) = 0 \quad \text{in} \ \ Q \\ u = 0 \quad \text{on} \ \ \Sigma \\ u(0) = u^0, \quad u'(0) = u^1 \quad \text{in} \ \ \Omega, \end{array} \right. \tag{3.22}$$

where $u^0 = \psi^1$ and $u^1 = \psi''(0)$.

Moreover, from (3.21) we obtain that:

$$u = 0 \quad \text{a.e. in} \quad \omega\times]0,T[\tag{3.23}$$

$$u' = 0 \quad \text{a.e. in} \quad \omega\times]0,T[. \tag{3.24}$$

Lemma 3.7 *Let u be the ultra-weak solution of (3.22) which satisfies (3.23) and (3.24). Then, we have that*

$$\{u^0, u^1\} = \{0,0\}.$$

Proof. Let us define the function $U : [-T,T] \to H$ given by:

$$U(t) = \left| \begin{array}{l} -3u(-t) + 4u(-\frac{t}{2}) \ \ \text{if} \ \ t\in[-T,0] \\ \\ u(t) \ \ \text{if} \ \ t\in[0,T], \end{array} \right.$$

and let us consider the mollifier sequence $\{\rho_\nu\}_{\nu\in N}$ such that:

$$\rho_\nu \in C_0^\infty(R), \ \ \rho_\nu \geq 0, \ \ supp(\rho_\nu) \subset]\frac{-1}{\nu}, 0[, \ \ \int_{\frac{-1}{\nu}}^0 \rho_\nu(t)dt = 1, \qquad \forall \nu \in \mathbf{N}.$$

Given $\varepsilon > 0$, we are going to define for large enough ν, the following sequence:

$$u_\nu(t) = (U * \rho_\nu)(t) = \int_{\frac{-1}{\nu}}^0 U(t-s)\rho_\nu(s)ds, \quad \forall t \in [0, T-\varepsilon].$$

As we have that $u \in C^0([0,T];\mathrm{H})$ it comes that

$$u_\nu\,, u_\nu^{'}\,, u_\nu^{''}\,, ... \in C^0([0,T-\varepsilon]; H), \quad \forall \nu \in \mathbf{N} \tag{3.25}$$

and u_ν satisfies

$$u_\nu^{''} - \mu\Delta u_\nu - (\lambda+\mu)\nabla(\alpha(x) div u_\nu) = 0 \quad in \;\; C^0([0,T-\varepsilon]; (H^{-2}(\Omega))^n).$$

Then for each $t \in [0, T-\varepsilon]$ we obtain that:

$$-\mu\Delta u_\nu(t) - (\lambda+\mu)\nabla(\alpha(x) div u_\nu(t)) = -u_\nu''(t) \in H.$$

So, observing the bilinear form given in (3.3), we obtain by Lax-Milgram's Lemma and by (3.25) that:

$$u_\nu(t) \in V, \;\; \forall t \in [0, T-\varepsilon] \tag{3.26}$$

and

$$u_\nu \in C^0([0,T-\varepsilon]; V), \;\; \forall \nu \in \mathbf{N}.$$

¿From (3.25) and (3.26) we obtain that:

$$u_\nu(0) \in V \quad and \quad u_\nu^{'}(0) \in H$$

and consequently u_ν is the weak solution of

$$\left| \begin{array}{l} u_\nu'' - \mu\Delta u_\nu - (\lambda+\mu)\nabla(\alpha(x) div u_\nu) = 0 \;\;\text{ in }\; C^0([0,T-\varepsilon]; H) \\ u_\nu = 0 \;\;\text{ on }\; \Gamma\times]0,T-\varepsilon[\\ u_\nu(0) = u_\nu^0, \;\; u_\nu'(0) = u_\nu^1 \;\;\text{ in }\; \Omega. \end{array} \right.$$

So, u_ν satisfies the inequality given in (3.14) for $]0, T-\varepsilon[\times\omega$, that is

$$||u_\nu^0||_V^2 + |u_\nu^1|_H^2 + |div u_\nu^0|^2_{L^2(\Omega)} \leq c_{12} \int_0^{T-\varepsilon} \int_\omega (\sum_{i=1}^n |u_{\nu_i}^{'}|^2 + \sum_{i=1}^n |u_{\nu_i}|^2) dx dt;$$

which in combination with (3.23) and (3.24) implies that:

$$u_\nu^0 = 0 \quad and \quad u_\nu^1 = 0 \;\; \forall\, \nu \in \mathbf{N}. \tag{3.27}$$

On the other hand, it comes by the definition of u_ν that

$$u_\nu^0 \to u^0 \;\; in \;\; H \quad and \quad u_\nu^1 \to u^1 \;\; in \;\; V'$$

and by (3.27) we have that:

$$u^0 = 0 \quad \text{and} \quad u^1 = 0$$

which proves Lemma 3.7. □

¿From the above lemma and from the fact that u is the ultra-weak solution of (3.22), we conclude that

$$u = 0 \quad in \quad C^0([0,T];H).$$

In other words, we have that

$$\psi'(x,t) = 0 \quad \forall t \in [0,T] \quad and \quad a.e. \ in \quad \Omega$$

or

$$\psi(x,t) = \psi(x) \quad \forall t \in [0,T] \quad and \quad a.e. \ in \quad \Omega.$$

Then, we are able to conclude that ψ^0 =0 and $\psi^1 = 0$, which combined with the fact that ψ is the solution of (2.3), implies that $\psi = 0$. This is a contradition with (3.16). So, we have proved inequality (3.15) and consequently inequality (3.2), which ends the proof of Theorem 3.1.

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