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A COVARIANT STINESPRING TYPE THEOREM FOR τ-MAPS

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Abstract. Let τ be a linear map from a unital C^*-algebra $\mathcal{A}$ to a von Neumann algebra $\mathcal{B}$ and let $\mathcal{C}$ be a unital C^*-algebra. A map T from a Hilbert $\mathcal{A}$-module E to a von Neumann $\mathcal{C}$-$\mathcal{B}$ module F is called a τ-map if

$$\langle T(x), T(y)\rangle = \tau(\langle x, y\rangle) \text{ for all } x, y \in E.$$

A Stinespring type theorem for τ-maps and its covariant version are obtained when τ is completely positive. We show that there is a bijective correspondence between the set of all τ-maps from E to F which are (u', u)-covariant with respect to a dynamical system (G, η, E) and the set of all (u', u)-covariant $\widetilde{\tau}$-maps from the crossed product $E \times_\eta G$ to F, where τ and $\widetilde{\tau}$ are completely positive.

1 Introduction

A linear mapping τ from a (pre-)C^*-algebra $\mathcal{A}$ to a (pre-)C^*-algebra $\mathcal{B}$ is called *completely positive* if

$$\sum_{i,j=1}^{n} b_j^* \tau(a_j^* a_i) b_i \geq 0$$

for each $n \in \mathbb{N}$, $b_1, b_2, \ldots, b_n \in \mathcal{B}$ and $a_1, a_2, \ldots, a_n \in \mathcal{A}$. The completely positive maps are used significantly in the theory of measurements, quantum mechanics, operator algebras etc. Paschke's Gelfand-Naimark-Segal (GNS) construction (cf. Theorem 5.2, [13]) characterizes completely positive maps between unital C^*-algebras, which is an abstraction of the Stinespring's theorem for operator valued completely positive maps (cf. Theorem 1, [22]). Now we define Hilbert C^*-modules which are a generalization of Hilbert spaces and C^*-algebras, were introduced by Paschke in the paper mentioned above and were also studied independently by Rieffel in [15].

Definition 1. *Let $\mathcal{B}$ be a (pre-)C^*-algebra and E be a vector space which is a right $\mathcal{B}$-module satisfying $\alpha(xb) = (\alpha x)b = x(\alpha b)$ for $x \in E, b \in \mathcal{B}, \alpha \in \mathbb{C}$. The space*

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E is called an inner-product $\mathcal{B}$-module *or a* pre-Hilbert $\mathcal{B}$-module *if there exists a mapping* $\langle\cdot,\cdot\rangle : E \times E \to \mathcal{B}$ *such that*

(i) $\langle x, x\rangle \geq 0$ *for* $x \in E$ *and* $\langle x, x\rangle = 0$ *only if* $x = 0$,

(ii) $\langle x, yb\rangle = \langle x, y\rangle b$ *for* $x, y \in E$ *and for* $b \in \mathcal{B}$,

(iii) $\langle x, y\rangle = \langle y, x\rangle^*$ *for* $x, y \in E$,

(iv) $\langle x, \mu y + \nu z\rangle = \mu\langle x, y\rangle + \nu\langle x, z\rangle$ *for* $x, y, z \in E$ *and for* $\mu, \nu \in \mathbb{C}$.

An inner-product $\mathcal{B}$-module E which is complete with respect to the norm

$$\|x\| := \|\langle x, x\rangle\|^{1/2} \text{ for } x \in E$$

is called a Hilbert $\mathcal{B}$-module *or* Hilbert C^*-module over $\mathcal{B}$. *It is said to be* full *if the closure of the linear span of* $\{\langle x, y\rangle : x, y \in E\}$ *equals* $\mathcal{B}$.

Hilbert C^*-modules are important objects to study the classification theory of C^*-algebras, the dilation theory of semigroups of completely positive maps, and so on. If a completely positive map takes values in any von Neumann algebra, then it gives us a von Neumann module by Paschke's GNS construction (cf. [19]). The von Neumann modules were recently utilized in [3] to explore Bures distance between two completely positive maps. Using the following definition of adjointable maps we define von Neumann modules: Let E and F be (pre-)Hilbert $\mathcal{A}$-modules, where $\mathcal{A}$ is a (pre-)C^*-algebra. A map $S : E \to F$ is called *adjointable* if there exists a map $S' : F \to E$ such that

$$\langle S(x), y\rangle = \langle x, S'(y)\rangle \text{ for all } x \in E, y \in F.$$

S' is unique for each S, henceforth we denote it by S^*. We denote the set of all adjointable maps from E to F by $\mathcal{B}^a(E, F)$ and we use $\mathcal{B}^a(E)$ for $\mathcal{B}^a(E, E)$. Symbols $\mathcal{B}(E, F)$ and $\mathcal{B}^r(E, F)$ represent the set of all bounded linear maps from E to F and the set of all bounded right linear maps from E to F, respectively.

Definition 2. *(cf. [18]) Let $\mathcal{B}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, i.e., strongly closed C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ containing the identity operator. Let E be a (pre-)Hilbert $\mathcal{B}$-module. The Hilbert space $E \bigodot \mathcal{H}$ is the interior tensor product of E and $\mathcal{H}$. For each $x \in E$ we get a bounded linear map from $\mathcal{H}$ to $E \bigodot \mathcal{H}$ defined as*

$$L_x(h) := x \odot h \text{ for all } h \in \mathcal{H}.$$

Note that $L_{x_1}^* L_{x_2} = \langle x_1, x_2\rangle$ for $x_1, x_2 \in E$. So we identify each $x \in E$ with L_x and consider E as a concrete submodule of $\mathcal{B}(\mathcal{H}, E \bigodot \mathcal{H})$. The module E is called a von Neumann $\mathcal{B}$-module *or a* von Neumann module over $\mathcal{B}$ *if E is strongly closed in $\mathcal{B}(\mathcal{H}, E \bigodot \mathcal{H})$. Let $\mathcal{A}$ be a unital (pre-)C^*-algebra. A von Neumann $\mathcal{B}$-module E is called a* von Neumann $\mathcal{A}$-$\mathcal{B}$ module *if there exists an adjointable left action of $\mathcal{A}$ on E.*

An alternate approach to the theory of von Neumann modules is introduced recently in [1] and an analogue of the Stinespring's theorem for von Neumann bimodules is discussed. The comparison of results coming from these two approach is provided by [20].

Let G be a locally compact group and let $M(\mathcal{A})$ denote the multiplier algebra of any C^*-algebra $\mathcal{A}$. An *action of G on $\mathcal{A}$* is defined as a group homomorphism $\alpha : G \to Aut(\mathcal{A})$. If $t \mapsto \alpha_t(a)$ is continuous for all $a \in \mathcal{A}$, then we call $(G, \alpha, \mathcal{A})$ a *C^*-dynamical system.*

Definition 3. *(cf. [11]) Let $\mathcal{A}$, $\mathcal{B}$ be unital (pre-)C^*-algebras and G be a locally compact group. Let $(G, \alpha, \mathcal{A})$ be a C^*-dynamical system and $u : G \to \mathcal{UB}$ be a unitary representation where $\mathcal{UB}$ is the group of all unitary elements of $\mathcal{B}$. A completely positive map $\tau : \mathcal{A} \to \mathcal{B}$ is called* u-covariant *with respect to $(G, \alpha, \mathcal{A})$ if*

$$\tau(\alpha_t(a)) = u_t \tau(a) u_t^* \text{ for all } a \in \mathcal{A} \text{ and } t \in G.$$

The existence of covariant completely positive liftings (cf. [4]) and a covariant version of the Stinespring's theorem for operator-valued u-covariant completely positive maps were obtained by Paulsen in [14], and they were used to provide three groups out of equivalence classes of covariant extensions. Later Kaplan (cf. [11]) extended this covariant version and as an application analyzed the completely positive lifting problem for homomorphisms of the reduced group C^*-algebras.

A map T from a (pre-)Hilbert $\mathcal{A}$-module E to a (pre-)Hilbert $\mathcal{B}$-module F is called *τ-map* (cf. [21]) if

$$\langle T(x), T(y) \rangle = \tau(\langle x, y \rangle) \text{ for all } x, y \in E.$$

Recently a Stinespring type theorem for τ-maps was obtained by Bhat, Ramesh and Sumesh (cf. [2]) for any operator valued completely positive map τ defined on a unital C^*-algebra. There are two covariant versions of this Stinespring type theorem see Theorem 3.4 of [9] and Theorem 3.2 of [8]. In Section 2, we give a Stinespring type theorem for τ-maps, when $\mathcal{B}$ is any von Neumann algebra and F is any von Neumann $\mathcal{B}$-module.

In [5] the notion of $\mathfrak{K}$-families is introduced, which is a generalization of the τ-maps, and several results are derived for covariant $\mathfrak{K}$-families. In [21] different characterizations of the τ-maps were obtained and as an application the dilation theory of semigroups of the completely positive maps was discussed. Extending some of these results for $\mathfrak{K}$-families, application to the dilation theory of semigroups of completely positive definite kernels is explored in [5].

In this article we get a covariant version of our Stinespring type theorem which requires the following notions: Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, E be a Hilbert $\mathcal{A}$-module, and let F be a Hilbert $\mathcal{B}$-module. A map $\Psi : E \to F$ is said to be a *morphism of Hilbert C^*-modules* if there exists a C^*-algebra homomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that

$$\langle \Psi(x), \Psi(y) \rangle = \psi(\langle x, y \rangle) \text{ for all } x, y \in E.$$

If E is full, then ψ is unique for Ψ. A bijective map $\Psi : E \to F$ is called an *isomorphism of Hilbert C^*-modules* if Ψ and Ψ^{-1} are morphisms of Hilbert C^*-modules. We denote the group of all isomorphisms of Hilbert C^*-modules from E to itself by $Aut(E)$.

Definition 4. *Let G be a locally compact group and let $\mathcal{A}$ be a C^*-algebra. Let E be a full Hilbert $\mathcal{A}$-module. A group homomorphism $t \mapsto \eta_t$ from G to $Aut(E)$ is called a* continuous action of G on E *if $t \mapsto \eta_t(x)$ from G to E is continuous for each $x \in E$. In this case we call the triple (G, η, E) a* dynamical system on the Hilbert $\mathcal{A}$-module E. *Any C^*-dynamical system $(G, \alpha, \mathcal{A})$ can be regarded as a dynamical system on the Hilbert $\mathcal{A}$-module $\mathcal{A}$.*

Let E be a full Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. Let F be a von Neumann $\mathcal{C}$-$\mathcal{B}$ module, where $\mathcal{C}$ is a unital C^*-algebra and $\mathcal{B}$ is a von Neumann algebra. We define covariant τ-maps with respect to (G, η, E) in Section 2, and develop a covariant version of our Stinespring type theorem. If (G, η, E) is a dynamical system on E, then there exists a crossed product Hilbert C^*-module $E \times_\eta G$ (cf. [6]). In Section 3, we prove that any τ-map from E to F which is (u', u)-covariant with respect to the dynamical system (G, η, E) extends to a (u', u)-covariant $\widetilde{\tau}$-map from $E \times_\eta G$ to F, where τ and $\widetilde{\tau}$ are completely positive. As an application we describe how covariant τ-maps on (G, η, E) and covariant $\widetilde{\tau}$-maps on $E \times_\eta G$ are related, where τ and $\widetilde{\tau}$ are completely positive maps. The approach in this article is similar to [2] and [9].

2 A Stinespring type theorem and its covariant version

Definition 5. *Let $\mathcal{A}$ and $\mathcal{B}$ be (pre-)C^*-algebras. Let E be a Hilbert $\mathcal{A}$-module and let F, F' be inner product $\mathcal{B}$-modules. A map $\Psi : E \to \mathfrak{B}^r(F, F')$ is called* quasi-representation *if there exists a $*$-homomorphism $\pi : \mathcal{A} \to \mathfrak{B}^a(F)$ satisfying*

$$\langle \Psi(y) f_1, \Psi(x) f_2 \rangle = \langle \pi(\langle x, y \rangle) f_1, f_2 \rangle \textit{ for all } x, y \in E \textit{ and } f_1, f_2 \in F.$$

In this case we say that Ψ is a quasi-representation of E on F and F', and π is associated to Ψ.

It is clear that Definition 5 generalizes the notion of representations of Hilbert C^*-modules on Hilbert spaces (cf. p.804 of [9]). The following theorem provides a decomposition of τ-maps in terms of quasi-representations. We use the symbol sot-lim for the limit with respect to the strong operator topology. Notation $[S]$ will be used for the norm closure of the linear span of any set S.

Theorem 6. *Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{B}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Let E be a Hilbert $\mathcal{A}$-module, E' be a von Neumann $\mathcal{B}$-module and let $\tau : \mathcal{A} \to \mathcal{B}$ be a completely positive map. If $T : E \to E'$ is a τ-map, then there exist*

(i) (a) a von Neumann $\mathcal{B}$-module F and a representation π of $\mathcal{A}$ to $\mathcal{B}^a(F)$,
(b) a map $V \in \mathcal{B}^a(\mathcal{B}, F)$ such that $\tau(a)b = V^*\pi(a)Vb$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$,

(ii) (a) a von Neumann $\mathcal{B}$-module F' and a quasi-representation $\Psi : E \to \mathcal{B}^a(F, F')$ such that π is associated to Ψ,
(b) a coisometry S from E' onto F' satisfying

$$T(x)b = S^*\Psi(x)Vb \text{ for all } x \in E \text{ and } b \in \mathcal{B}.$$

Proof. Let $\langle\ ,\ \rangle$ be a $\mathcal{B}$-valued positive definite semi-inner product on $\mathcal{A}\bigotimes_{alg}\mathcal{B}$ defined by

$$\langle a \otimes b, c \otimes d\rangle := b^*\tau(a^*c)d \text{ for } a, c \in \mathcal{A} \text{ and } b, d \in \mathcal{B}.$$

Using Cauchy-Schwarz inequality we deduce that $K = \{x \in \mathcal{A}\bigotimes_{alg}\mathcal{B} : \langle x, x\rangle = 0\}$ is a submodule of $\mathcal{A}\bigotimes_{alg}\mathcal{B}$. Therefore $\langle\ ,\ \rangle$ extends naturally on the quotient module $\left(\mathcal{A}\bigotimes_{alg}\mathcal{B}\right)/K$ as a $\mathcal{B}$-valued inner product. We get a Stinespring triple (π_0, V, F_0) associated to τ, construction is similar to Proposition 1 of [11], where F_0 is the completion of the inner-product $\mathcal{B}$-module $\left(\mathcal{A}\bigotimes_{alg}\mathcal{B}\right)/K$, $\pi_0 : \mathcal{A} \to \mathcal{B}^a(F_0)$ is a $*$-homomorphism defined by

$$\pi_0(a')(a \otimes b + K) := a'a \otimes b + K \text{ for all } a, a' \in \mathcal{A} \text{ and } b \in \mathcal{B},$$

and a mapping $V \in \mathcal{B}^a(\mathcal{B}, F_0)$ is defined by

$$V(b) = 1 \otimes b + K \text{ for all } b \in \mathcal{B}.$$

Indeed, $[\pi_0(\mathcal{A})V\mathcal{B}] = F_0$. Let F be the strong operator topology closure of F_0 in $\mathcal{B}(\mathcal{H}, F_0 \bigodot \mathcal{H})$. Without loss of generality we can consider $V \in \mathcal{B}^a(\mathcal{B}, F)$. Adjointable left action of $\mathcal{A}$ on F_0 extends to an adjointable left action of $\mathcal{A}$ on F as follows:

$$\pi(a)(f) := \text{sot-}\lim_{\alpha} \pi_0(f^0_\alpha) \text{ where } a \in \mathcal{A},\ f\text{=sot-}\lim_{\alpha} f^0_\alpha \in F \text{ with } f^0_\alpha \in F_0.$$

For all $a \in \mathcal{A}$; f=sot-$\lim\limits_{\alpha} f^0_\alpha$, g=sot-$\lim\limits_{\beta} g^0_\beta \in F$ with $f^0_\alpha, g^0_\beta \in F_0$ we have

$$\begin{aligned}\langle \pi(a)f, g\rangle &= \text{sot-}\lim_{\beta}\langle \pi(a)f, g^0_\beta\rangle = \text{sot-}\lim_{\beta}(\text{sot-}\lim_{\alpha}\langle g^0_\beta, \pi_0(a)f^0_\alpha\rangle)^* \\ &= \text{sot-}\lim_{\beta}(\text{sot-}\lim_{\alpha}\langle \pi_0(a)^*g^0_\beta, f^0_\alpha\rangle)^* = \langle f, \pi(a^*)g\rangle.\end{aligned}$$

The triple (π, V, F) satisfies all the conditions of the statement (i).

Let F'' be the Hilbert $\mathcal{B}$-module $[T(E)\mathcal{B}]$. For $x \in E$, define $\Psi_0(x) : F_0 \to F''$ by

$$\Psi_0(x)(\sum_{j=1}^{n} \pi_0(a_j)Vb_j) := \sum_{j=1}^{n} T(xa_j)b_j \ \text{ for all } a_j \in \mathcal{A}, b_j \in \mathcal{B}.$$

It follows that

$$\begin{aligned}
&\langle \Psi_0(y)(\sum_{j=1}^{n} \pi_0(a_j)Vb_j), \Psi_0(x)(\sum_{i=1}^{m} \pi_0(a_i')Vb_i')\rangle = \sum_{j=1}^{n}\sum_{i=1}^{m} b_j^*\langle T(ya_j), T(xa_i')\rangle b_i' \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} b_j^*\tau(\langle ya_j, xa_i'\rangle)b_i' = \sum_{j=1}^{n}\sum_{i=1}^{m}\langle \pi_0(a_i')^*\pi_0(\langle x, y\rangle)\pi_0(a_j)Vb_j, Vb_i'\rangle \\
&= \langle \pi_0(\langle x, y\rangle)(\sum_{j=1}^{n} \pi_0(a_j)Vb_j), \sum_{i=1}^{m} \pi_0(a_i')Vb_i'\rangle
\end{aligned}$$

for all $x, y \in E, a_i', a_j \in \mathcal{A}, b_i', b_j \in \mathcal{B}$ where $1 \le j \le n$ and $1 \le i \le m$. This computation proves that $\Psi_0(x) \in \mathcal{B}^r(F_0, F'')$ for each $x \in E$ and also that $\Psi_0 : E \to \mathcal{B}^r(F_0, F'')$ is a quasi-representation. We denote by F' the strong operator topology closure of F'' in $\mathcal{B}(\mathcal{H}, E' \bigodot \mathcal{H})$. Let $x \in E$, and let $\Psi(x) : F \to F'$ be a mapping defined by

$$\Psi(x)(f) := \text{sot-}\lim_{\alpha} \Psi_0(x)f_\alpha^0 \text{ where } f\text{=sot-}\lim_{\alpha} f_\alpha^0 \in F \text{ for } f_\alpha^0 \in F_0.$$

For all f=sot-$\lim\limits_{\alpha} f_\alpha^0 \in F$ with $f_\alpha^0 \in F_0$ and for all $x, y \in E$ we have

$$\langle \Psi(x)f, \Psi(y)f\rangle = \text{sot-}\lim_{\alpha}\{\text{sot-}\lim_{\beta}\langle \Psi_0(y)f_\alpha^0, \Psi_0(x)f_\beta^0\rangle\}^* = \langle f, \pi(\langle x, y\rangle)f\rangle.$$

Since F is a von Neumann $\mathcal{B}$-module, this proves that $\Psi : E \to \mathcal{B}^a(F, F')$ is a quasi-representation. Since, F' is a von Neumann $\mathcal{B}$-submodule of E', there exists an orthogonal projection from E' onto F' (cf. Theorem 5.2 of [18]) which we denote by S. Eventually

$$S^*\Psi(x)Vb = \Psi(x)Vb = \Psi(x)(\pi(1)Vb) = T(x)b \text{ for all } x \in E, b \in \mathcal{B}.$$

$\square$

Let E be a (pre-)Hilbert $\mathcal{A}$-module, where $\mathcal{A}$ is a (pre-)C^*-algebra $\mathcal{A}$. A map $u \in \mathcal{B}^a(E)$ is said to be *unitary* if $u^*u = uu^* = 1_E$ where 1_E is the identity operator on E. We denote the set of all unitaries in $\mathcal{B}^a(E)$ by $\mathcal{U}\mathcal{B}^a(E)$.

Definition 7. *Let $\mathcal{B}$ be a (pre-)C^*-algebra, $(G, \alpha, \mathcal{A})$ be a C^*-dynamical system of a locally compact group G, and let F be a (pre-)Hilbert $\mathcal{B}$-module. A representation $\pi : \mathcal{A} \to \mathbb{B}^a(F)$ is called* v-covariant *with respect to $(G, \alpha, \mathcal{A})$ and with respect to a unitary representation $v : G \to \mathcal{U}\mathbb{B}^a(F)$ if*

$$\pi(\alpha_t(a)) = v_t \pi(a) v_t^* \text{ for all } a \in \mathcal{A}, t \in G.$$

In this case we write (π, v) is a covariant representation of $(G, \alpha, \mathcal{A})$.

Let E be a full Hilbert $\mathcal{A}$-module and let G be a locally compact group. If (G, η, E) is a dynamical system on E, then there exists a unique C^*-dynamical system $(G, \alpha^\eta, \mathcal{A})$ (cf. p.806 of [9]) such that

$$\alpha_t^\eta(\langle x, y\rangle) = \langle \eta_t(x), \eta_t(y)\rangle \text{ for all } x, y \in E \text{ and } t \in G.$$

We denote by $(G, \alpha^\eta, \mathcal{A})$ the C^*-dynamical system coming from the dynamical system (G, η, E). For all $x \in E$ and $a \in \mathcal{A}$ we infer that $\eta_t(xa) = \eta_t(x)\alpha_t^\eta(a)$, for

$$\begin{aligned} \|\eta_t(xa) - \eta_t(x)\alpha_t^\eta(a)\|^2 =& \|\langle \eta_t(xa), \eta_t(xa)\rangle - \langle \eta_t(xa), \eta_t(x)\alpha_t^\eta(a)\rangle \\ &- \langle \eta_t(x)\alpha_t^\eta(a), \eta_t(xa)\rangle + \langle \eta_t(x)\alpha_t^\eta(a), \eta_t(x)\alpha_t^\eta(a)\rangle\| \\ =& \|\alpha_t^\eta(\langle xa, xa\rangle) - \langle \eta_t(xa), \eta_t(x)\rangle \alpha_t^\eta(a) \\ &- \alpha_t^\eta(a^*)\langle \eta_t(x), \eta_t(xa)\rangle + \alpha_t^\eta(a^*)\langle \eta_t(x), \eta_t(x)\rangle \alpha_t^\eta(a)\| = \quad 0. \end{aligned}$$

Definition 8. *Let $\mathcal{B}$ and $\mathcal{C}$ be unital (pre-)C^*-algebras. A* (pre-)C^*-correspondence from $\mathcal{C}$ to $\mathcal{B}$ *is defined as a (pre-)Hilbert $\mathcal{B}$-module F together with a $*$-homomorphism $\pi' : \mathcal{C} \to \mathbb{B}^a(F)$. The adjointable left action of $\mathcal{C}$ on F induced by π' is defined as*

$$cy := \pi'(c)y \text{ for all } c \in \mathcal{C}, y \in F.$$

In the remaining part of this section a covariant version of Theorem 6 is derived, which finds applications in the next section. For that we first define covariant τ-maps using the notion of (pre-)C^*-correspondence. Every von Neumann $\mathcal{B}$-module E can be considered as a (pre-)C^*-correspondence from $\mathbb{B}^a(E)$ to $\mathcal{B}$.

Definition 9. *(cf. [9]) Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{B}$, $\mathcal{C}$ be unital (pre-)C^*-algebras. Let E be a Hilbert $\mathcal{A}$-module and let F be a (pre-)C^*-correspondence from $\mathcal{C}$ to $\mathcal{B}$. Let $u : G \to \mathcal{U}\mathcal{B}$ and $u' : G \to \mathcal{U}\mathcal{C}$ be unitary representations on a locally compact group G. A τ-map, $T : E \to F$, is called* (u', u)-covariant *with respect to the dynamical system (G, η, E) if*

$$T(\eta_t(x)) = u_t' T(x) u_t^* \text{ for all } x \in E \text{ and } t \in G.$$

If E is full and $T : E \to F$ is a τ-map which is (u', u)-covariant with respect to (G, η, E), then the map τ is u-covariant with respect to the induced C^*-dynamical system $(G, \alpha^\eta, \mathcal{A})$, because

$$\begin{aligned}\tau(\alpha_t^\eta(\langle x, y\rangle)) &= \tau(\langle \eta_t(x), \eta_t(y)\rangle) = \langle T(\eta_t(x)), T(\eta_t(y))\rangle = \langle u_t' T(x) u_t^*, u_t' T(y) u_t^*\rangle \\ &= \langle T(x)u_t^*, T(y)u_t^*\rangle = u_t\langle T(x), T(y)\rangle u_t^* = u_t \tau(\langle x, y\rangle)u_t^*\end{aligned}$$

for all $x, y \in E$ and $t \in G$.

Definition 10. *Let (G, η, E) be a dynamical system on a Hilbert $\mathcal{A}$-module E, where $\mathcal{A}$ is a C^*-algebra. Let F and F' be Hilbert $\mathcal{B}$-modules over a (pre-)C^*-algebra $\mathcal{B}$. $w : G \to \mathcal{U}\mathcal{B}^a(F')$ and $v : G \to \mathcal{U}\mathcal{B}^a(F)$ are unitary representations on a locally compact group G. A quasi-representation of E on F and F' is called* (w, v)-covariant with respect to (G, η, E) *if*

$$\Psi(\eta_t(x)) = w_t \Psi(x) v_t^* \text{ for all } x \in E \text{ and } t \in G.$$

In this case we say that (Ψ, v, w, F, F') is a covariant quasi-representation of (G, η, E). Any v-covariant representation of a C^*-dynamical system $(G, \alpha, \mathcal{A})$ can be regarded as a (v, v)-covariant representation of a dynamical system on the Hilbert $\mathcal{A}$-module $\mathcal{A}$.

Let $\mathcal{A}$ be a C^*-algebra and let G be a locally compact group. Let E be a full Hilbert $\mathcal{A}$-module, and let F and F' be Hilbert $\mathcal{B}$-modules over a (pre-)C^*-algebra $\mathcal{B}$. If (Ψ, v, w, F, F') is a covariant quasi-representation with respect to (G, η, E), then the representation of $\mathcal{A}$ associated to Ψ is v-covariant with respect to $(G, \alpha^\eta, \mathcal{A})$. Moreover, if π is the representation associated to Ψ, then

$$\begin{aligned}\langle \pi(\alpha_t^\eta(\langle x, y\rangle))f, f'\rangle &= \langle \pi(\langle \eta_t(x), \eta_t(y)\rangle)f, f'\rangle = \langle \Psi(\eta_t(y))f, \Psi(\eta_t(x))f'\rangle \\ &= \langle w_t\Psi(y)v_t^* f, w_t\Psi(x)v_t^* f'\rangle = \langle v_t \pi(\langle x, y\rangle)v_t^* f, f'\rangle\end{aligned}$$

for all $x, y \in E$, $t \in G$ and $f, f' \in F$.

Theorem 11. *Let $\mathcal{A}$, $\mathcal{C}$ be unital C^*-algebras and let $\mathcal{B}$ be a von Neumann algebra acting on $\mathcal{H}$. Let $u : G \to \mathcal{U}\mathcal{B}$, $u' : G \to \mathcal{U}\mathcal{C}$ be unitary representations of a locally compact group G. Let E be a full Hilbert $\mathcal{A}$-module and E' be a von Neumann $\mathcal{C}$-$\mathcal{B}$ module. If $T : E \to E'$ is a τ-map which is (u', u)-covariant with respect to (G, η, E) and if $\tau : \mathcal{A} \to \mathcal{B}$ is completely positive, then there exists*

- *(i) (a) a von Neumann $\mathcal{B}$-module F with a covariant representation (π, v) of $(G, \alpha^\eta, \mathcal{A})$ to $\mathcal{B}^a(F)$,*
  - *(b) a map $V \in \mathcal{B}^a(\mathcal{B}, F)$ such that*
    - *(1) $\tau(a)b = V^*\pi(a)Vb$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$,*

(2) $v_t V b = V u_t b$ for all $t \in G$, $b \in \mathcal{B}$,

(ii) (a) a von Neumann $\mathcal{B}$-module F' and a covariant quasi-representation (Ψ, v, w, F, F') of (G, η, E) such that π is associated to Ψ,

(b) a coisometry S from E' onto F' such that

(1) $T(x)b = S^*\Psi(x)Vb$ for all $x \in E$, $b \in \mathcal{B}$,

(2) $w_t S y = S u'_t y$ for all $t \in G$, $y \in E'$.

Proof. By part (i) of Theorem 6 we obtain the triple (π, V, F) associated to τ. Here F is a von Neumann $\mathcal{B}$-module, $V \in \mathcal{B}^a(\mathcal{B}, F)$, and π is a representation of $\mathcal{A}$ to $\mathcal{B}^a(F)$ such that

$$\tau(a)b = V^*\pi(a)Vb \text{ for all } a \in \mathcal{A},\ b \in \mathcal{B}.$$

Recall the proof, using the submodule K we have constructed the triple (π_0, V, F_0) with $[\pi_0(\mathcal{A})V\mathcal{B}] = F_0$. Define $v^0 : G \to \mathcal{B}^a(F_0)$ (cf. Theorem 3.1, [7]) by

$$v_t^0(a \otimes b + K) := \alpha_t(a) \otimes u_t(b) + K \text{ for all } a \in \mathcal{A},\, b \in \mathcal{B} \text{ and } t \in G.$$

Since τ is u-covariant with respect to $(G, \alpha^\eta, \mathcal{A})$, for $a, a' \in \mathcal{A}$, $b, b' \in \mathcal{B}$ and $t \in G$ it follows that

$$\begin{aligned}\langle v_t^0 a \otimes b + K, v_t^0 a' \otimes b' + K\rangle &= \langle \alpha_t(a) \otimes u_t b, \alpha_t(a') \otimes u_t b'\rangle = (u_t b)^*\tau(\alpha_t(a^*a'))u_t b' \\ &= b^*\tau(a^*a')b' = \langle a \otimes b + K, a' \otimes b' + K\rangle.\end{aligned}$$

This map v_t^0 extends as a unitary on F_0 for each $t \in G$ and further we get a group homomorphism $v^0 : G \to \mathcal{U}\mathcal{B}^a(\mathcal{F}_0)$. The continuity of $t \mapsto \alpha_t^\eta(b)$ for each $b \in \mathcal{B}$, the continuity of u and the fact that v_t^0 is a unitary for each $t \in G$ together implies the continuity of v^0. Thus $v^0 : G \to \mathcal{U}\mathcal{B}^a(\mathcal{F}_0)$ becomes a unitary representation. For each $t \in G$ define $v_t : F \to F$ by

$$v_t(\text{sot-}\lim_\alpha f_\alpha^0) := \text{sot-}\lim_\alpha v_t^0(f_\alpha^0) \text{ where } f = \text{sot-}\lim_\alpha f_\alpha^0 \in F \text{ for } f_\alpha^0 \in F_0.$$

It is clear that $v : G \to \mathcal{B}^a(F)$ is a unitary representation of G on F and moreover it satisfies the condition (i)(b)(2) of the statement.

Notation F'' will be used for $[T(E)\mathcal{B}]$ which is a Hilbert $\mathcal{B}$-module. Let F' be the strong operator topology closure of F'' in $\mathcal{B}(\mathcal{H}, E' \bigodot \mathcal{H})$. For each $x \in E$, define $\Psi_0(x) : F_0 \to F''$ by

$$\Psi_0(x)(\sum_{j=1}^n \pi(a_j)Vb_j) := \sum_{j=1}^n T(xa_j)b_j \text{ for all } a_j \in \mathcal{A}, b_j \in \mathcal{B}$$

and define $\Psi(x) : F \to F'$ by

$$\Psi(x)(f) := \text{sot-}\lim_\alpha \Psi_0(x)f_\alpha^0 \text{ where } f = \text{sot-}\lim_\alpha f_\alpha^0 \in F \text{ for } f_\alpha^0 \in F_0.$$

$\Psi_0 : E \to \mathcal{B}^r(F_0, F'')$ and $\Psi : E \to \mathcal{B}^a(F, F')$ are quasi-representations (see part (ii) of Theorem 6). Indeed, there exists an orthogonal projection S from E' onto F' such that

$$T(x)b = S^*\Psi(x)Vb \text{ for all } x \in E \text{ and } b \in \mathcal{B}.$$

Since T is (u', u)-covariant, we have

$$u'_t(\sum_{i=1}^{n} T(x_i)b_i) = \sum_{i=1}^{n} T(\eta_t(x_i))u_t b_i \text{ for all } t \in G, x_i \in E, b_i \in \mathcal{B}, i = 1, 2, \ldots, n.$$

From this computation it is clear that F'' is invariant under u'. For each $t \in G$ define $w^0_t := u'_t|_{F''}$, the restriction of u'_t to F''. In fact, $t \mapsto w^0_t$ is a unitary representation of G on F''. Further

$$\Psi_0(\eta_t(x))(\sum_{i=1}^{n} \pi_0(a_i)Vb_i) = \sum_{i=1}^{n} T(\eta_t(x)\alpha^\eta_t\alpha^\eta_{t^{-1}}(a_i))b_i = \sum_{i=1}^{n} T(\eta_t(x\alpha^\eta_{t^{-1}}(a_i)))b_i$$
$$= \sum_{i=1}^{n} u'_t T(x\alpha^\eta_{t^{-1}}(a_i))u_{t^{-1}}b_i = w^0_t\Psi_0(x)(\sum_{i=1}^{n} \pi_0(\alpha^\eta_{t^{-1}}(a_i))Vu_{t^{-1}}b_i)$$
$$= w^0_t\Psi_0(x)v_{t^{-1}}(\sum_{i=1}^{n} \pi_0(a_i)Vb_i)$$

for all $a_1, a_2, \ldots, a_n \in \mathcal{A}, b_1, b_2, \ldots, b_n \in \mathcal{B}, x \in E, t \in G$. Therefore $(\Psi_0, v^0, w^0, F_0, F'')$ is a covariant quasi-representation of (G, η, E) and π_0 is associated to Ψ_0. For each $t \in G$ define $w_t : F' \to F'$ by

$$w_t(\text{sot-}\lim_\alpha f''_\alpha) := \text{sot-}\lim_\alpha u'_t f''_\alpha \text{ where all } f''_\alpha \in F''.$$

It is evident that the map $t \mapsto w_t$ is a unitary representation of G on F'. S is the orthogonal projection of E' onto F' so we obtain $w_t S = S u'_t$ on F for all $t \in G$. Finally

$$\Psi(\eta_t(x))f = \text{sot-}\lim_\alpha \Psi_0(\eta_t(x))f^0_\alpha = \text{sot-}\lim_\alpha w^0_t\Psi_0(x)v^0_{t^{-1}}f^0_\alpha = w_t\Psi(x)v_{t^{-1}}f$$

for all $x \in E$, $t \in G$ and f=sot-$\lim_\alpha f^0_\alpha \in F$ for $f^0_\alpha \in F_0$. Whence (Ψ, v, w, F, F') is a covariant quasi-representation of (G, η, E) and observe that π is associated to Ψ. □

3 τ-maps from the crossed product of Hilbert C^*-modules

Let (G, η, E) be a dynamical system on E, which is a full Hilbert C^*-module over $\mathcal{A}$, where G is a locally compact group. The *crossed product* Hilbert C^*-module $E \times_\eta G$

(cf. Proposition 3.5, [6]) is the completion of an inner-product $\mathcal{A} \times_{\alpha^\eta} G$-module $C_c(G,E)$ such that the module action and the $\mathcal{A} \times_{\alpha^\eta} G$-valued inner product are given by

$$lg(s) = \int_G l(t)\alpha_t^\eta(g(t^{-1}s))dt,$$
$$\langle l,m\rangle_{\mathcal{A}\times_{\alpha^\eta}G}(s) = \int_G \alpha_{t^{-1}}^\eta(\langle l(t), m(ts)\rangle)dt$$

respectively for $s \in G$, $g \in C_c(G,\mathcal{A})$ and $l,m \in C_c(G,E)$. The following lemma shows that any covariant quasi-representation $(\Psi_0, v^0, w^0, F_0, F')$ with respect to (G,η,E) provides a quasi-representation $\Psi_0 \times v^0$ of $E \times_\eta G$ on F_0 and F' satisfying

$$(\Psi_0 \times v^0)(l) = \int_G \Psi_0(l(t))v_t^0 dt \text{ for all } l \in C_c(G,E).$$

Moreover, it says that if π_0 is associated to Ψ_0, then the integrated form of the covariant representation (π_0, v^0, F_0) with respect to $(G, \alpha^\eta, \mathcal{A})$ is associated to $\Psi_0 \times v^0$.

Lemma 12. *Let (G,η,E) be a dynamical system on a full Hilbert $\mathcal{A}$-module E, where $\mathcal{A}$ is a unital C^*-algebra and G is a locally compact group. Let F_0 and F' be Hilbert $\mathcal{B}$-modules, where $\mathcal{B}$ is a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. If $(\Psi_0, v^0, w^0, F_0, F')$ is a covariant quasi-representation with respect to (G,η,E), then $\Psi_0 \times v^0$ is a quasi-representation of $E \times_\eta G$ on F_0 and F'.*

Proof. For $l \in C_c(G,E)$ and $g \in C_c(G,\mathcal{A})$, we get

$$\begin{aligned}(\Psi_0 \times v^0)(lg) &= \int_G\int_G \Psi_0(l(t)\alpha_t^\eta(g(t^{-1}s))v_s^0 dsdt\\ &= \int_G\int_G \Psi_0(l(t))\pi_0(\alpha_t^\eta(g(t^{-1}s))v_s^0 dsdt\\ &= \int_G\int_G \Psi_0(l(t))v_t^0\pi_0(g(t^{-1}s)v_t^{0*}v_s^0 dsdt\\ &= (\Psi_0 \times v^0)(l)(\pi_0 \times v^0)(g).\end{aligned}$$

For $l,m \in C_c(G,E)$ and $f_0, f_0' \in F_0$ we have

$$\begin{aligned}\langle(\pi_0 \times v^0)(\langle l,m\rangle)f_0, f_0'\rangle &= \left\langle \int_G \pi_0(\langle l,m\rangle(s))v_s^0 f_0 ds, f_0'\right\rangle\\ &= \left\langle \int_G\int_G v_t^{0*}\pi_0(\langle l(t), m(ts)\rangle)v_{ts}^0 f_0 dtds, f_0'\right\rangle\\ &= \int_G\int_G \langle\Psi_0(m(ts))v_{ts}^0 f_0, \Psi_0(l(t))v_t^0 f_0'\rangle dtds\\ &= \left\langle \int_G \Psi_0(m(s))v_s^0 f_0 ds, \int_G \Psi_0(l(t))v_t^0 f_0' dt\right\rangle\end{aligned}$$

$$= \langle(\Psi_0 \times v^0)(m)f_0, (\Psi_0 \times v^0)(l)f_0'\rangle.\square$$

Definition 13. *(cf. [9]) Let G be a locally compact group with the modular function $\triangle$. Let $u : G \to \mathcal{UB}$ and $u' : G \to \mathcal{UC}$ be unitary representations of G on unital (pre-)C^*-algebras $\mathcal{B}$ and $\mathcal{C}$, respectively. Let F be a (pre-)C^*-correspondence from $\mathcal{C}$ to $\mathcal{B}$ and let (G, η, E) be a dynamical system on a Hilbert $\mathcal{A}$-module E, where $\mathcal{A}$ is a unital C^*-algebra. A τ-map, $T : E \times_\eta G \to F$, is called* (u', u)-covariant *if*

(a) $T(\eta_t \circ m_t^l) = u_t' T(m)$ where $m_t^l(s) = m(t^{-1}s)$ for all $s, t \in G$, $m \in C_c(G, E)$;

(b) $T(m_t^r) = T(m)u_t$ where $m_t^r(s) = \triangle(t)^{-1}m(st^{-1})$ for all $s, t \in G$, $m \in C_c(G, E)$.

Proposition 14. *Let $\mathcal{B}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, $\mathcal{C}$ be a unital C^*-algebra, and let F be a von Neumann $\mathcal{C}$-$\mathcal{B}$ module. Let (G, η, E) be a dynamical system on a full Hilbert $\mathcal{A}$-module E, where $\mathcal{A}$ is a unital C^*-algebra and G is a locally compact group. Let $u : G \to \mathcal{UB}$, $u' : G \to \mathcal{UC}$ be unitary representations and let $\tau : \mathcal{A} \to \mathcal{B}$ be a completely positive map. If $T : E \to F$ is a τ-map which is (u', u)-covariant with respect to (G, η, E), then there exist a completely positive map $\widetilde{\tau} : \mathcal{A} \times_{\alpha^\eta} G \to \mathcal{B}$ and a (u', u)-covariant map $\widetilde{T} : E \times_\eta G \to F$ which is a $\widetilde{\tau}$-map. Indeed, $\widetilde{T}$ satisfies*

$$\widetilde{T}(l) = \int_G T(l(s))u_s ds \text{ for all } l \in C_c(G, E).$$

Proof. By Theorem 11 there exists the Stinespring type construction $(\Psi, \pi, v, w, V, S, F, F')$, associated to T, based on the construction $(\Psi_0, \pi_0, v^0, T, F_0, F'')$. Define a map $\widetilde{T} : E \times_\eta G \to F$ by

$$\widetilde{T}(l) := S^*(\Psi_0 \times v^0)(l)V, \text{ for all } l \in C_c(G, E).$$

Indeed, for all $l \in C_c(G, E)$ we obtain

$$\begin{aligned}\widetilde{T}(l) &= S^*(\Psi_0 \times v^0)(l)V = S^* \int_G \Psi_0(l(s))v_s^0 ds V = \int_G S^*\Psi_0(l(s))V u_s ds \\ &= \int_G T(l(s))u_s ds.\end{aligned}$$

It is clear that $(\pi_0 \times v^0, V, F_0)$ is the Stinespring triple (cf. Theorem 6) associated to the completely positive map $\widetilde{\tau} : \mathcal{A} \times_{\alpha^\eta} G \to \mathcal{B}$ defined by

$$\widetilde{\tau}(h) := \int_G \tau(f(t))v_t^0 dt \text{ for all } f \in C_c(G, \mathcal{A}); b, b' \in \mathcal{B}.$$

We have

$$\langle\widetilde{T}(l), \widetilde{T}(m)\rangle b = \langle S^*(\Psi_0 \times v^0)(m)V, S^*(\Psi_0 \times v^0)(l)V\rangle b = \widetilde{\tau}(\langle l, m\rangle)b$$

for all $l, m \in E \times_\eta G,\ b \in \mathcal{B}$. Hence $\widetilde{T}$ is a $\widetilde{\tau}$-map. Further,

$$\begin{aligned}
\widetilde{T}(\eta_t \circ m^l_t) &= S^* \int_G \Psi_0(\eta_t(m(t^{-1}s)))v^0_s ds V = S^* \int_G w^0_t \Psi_0(m(t^{-1}s))v^0_{t^{-1}s} ds V \\
&= u'_t \widetilde{T}(m); \\
\widetilde{T}(m^r_t) &= S^* \int_G \triangle(t)^{-1} \Psi_0(m(st^{-1}))v^0_s ds V = S^* \int_G \Psi_0(m(g))v^0_g v^0_t dg V \\
&= \widetilde{T}(m)u_t \text{ where } t \in G,\ m \in C_c(G, E).
\end{aligned}$$

□

Proposition 14 gives us a map $T \mapsto \widetilde{T}$ where $T : E \to F$ is a τ-map which is (u', u)-covariant with respect to (G, η, E) and $\widetilde{T} : E \times_\eta G \to F$ is (u', u)-covariant $\widetilde{\tau}$-map such that τ and $\widetilde{\tau}$ are completely positive. This map is actually a one-to-one correspondence. To prove this result we need the following terminologies:

We identify $M(\mathcal{A})$ with $\mathcal{B}^a(\mathcal{A})$ (cf. Theorem 2.2 of [12]), here $\mathcal{A}$ is considered as a Hilbert $\mathcal{A}$-module in the natural way. The *strict topology* on $\mathcal{B}^a(E)$ is the topology given by the seminorms $a \mapsto \|ax\|$, $a \mapsto \|a^*y\|$ for each $x, y \in E$. For each C^*-dynamical system $(G, \alpha, \mathcal{A})$ we get a non-degenerate faithful homomorphism $i_\mathcal{A} : \mathcal{A} \to M(\mathcal{A} \times_\alpha G)$ and an injective strictly continuous homomorphism $i_G : G \to \mathcal{U}M(\mathcal{A} \times_\alpha G)$ (cf. Proposition 2.34 of [23]) defined by

$$i_\mathcal{A}(a)(f)(s) := af(s) \text{ for } a \in \mathcal{A},\ \ s \in G,\ \ f \in C_c(G, \mathcal{A});$$

$$i_G(r)f(s) := \alpha_r(f(r^{-1}s)) \text{ for } r, s \in G,\ \ f \in C_c(G, \mathcal{A}).$$

Let E be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$. Define the *multiplier module* $M(E) := \mathcal{B}^a(\mathcal{A}, E)$. $M(E)$ is a Hilbert C^*-module over $M(\mathcal{A})$ (cf. Proposition 1.2 of [16]). For a dynamical system (G, η, E) on E we get a non-degenerate morphism of modules i_E from E to $M(E \times_\eta G)$ (cf. Theorem 3.5 of [10]) as follows: For each $x \in E$ define $i_E(x) : C_c(G, \mathcal{A}) \to C_c(G, E)$ by

$$i_E(x)(f)(s) := xf(s) \text{ for all } f \in C_c(G, \mathcal{A}),\ \ s \in G.$$

Note that i_E is an $i_\mathcal{A}$-map.

Theorem 15. *Let $\mathcal{A}$, $\mathcal{C}$ be unital C^*-algebras, and let $\mathcal{B}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Let $u : G \to \mathcal{UB}$, $u' : G \to \mathcal{UC}$ be unitary representations of a locally compact group G. If (G, η, E) is a dynamical system on a full Hilbert $\mathcal{A}$-module E, and if F is a von Neumann $\mathcal{C}$-$\mathcal{B}$ module, then there exists a bijective correspondence $\mathfrak{I}$ from the set of all τ-maps, $T : E \to F$, which are (u', u)-covariant with respect to (G, η, E) onto the set of all maps $\widetilde{T} : E \times_\eta G \to F$ which are (u', u)-covariant $\widetilde{\tau}$-maps such that $\tau : \mathcal{A} \to \mathcal{B}$ and $\widetilde{\tau} : \mathcal{A} \times_{\alpha^\eta} G \to \mathcal{B}$ are completely positive maps.*

Proof. Proposition 14 ensures that the map $\mathfrak{I}$ exists and is well-defined. Let $T : E \times_\eta G \to F$ be a (u', u)-covariant τ-map, where $\tau : \mathcal{A} \times_{\alpha^\eta} G \to \mathcal{B}$ is a completely positive map. Suppose $(\Psi_0, \pi_0, V, F_0, F'')$ and (Ψ, π, V, S, F, F') are the Stinespring type constructions associated to T as in the proof of Theorem 6. Let $\{e_i\}_{i\in\mathcal{I}}$ be an approximate identity for $\mathcal{A} \times_{\alpha^\eta} G$. Then there exists a representation $\overline{\pi_0} : M(\mathcal{A} \times_{\alpha^\eta} G) \to \mathcal{B}^a(F_0)$ (cf. Proposition 2.39 of [23]) defined by

$$\overline{\pi_0}(a)x := \lim_i \pi_0(ae_i)x \text{ for all } a \in M(\mathcal{A} \times_{\alpha^\eta} G) \text{ and } x \in F_0.$$

A mapping $\overline{\Psi_0} : M(E \times_\eta G) \to \mathcal{B}^r(F_0, F'')$ defined by

$$\overline{\Psi_0}(h)x := \lim_i \Psi_0(he_i)x \text{ for all } h \in M(E \times_\eta G) \text{ and } x \in F_0,$$

is a quasi representation and $\overline{\pi_0}$ is associated to $\overline{\Psi_0}$. If $\widetilde{\pi_0} := \overline{\pi_0} \circ i_{\mathcal{A}}$, then we further get a quasi-representation $\widetilde{\Psi_0} : E \to \mathcal{B}^r(F_0, F'')$ defined as $\widetilde{\Psi_0} := \overline{\Psi_0} \circ i_E$ such that $\widetilde{\pi_0}$ is associated to $\widetilde{\Psi_0}$. Define maps $T_0 : E \to F$ and $\tau_0 : \mathcal{A} \to \mathcal{B}$ by

$$T_0(x)b := S^*\widetilde{\Psi_0}(x)Vb \text{ for } b \in \mathcal{B},\ x \in E \textit{ and}$$

$$\tau_0(a) := V^*\widetilde{\pi_0}(a)V \text{ for all } a \in \mathcal{A}.$$

It follows that τ_0 is a completely positive map and T_0 is a τ_0-map.

Let $v^0 : G \to \mathcal{UB}^a(F_0)$ be a unitary representation defined by $v^0 := \overline{\pi_0} \circ i_G$ where

$$i_G(t)(f)(s) := \alpha_t(f(t^{-1}s)) \text{ for all } t, s \in G,\ f \in C_c(G, \mathcal{A}).$$

Observe that $\widetilde{\pi_0} : \mathcal{A} \to \mathcal{B}^a(F_0)$ is a v^0-covariant and $\widetilde{\pi_0} \times v^0 = \pi_0$ (cf. Proposition 2.39, [23]). We extend v^0 to a unitary representation $v : G \to \mathcal{UB}^a(F)$ as in the proof of Theorem 11. It is easy to verify that

$$\alpha_t^\eta \circ \langle m, m'\rangle_t^l = \langle m_{t^{-1}}^r, m'\rangle \text{ for all } m, m' \in C_c(G, E).$$

Using the fact that T is (u', u)-covariant we get

$$\tau(\alpha_t^\eta \circ \langle m, m'\rangle_t^l) = \tau(\langle m_{t^{-1}}^r, m'\rangle) = \langle T(m)u_{t^{-1}}, T(m')\rangle = u_t\tau(\langle m, m'\rangle),$$

for all $m, m' \in C_c(G, E)$. Therefore we have

$$\begin{aligned}
\langle v_t(\pi_0(f)Vb), Vb'\rangle &= \langle v_t^0((\widetilde{\pi_0} \times v^0)(f)Vb, Vb'\rangle = \left\langle \int_G \widetilde{\pi_0}(\alpha_t^\eta(f(s)))v_{ts}^0 Vbds, Vb'\right\rangle \\
&= \langle(\widetilde{\pi_0} \times v^0)(\alpha_t^\eta \circ f_t^l)Vb, Vb'\rangle = \langle\tau(\alpha_t^\eta \circ f_t^l)b, b'\rangle \\
&= \langle(\pi_0(f)Vb), Vu_{t^{-1}}b'\rangle
\end{aligned}$$

for all $t \in G$, $b, b' \in \mathcal{B}$ and $f \in C_c(G, \mathcal{A})$. This implies that $v_tV = Vu_t$ for each $t \in G$. For each $t \in G$ define $w_t^0 : [\widetilde{\Psi_0}(E)V\mathcal{B}] \to [\widetilde{\Psi_0}(E)V\mathcal{B}]$ by

$$w_t^0(\widetilde{\Psi_0}(x)Vb) := \widetilde{\Psi_0}(\eta_t(x))Vu_tb \text{ for all } x \in E,\ b \in \mathcal{B}.$$

Let $t \in G$, $x, y \in E$ and $b, b' \in \mathcal{B}$. Then

$$\begin{aligned}
&\langle \widetilde{\Psi_0}(\eta_t(x))Vu_t b, \widetilde{\Psi_0}(\eta_t(y))Vu_t b'\rangle \\
=&\langle \widetilde{\pi_0}(\langle \eta_t(y), \eta_t(x)\rangle)Vu_t b, Vu_t b'\rangle = \langle \widetilde{\pi_0}(\alpha_t^{\eta}(\langle y, x\rangle))Vu_t b, Vu_t b'\rangle \\
=&\langle v_t^0 \widetilde{\pi_0}(\langle y, x\rangle)v_{t^{-1}}^0 Vu_t b, Vu_t b'\rangle = \langle \widetilde{\pi_0}(\langle y, x\rangle)Vb, Vb'\rangle \\
=&\langle \widetilde{\Psi_0}(x)Vb, \widetilde{\Psi_0}(y)Vb'\rangle.
\end{aligned}$$

Indeed, for fix $t \in G$, the continuity of the maps $t \mapsto \eta_t(x)$ and $t \mapsto u_t b$ for $b \in \mathcal{B}$, $x \in E$ provides the fact that the map $t \mapsto w_t^0(z)$ is continuous for each $z \in \widetilde{\Psi_0}(E)V\mathcal{B}$. Therefore w^0 is a unitary representation of G on $[\widetilde{\Psi_0}(E)V\mathcal{B}]$ and hence it naturally extends to a unitary representation of G on the strong operator topology closure of $[\widetilde{\Psi_0}(E)V\mathcal{B}]$ in $\mathcal{B}(\mathcal{H}, F \bigodot \mathcal{H})$, which we denote by w.

Note that $E \otimes C_c(G)$ is dense in $E \times_\eta G$ (cf. Theorem 3.5 of [10]). For $x \in E$ and $f \in C_c(G)$ we have

$$\begin{aligned}
(\widetilde{\Psi_0} \times v^0)(x \otimes f) &= \int_G \widetilde{\Psi_0}(xf(t))v_t^0 dt = \int_G \overline{\Psi_0}(i_E(xf(t)))\overline{\pi_0}(i_G(t))dt \\
&= \overline{\Psi_0}(i_E(x)\int_G f(t)i_G(t)dt) = \overline{\Psi_0}(i_E(y)i_{\mathcal{A}}(\langle y, y\rangle)\int_G f(t)i_G(t)dt) \\
&= \overline{\Psi_0}(i_E(y)(\langle y, y\rangle \otimes f)) = \overline{\Psi_0}(y\langle y, y\rangle \otimes f) = \Psi_0(x \otimes f)
\end{aligned}$$

where $x = y\langle y, y\rangle$ for some $y \in E$ (cf. Proposition 2.31 [17]). Also the 3rd last equality follows from Corollary 2.36 of [23]. This proves $\widetilde{\Psi_0} \times v^0 = \Psi_0$ on $E \times_\eta G$. Also for all $m \in C_c(G, E)$ and $b \in \mathcal{B}$ we get

$$\begin{aligned}
Su_t'(T(m)b) &= ST(\eta_t \circ m_t^l)b = SS^*\Psi_0(\eta_t \circ m_t^l)Vb = \Psi_0(\eta_t \circ m_t^l)Vb \\
&= \int_G \widetilde{\Psi_0}(\eta_t(m(t^{-1}s)))v_s^0 Vbds = w_t \int_G \widetilde{\Psi_0}(m(t^{-1}s))v_{t^{-1}s}^0 Vbds \\
&= w_t \Psi_0(m)Vb = w_t ST(m)b.
\end{aligned}$$

As T is (u', u)-covariant, it satisfies $T(\eta_t \circ m_t^l) = u_t' T(m)$, where $m_t^l(s) = m(t^{-1}s)$ for all $s, t \in G$, $m \in C_c(G, E)$. Thus the strong operator topology closure of $[T(E \times_\eta G)\mathcal{B}]$ in $\mathcal{B}(\mathcal{H}, F \bigodot \mathcal{H})$, say F_T, is invariant under u'. This together with the fact that S is an orthogonal projection onto F_T provides $Su_t'z = w_t Sz$ for all $z \in F_T^{\perp}$. So we obtain the equality $Su_t'y = w_t Sy$ for all $y \in F$. Hence

$$T_0(\eta_t(x))b = S^*\widetilde{\Psi_0}(\eta_t(x))Vb = S^* w_t \widetilde{\Psi_0}(x)Vu_{t^{-1}}b = u_t' T_0(x)u_t^* b$$

for all $t \in G$, $x \in E$ and $b \in \mathcal{B}$. Moreover,

$$\widetilde{T_0}(m)b = S^* \int_G \widetilde{\Psi_0}(m(t))Vu_t bdt = S^*\Psi_0(m)Vb = T(m)b$$

for all $m \in C_c(G,E)$, $b \in \mathcal{B}$. This gives $\widetilde{T_0} = T$ and proves that the map $\mathfrak{I}$ is onto.

Let $\tau_1 : \mathcal{A} \to \mathcal{B}$ be a completely positive map and let $T_1 : E \to F$ be a (u', u)-covariant τ_1-map satisfying $\widetilde{T_1} = T$. If $(\Psi_1, \pi_1, V_1, S_1, F_1, F_1')$ is the (w_1, v_1)-covariant Stinespring type construction associated to T_1 coming from Theorem 11, then we show that $(\Psi_1 \times v_1, V_1, S_1, F_1, F_1')$ is unitarily equivalent to the Stinespring type construction associated to T. Indeed, from Proposition 14, there exists a decomposition

$$\widetilde{T_1}(m) = S_1^*(\Psi_1 \times v_1)(m)V_1 \text{ for all } m \in C_c(G,E).$$

This implies that for all $m, m' \in C_c(G,E)$ we get

$$\begin{aligned}\tau(\langle m, m'\rangle) &= \langle T(m), T(m')\rangle = \langle \widetilde{T_1}(m), \widetilde{T_1}(m')\rangle \\ &= \langle S_1^*(\Psi \times v_1)(m)V_1, S_1^*(\Psi \times v_1)(m')V_1\rangle \\ &= \langle (\pi_1 \times v_1)(\langle m, m'\rangle)V_1, V_1\rangle.\end{aligned}$$

E is full gives $E \times_\eta G$ is full (cf. the proof of Proposition 3.5, [6]) and hence $\tau(f) = \langle(\pi_1 \times v_1)(f)V_1, V_1\rangle$ for all $f \in C_c(G, \mathcal{A})$. Using this fact we deduce that

$$\begin{aligned}\langle \pi(f)Vb, \pi(f')Vb'\rangle &= \langle \pi(f'^*f)Vb, Vb'\rangle = b^*\tau(f'^*f)b' \\ &= \langle \pi_1 \times v_1(f)V_1b, \pi_1 \times v_1(f')V_1b'\rangle\end{aligned}$$

for all $f, f' \in C_c(G, \mathcal{A})$ and $b, b' \in \mathcal{B}$. Thus we get a unitary $U_1 : F \to F_1$ defined by

$$U_1(\pi(f)Vb) := \pi_1 \times v_1(f)V_1b \text{ for } f \in C_c(G, \mathcal{A}),\ b \in \mathcal{B}$$

and which satisfies $V_1 = U_1V$, $\pi_1 \times v_1(f) = U_1\pi(f)U_1^*$ for all $f \in C_c(G, \mathcal{A})$. Another computation

$$\begin{aligned}\|\Psi(m)Vb\|^2 &= \|\langle \Psi(m)Vb, \Psi(m)Vb\rangle\| = \|\langle \pi(\langle m,m\rangle)Vb, Vb\rangle\| = \|b^*\tau(\langle m,m\rangle)b\| \\ &= \|b^*\langle \pi_1 \times v_1(\langle m,m\rangle)V_1, V_1\rangle b\| = \|\langle \Psi_1 \times v_1(m)V_1b, \Psi_1 \times v_1(m)V_1b\rangle\| \\ &= \|\Psi_1 \times v_1(m)V_1b\|^2\end{aligned}$$

for all $m \in C_c(G,E)$, $b \in \mathcal{B}$ provides a unitary $U_2 : F' \to F_1'$ defined as

$$U_2(\Psi(m)Vb) := \Psi_1 \times v_1(m)V_1b \text{ for } m \in C_c(G,E),\ b \in \mathcal{B}.$$

Further, it satisfies conditions $S_1 = U_2S$ and $U_2\Psi(m) = \Psi_1 \times v_1(m)U_1$ for all $m \in C_c(G,E)$. This implies $U_2\overline{\widetilde{\Psi} \times v}(z') = \overline{\Psi_1 \times v_1}(z')U_1$ for all $z' \in M(E \times_\eta G)$ and so $U_2\widetilde{\Psi}(x) = \Psi_1 \times v_1(x)U_1$ for all $x \in E$. Using it we have

$$T_0(x) = S^*\widetilde{\Psi}(x)V = S_1^*U_2\widetilde{\Psi}(x)U_1^*V_1 = S_1^*U_2U_2^*(\Psi_1 \times v_1)(x)U_1U_1^*V_1 = T_1(x)$$

for all $x \in E$ and $b \in \mathcal{B}$. Hence $\mathfrak{I}$ is injective. □

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