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COMPARING THE RADII OF SOME BALLS APPEARING IN CONNECTION TO THREE LOCAL CONVERGENCE THEOREMS FOR NEWTON'S METHOD

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Abstract. We provide three local convergence theorems for Newton's method in Banach space. These theorems use information on the first, second, and first-second Fréchet derivatives of the operator involved, respectively. In all cases, we show that for any starting point inside a ball centered at the solution of the nonlinear equation, Newton's method converges to that solution. Our goal is to obtain the smallest and the largest convergence region for method (2). Our results apply to solve autonomous differential equations. A numerical example is also provided.

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1. INTRODUCTION.

In this study we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 3D0, \tag{1}$$

where F is a twice Fréchet-differentiable operator defined on a convex subset D of a Banach space E_1 with values in a Banach space E_2.

The most popular method for generating a sequence $\{x_n\}$ $(n \geq 0)$ converging to x^* is undoubtedly Newton's method

$$x_{n+1} = 3Dx_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0)\ (x_0 \in D). \tag{2}$$

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Here $F'(x) \in L(E_1, E_2)$ $(x \in D)$ the space of bounded linear operators from E_1 into E_2. Sufficient convergence conditions for the convergence of Newton's method (2) to x^* have been given by many authors [1], [3], [5].

In this study, we assume the existence of a solution $x^* \in D$ of equation (1), and using Lipschitz conditions on the first and second Fréchet-derivative we provide three local convergence theorems for method (2). Each theorem guarantees the convergence of Newton's method (2) to x^* as long as the starting point x_0 is inside a ball centered at x^* and of a certain radius. Our goal is to obtain the smallest and the largest convergence region for method (2). Our results have theoretical value, but also practical use, since they can be applied to solve autonomous differential equations [1], [2], [4].

Finally, we complete this study with a numerical example.

2. CONVERGENCE ANALYSIS.

Let $x^* \in D$ be a simple solution of equation $F(x) = 3D0$ in the sense that $F'(x^*)$ has a bounded inverse. We denote by $U(x^*, r)$ the open ball with center x^* and radius $r > 0$. Let also α, β, γ be constants such that

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq \alpha \|x - y\|, \tag{3}$$

$$\|F'(x^*)^{-1}F''(x^*)\| \leq \beta, \tag{4}$$

and

$$\|F'(x^*)^{-1}(F''(x) - F''(y))\| \leq \gamma \|x - y\| \tag{5}$$

for all $x, y \in D$. Define parameters r_1, r_2, r_3 by

$$r_1 = 3D\frac{2}{3\alpha}, \tag{6}$$

$$r_2 = 3D\frac{12}{9\beta + \sqrt{81\beta^2 + 96\gamma}}, \tag{7}$$

$$r_3 = 3D\frac{12}{3(\beta + 2\alpha) + \sqrt{9(\beta + 2\alpha)^2 + 24\gamma}}, \tag{8}$$

and functions f_1, f_2, f_3 on $[0, +\infty)$ by

$$f_1(r) = 3D3ar - 2, \tag{9}$$

$$f_2(r) = 3D4\gamma r^2 + 9\beta - 6 \tag{10}$$

$$f_3(r) = 3D\gamma r^2 + 3(\beta + 2\alpha)r - 6. \tag{11}$$

Using (2) we can have the approximations:

$$\begin{aligned} x_{n+1} - x^* = 3D[F'(x_n)^{-1}F'(x^*)]\bigg\{\int_0^1 F'(x^*)^{-1}[F'(x^* + t(x_n - x^*)) \\ - F'(x_n)]\, dt(x_n - x^*)\bigg\}, \end{aligned} \tag{12}$$

(13)

$$x_{n+1} - x^* = 3D[F'(x_n)^{-1}F'(x^*)]\Big\{\int_0^1 [F'(x^*)^{-1}(F''(x_n + t(x^* - x_n)) - F''(x_0))(1-t)\,dt(x^* - x_n)^2 + \frac{1}{2}F'(x^*)^{-1}F''(x_0)(x^* - x_n)^2\Big\}.$$

Consider also the identities

$$F'(x^*)^{-1}F'(x) = 3DI + F'(x^*)^{-1}(F'(x) - F'(x^*)) \tag{14}$$

and

$$\int_{x^*}^{x} F'(x^*)^{-1}[F''(y) - F''(x^*)]\,dy = 3D\ F'(x^*)^{-1}F'(x) - I - F'(x^*)^{-1}F''(x^*)(x - x^*). \tag{15}$$

Then we can obtain the following local convergence theorems for Newton's method (2).

Theorem 1. *Let $F\colon D \subseteq E_1 \to E_2$ be Fréchet differentiable, and x^* be a simple solution of equation $F(x) = 3D0$. Assume Lipschitz condition* (3) *holds, and $U(x^*, r_1) \subseteq D$, where r_1 is given by* (6). *Then, for any $x_0 \in U(x^*, r_1)$, Newton's method* (2) *converges to x^*. Moreover the following error bounds hold for all $n \geq 0$*

$$\|x_{n+1} - x^*\| \leq \frac{\alpha}{2(1 - \alpha\|x_n - x^*\|)}\|x_n - x^*\|^2. \tag{16}$$

Proof. The proof was essentially given in [6] and uses (3), (12), (14). However it was not given in affine invariant form as it is here. The advantages of such a form have been explained in some detail in [1], [3].

Theorem 2. *Let $F\colon D \subseteq E_1 \to E_2$ be twice Fréchet-differentiable, and x^* be a simple solution of equation $F(x) = 3D0$. Assume conditions* (4), (5) *hold, and $U(x_0, r_2) \subseteq D$, where r_2 is given by* (7). *Then, for any $x_0 \in U(x_0, r_2)$, Newton's method* (2) *converges to x^*. Moreover the following error bounds hold for all $n \geq 0$*

$$\|x_{n+1} - x^*\| \leq \frac{\frac{\gamma}{3}\|x_n - x^*\| + \beta}{2\big[1 - \big(\frac{1}{2}\gamma\|x_n - x^*\|^2 + \beta\|x_n - x^*\|\big)\big]}\|x_n - x^*\|^2. \tag{17}$$

Proof. For $x_0 \in U(x_0, r_2)$ using induction on $n \geq 0$, (14), and the Banach lemma on invertible operators [2], [5], we obtain $F'(x_n)^{-1}$ exists for all $n \geq 0$,

$$\|F'(x_n)^{-1}F'(x^*)\| \leq \left[1 - \left(\frac{1}{2}\gamma\|x_n - x^*\| + \beta\right)\|x_n - x^*\|\right]^{-1} \tag{18}$$

and

$$\left\|\int_0^1 F'(x^*)^{-1}[F''(x_n + t(x^* - x_n))(1-t)\,dt(x^* - x_n)^2\right\| \leq \frac{1}{2}\left(\frac{\gamma}{3}\|x_n - x^*\| + \beta\right)\|x_n - x^*\|^2. \tag{19}$$

By (13), (18) and (19) we obtain (17). The choice of r_2 guarantees that there exists $c \in [0,1]$ such that

$$\|x_{n+1} - x^*\| \leq c\|x_n - x^*\| \leq c^{n+1} r_2 \to 0 \quad \text{as } n \to \infty.$$

Hence, we get $\lim_{n\to\infty} x_{n+1} = 3Dx^*$.

That completes the proof of the theorem. ♠

Theorem 3. *Let $F\colon D \subseteq E_1 \to E_2$ be twice Fréchet-differentiable, and x^* be a simple solution of equation $F(x) = 3D0$. Assume conditions* (3)–(5) *hold, and $U(x_0, r_3) \subseteq D$, where r_3 is given by* (18)*. Then for any $x_0 \in U(x_0, r_3)$, Newton's method* (2) *converges to x^*. Moreover the following error bounds hold for all $n \geq 0$:*

$$\|x_{n+1} - x^*\| \leq \frac{\frac{\gamma}{3}\|x_n - x^*\| + \beta}{2(1 - a\|x_n - x^*\|)}\|x_n - x^*\|^2. \tag{20}$$

Proof. As in the proof of Theorem 2 but we use (14) instead of (15).

That completes the proof of Theorem 3. ♠

Remark 1. It follows from (3)–(8) that when $\gamma = 3D0$, then $\beta = 3D\alpha$ and $r_1 = 3Dr_2 = 3Dr_3$. Many other possibilities exist. For example $f_2(r_1) < 0$, $f_2(r_3) < 0$ and $f_3(r_1) < 0$ imply $r_1 < r_3 < r_2$ (see the example that follows).

Remark 2. The results obtained here have theoretical and practical value. Indeed, in practice they can be used to solve autonomous differential equations of the form

$$F'(x) = 3DT(F(x)) \quad (x \in D), \tag{21}$$

where T is a known operator [2], [4].

We note that (21) gives $F'(x^*) = 3DT(F(x^*)) = 3DT(0)$, which is known. So we can use Theorems 1–3, even when the solution x^* is not known.

We conclude this study with an example involving Theorems 1–3 and Remarks 1 and 2.

Example. Let $E_1 = 3DE_2 = 3D\mathbf{R}$, $D = 3D[-1,1]$. Define the function F by

$$F(x) = 3De^x - 1, \quad (x \in D). \tag{22}$$

Choose $T(x) = 3Dx + 1$ $(x \in D)$. Then F so defined satisfies (21). Using (3)–(8), (22), we get $F'(x^*) = 3DT(0) = 3D1$, $\alpha = 3D\gamma = 3De$, $\beta = 3D1$,

$$r_1 = 3D.245253, \quad r_2 = 3D.4364902 \quad \text{and} \quad r_3 = 3D.2982063.$$

Therefore the conclusions of Theorems 1–3 hold. In particular, for $x_0 \in U(x_0, s)$ with $s = 3Dr_1$ or r_2 or r_3, Newton's method converges to the solution $x^* = 3D0$ of equation $F(x) = 3D0$, where F is given by (22).

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