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TOEPLITZ OPERATORS AND MULTIPLICATION OPERATORS IN THE COMMUTANT OF A COMPOSITION OPERATOR ON WEIGHTED BERGMAN SPACES

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Abstract. Let φ be an analytic self-map of $\mathbb{D}$. We investigate which Toeplitz operators and multiplication operators commute with a given composition operator C_φ on $A^p_\alpha(\mathbb{D})$ for $1 < p < \infty$ and $-1 < \alpha < \infty$. Let S be a bounded linear operator in the commutant of C_φ. We show that under a certain condition on S, S is a polynomial in C_φ.

1 Introduction

Let $\mathbb{D}$ denote the open unit disc in the complex plane and let dA be the normalized area measure on $\mathbb{D}$. For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space $A^p_\alpha(\mathbb{D}) = A^p_\alpha$ is the space of analytic functions in $L^p(\mathbb{D}, dA_\alpha)$, where

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z).$$

If f is in $L^p(\mathbb{D}, dA_\alpha)$, we note that

$$\|f\|_{p.\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z)\right)^{\frac{1}{p}}.$$

When $1 \leq p < \infty$, the space $L^p(\mathbb{D}, dA_\alpha)$ is a Banach space and the weighted Bergman space A^p_α is closed in $L^p(\mathbb{D}, dA_\alpha)$. So A^p_α is a Banach space. Let $L^\infty(\mathbb{D})$ denote the space of essentially bounded functions on $\mathbb{D}$. For $f \in L^\infty(\mathbb{D})$, we define

$$\|f\|_\infty = \text{esssup}\{|f(z)| : z \in \mathbb{D}\}.$$

The space $L^\infty(\mathbb{D})$ is a Banach space with the above norm. As usual, let $H^\infty(\mathbb{D}) = H^\infty$ denote the space of bounded analytic functions on $\mathbb{D}$. It is clear that H^∞ is closed in $L^\infty(\mathbb{D})$ and hence is a Banach space.

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Let φ be an analytic self-map of the unit disc, $1 < p < \infty$ and $-1 < \alpha < \infty$. The composition operator C_φ on A^p_α, is defined by the rule $C_\varphi(f) = f \circ \varphi$. Every composition operator C_φ on A^p_α is bounded (see, e.g., [9]).

Let for each $1 < p < \infty$, $P_\alpha : L^p(\mathbb{D}, dA_\alpha) \to A^p_\alpha$ be the Bergman projection. We note that P_α is an integral operator represented by

$$P_\alpha g(z) = \int_{\mathbb{D}} K(z,w)g(w)dA_\alpha(w),$$

where

$$\begin{aligned} K(z,w) &= \frac{1}{(1-z\overline{w})^{2+\alpha}} \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}(z\overline{w})^n. \end{aligned}$$

For each $f \in L^\infty(\mathbb{D})$ and $1 < p < \infty$, we define the Toeplitz operator T_f on A^p_α with symbol f by $T_f(g) = P_\alpha(fg)$. If we define $M_f : L^p(\mathbb{D}, dA_\alpha) \to L^p(\mathbb{D}, dA_\alpha)$ by $M_f(g) = fg$, it is obvious that M_f is bounded. Since the Bergman projection is bounded (see, e.g., [8]), we conclude that T_f is a bounded operator.

If f is a bounded complex valued harmonic function defined on $\mathbb{D}$, then there are holomorphic functions f_1 and f_2 such that $f = f_1 + \overline{f_2}$. This decomposition is unique if we require $f_2(0) = 0$. Of course f_1 and f_2 are not necessarily bounded, but they are certainly Bloch functions and they are in A^p_α for $1 \le p \le \infty$ (see, e.g., [1]).

Throughout this paper, we write $\varphi^{[j]}$ to denote the jth iterate of φ, that is, $\varphi^{[0]}$ is the identity map on $\mathbb{D}$ and $\varphi^{[j+1]} = \varphi \circ \varphi^{[j]}$.

Suppose that φ is an analytic self-map of $\mathbb{D}$ which is not the identity and not an elliptic disc automorphism. Then there is a point a in $\overline{\mathbb{D}}$ such that iterates of φ converges to a uniformly on compact subsets of $\mathbb{D}$. We note that for each fixed positive integer l, $\{(\varphi^{[n]})^l\}$ converges weakly to a^l as $n \to \infty$ (see, e.g., [6]). For each $1 < p < \infty$ and w in $\mathbb{D}$, let λ_w be the point evaluation function at w, that is, $\lambda_w(g) = g(w)$, where $g \in A^p_\alpha$. It is well-known that point evaluations at the points of $\mathbb{D}$ are all continuous on A^p_α (see, e.g., [8]).

Given a fixed operator A, we say that an operator B commutes with A if $AB = BA$. The set of all operators which commute with a fixed operator A is called the commutant of A. The commutant of a particular operator is known in a few cases. For further information about commutant of a composition operator, see [2], [3] and [7]. Also in [5], Carl Cowen showed that if f is a covering map of $\mathbb{D}$ onto a bounded domain in the complex plane, then the commutant of the Toeplitz operator T_f is generated by composition operators induced by linear fractional transformation φ

that satisfy $f \circ \varphi = f$ and by Toeplitz operators. Also in [4], Bruce Clod determined which Toeplitz operators are in the commutant of a given composition operator C_φ on H^2.

In this paper, under certain conditions on φ we investigate which Toeplitz operators and Multiplication operators commute with C_φ on A^p_α for $1 < p < \infty$.

2 Toeplitz operators in the commutant of a composition operator

Throughout this section, C_φ denotes a bounded composition operator on A^p_α for $1 < p < \infty$ and $-1 < \alpha < \infty$. Our goal is to find information about the commutant of C_φ.

Theorem 1. *Let f be a harmonic function in $L^\infty(\mathbb{D})$, and let φ be an analytic self-map of $\mathbb{D}$ which is neither an elliptic disc automorphism of finite periodicity nor the identity mapping. If $C_\varphi T_f = T_f C_\varphi$, then f is an analytic function.*

Proof. Let $f = f_1 + \overline{f_2}$ such that f_1 and f_2 belong to A^p_α, $f_2(0) = 0$, $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=1}^{\infty} b_n z^n$. Since φ is an analytic map which is not an elliptic disc automorphism of finite periodicity, φ is a constant function or φ is an elliptic automorphism of infinite periodicity or φ is neither an elliptic disc automorphism nor a constant.

Case(1): Let φ be a constant. Then $\varphi(z) = b$ for all $z \in \mathbb{D}$, where $|b| < 1$. Since $T_f C_\varphi(1) = C_\varphi T_f(1)$, we have $f_1(z) = f_1(b)$. Thus f_1 is a constant, let $f_1 = c$. For every g in A^p_α, $T_f C_\varphi(g) = C_\varphi T_f(g)$ which implies that

$$cg(b) = P(\overline{f}_2 g)(b) + cg(b).$$

So $P(\overline{f}_2 g)(b) = 0$. In particular, if $g(z) = z^k$, then $b_k = 0$ for all $k \in \mathbb{N}$. Hence $f = f_1 = c$ is analytic.

Case(2): Suppose that φ is an elliptic disc automorphism of infinite periodicity. If $\varphi(0) = 0$, then Schwarz's Lemma implies that $\varphi(z) = e^{i\theta} z$, where $e^{in\theta} \neq 1$ for all integers $n \neq 0$. Since $C_\varphi T_f(1) = T_f C_\varphi(1)$, we have $f_1(e^{i\theta} z) = f_1(z)$ and so $f_1 = a_0$. Now by induction, we show that $f_2 = 0$. Since $T_f C_\varphi(z) = C_\varphi T_f(z)$, we have $\overline{b_1} = e^{i\theta} \overline{b_1}$, so $b_1 = 0$. Let $b_1 = b_2 = \cdots = b_{l-1} = 0$. We show that $b_l = 0$. Since $C_\varphi T_f(z^l) = T_f C_\varphi(z^l)$, we have $\overline{b_l} = e^{il\theta} \overline{b_l}$ and so $b_l = 0$. Hence f must be a constant function.

Now let $b \neq 0$ be the fixed point of φ. Since $T_f C_\varphi(1) = C_\varphi T_f(1)$, we have $f_1 = f_1 \circ \varphi$. Since φ has infinite periodicity, we conclude that f_1 is a constant. Hence f_2 induces a Toeplitz operator which commutes with C_φ. We claim that

$f_2 = 0$. Let $\alpha(z) = \frac{b-z}{1-\bar{b}z}$, note that $\alpha^{-1} = \alpha$. Since $T_{\overline{f}_2}$ commutes with C_φ, $A = C_\alpha T_{\overline{f}_2} C_\alpha$ commutes with $C_\alpha C_\varphi C_\alpha = C_{\alpha\circ\varphi\circ\alpha}$. The function $\alpha\circ\varphi\circ\alpha$ is an elliptic disc automorphism of infinite periodicity with fixed point 0. Thus there exists $\{\lambda_n\}_{n=1}^\infty$ such that $A(z^n) = \lambda_n z^n$ and $T_{\overline{f}_2} = C_\alpha A C_\alpha$ (If $C_\varphi T = TC_\varphi$ and $\varphi(z) = e^{i\theta}z$, then there exists $\{\lambda_n\}_{n=1}^\infty$ such that $T(z^n) = \lambda_n z^n$). Set $g = A(\alpha)$, we have

$$g(z) = \lambda_0 b + \sum_{k=1}^{\infty} \lambda_k (\bar{b})^{k-1}(|b|^2 - 1)z^k.$$

Since $T_{\overline{f}_2}(z) = \frac{2}{2+\alpha}\overline{b_1}$, we see that $g\circ\alpha$ is a constant. Hence g is a constant which implies that $\lambda_k = 0$ for $k \geq 1$. On the other hand, $\lambda_0 = 0$. Thus $A = 0$ and hence $f_2 = 0$.

Case(3): Let φ be neither an elliptic disc automorphism nor a constant. Suppose that a is the Denjoy-Wolff point of φ. Since $T_f C_\varphi = C_\varphi T_f$, we have

$$T_f C_{\varphi^{[n]}}(z) = C_{\varphi^{[n]}} T_f(z).$$

Therefore

$$\begin{aligned} C_{\varphi^{[n]}} T_f(z) &= C_{\varphi^{[n]}} P(zf_1 + z\overline{f}_2) \\ &= \left(\frac{2}{2+\alpha}\overline{b_1} + zf_1\right)\circ\varphi^{[n]}, \end{aligned}$$

and $T_f C_\varphi(1) = C_\varphi T_f(1)$ which implies that $f_1\circ\varphi = f_1$. Hence

$$T_f C_{\varphi^{[n]}}(z) = \frac{2}{2+\alpha}\overline{b_1} + f_1\varphi^{[n]}.$$

Now if we apply λ_0 on $T_f C_{\varphi^{[n]}}$, then we obtain

$$\lambda_0(T_f C_{\varphi^{[n]}}(z)) = \frac{2}{2+\alpha}\overline{b_1} + a_0\varphi^{[n]}(0).$$

Hence $\{\lambda_0(T_f C_{\varphi^{[n]}})\}$ converges to $\frac{2}{2+\alpha}\overline{b_1} + a_0 a$ as $n\to\infty$. Since $\{\varphi^{[n]}\}$ converges weakly to a as $n\to\infty$, $\{T_f(\varphi^{[n]})\}$ converges weakly to $T_f(a) = af_1$ as $n\to\infty$. So $\{\lambda_0(T_f C_{\varphi^{[n]}})\}$ converges to $a_0 a$ as $n\to\infty$. Thus $b_1 = 0$.

Now let $b_1 = b_2 = \cdots = b_{l-1} = 0$. Consider $T_f(z^l)$ in the above argument, we have

$$T_f((\varphi^{[n]})^l) = \frac{\Gamma(l+1)\Gamma(\alpha+2)}{\Gamma(l+2+\alpha)}\overline{b_l} + f_1(\varphi^{[n]})^l.$$

By applying λ_0 on $T_f((\varphi^{[n]})^l)$ and since $\{T_f((\varphi^{[n]})^l)\}$ converges weakly to $T_f(a^l)$ as $n \to \infty$, we get

$$a^l a_0 = \frac{\Gamma(l+1)\Gamma(\alpha+2)}{\Gamma(l+2+\alpha)} \bar{b}_l + a^l a_0.$$

Thus $b_l = 0$. Hence by the strong induction, $b_n = 0$ for all $n \geq 1$, that is, f is analytic. □

Remark 2. *If $\varphi(z) = \frac{1}{2}z$, then φ is loxodromic and φ is not an elliptic disc automorphism. Also let $f(z) = |z|^2$, we have f is bounded and f is not a harmonic function. Since for every $n \in \mathbb{N}$,*

$$T_f C_\varphi(z^n) = C_\varphi T_f(z^n) = \frac{n+1}{2^n(n+2+\alpha)} z^n,$$

we have $C_\varphi T_f = T_f C_\varphi$ and f is not analytic. This example shows that Theorem 1 is not true in general without f being harmonic.

The following theorem shows that Theorem 1 is not true for all elliptic disc automorphisms.

Theorem 3. *Let f be a harmonic function in $L^\infty(\mathbb{D})$, and let φ be an elliptic disc automorphism of period q, where $q \geq 2$ with $\varphi(0) = 0$. Then $T_f C_\varphi = C_\varphi T_f$ if and only if $f(z) = \sum_{n=0}^{\infty} a_{nq} z^{nq} + \sum_{n=1}^{\infty} \bar{b}_{nq} \bar{z}^{nq}$.*

Proof. By hypothesis, $\varphi(z) = e^{i\theta} z$ with $\theta = 2\pi\frac{p}{q}$, where p is an integer, q is a natural number and $g.c.d(p,q) = 1$. Let $f = f_1 + \overline{f_2}$ such that f_1 and f_2 belong to A^p_α, $f_2(0) = 0$, $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=1}^{\infty} b_n z^n$. Since $T_f C_{e^{2\pi i \frac{p}{q}} z}(1) = C_{e^{2\pi i \frac{p}{q}} z} T_f(1)$, we have $f_1(z) = f_1(e^{2\pi i \frac{p}{q}} z)$. Thus

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n (e^{2\pi i \frac{p}{q}})^n z^n.$$

So if $q \nmid n$, $a_n = 0$. Hence $f_1(z) = \sum_{n=0}^{\infty} a_{nq} z^{nq}$. Since $T_f C_{e^{2\pi i \frac{p}{q}} z}(z) = C_{e^{2\pi i \frac{p}{q}} z} T_f(z)$, we have

$$\frac{2}{2+\alpha} \bar{b}_1 e^{2\pi i \frac{p}{q}} + z e^{2\pi i \frac{p}{q}} f_1(z) = z e^{2\pi i \frac{p}{q}} f_1(z) + \frac{2}{2+\alpha} \bar{b}_1.$$

Therefore $b_1 = 0$. For n such that $q \nmid n$ assume by induction that if $m < n$ and $q \nmid m$, then $b_m = 0$. Since

$$T_f C_{e^{2\pi i \frac{p}{q}} z}(z^n) = C_{e^{2\pi i \frac{p}{q}} z} T_f(z^n),$$

by a similar argument, we can prove that $b_n = 0$ which we omit the details.

Conversely, if $f(z) = \sum_{n=0}^{\infty} a_{nq} z^{nq} + \sum_{n=1}^{\infty} \bar{b}_{nq} \bar{z}^{nq}$, then by straightforward calculation T_f commutes with C_φ. □

In Theorems 1 and 3 we have shown that except for elliptic disc automorphisms of finite periodicity, the Toeplitz operators which commute with C_φ must be analytic, that is, symbol of the Toeplitz operator must be analytic. Now let f be in H^∞. Then $T_f = M_f$ and in this case M_f commutes with C_φ is equivalent to $f \circ \varphi = f$. We will determine which multiplication operators commute with C_φ for certain composition operator C_φ.

Lemma 4. *Let f be in H^∞, and let α be a disc automorphism. Then $C_\alpha M_f C_{\alpha^{-1}} = M_{f\circ\alpha}$.*

Proof. Let g be in A^p_α. Then

$$\begin{aligned} C_\alpha M_f C_{\alpha^{-1}}(g) &= C_\alpha M_f(g\circ\alpha^{-1}) \\ &= C_\alpha(g\circ\alpha^{-1}.f) \\ &= (g\circ\alpha^{-1}.f)\circ\alpha \\ &= g.f\circ\alpha \\ &= M_{f\circ\alpha}(g). \end{aligned}$$

□

Proposition 5. *Let φ be an elliptic disc automorphism with fixed point b, and let $f \in H^\infty$. Then*

(a) If φ is of infinite periodicity, then the multiplication operator M_f commutes with C_φ if and only if f is a constant.

(b) If φ is of period q, then M_f commutes with C_φ if and only if f is of the form $f(z) = \sum_{n=0}^{\infty} a_{nq}\left(\frac{b-z}{1-\bar{b}z}\right)^{nq}$.

Proof. (a) The proof follows from Theorem 1 case (2).

(b) If $f \in H^\infty$ and $\alpha(z) = \frac{b-z}{1-\bar{b}z}$, then $\alpha\circ\varphi\circ\alpha$ is an elliptic disc automorphism of period q, with fixed point 0 and we have M_f commutes with C_φ if and only if $C_\alpha M_f C_\alpha$ commutes with $C_\alpha C_\varphi C_\alpha = C_{\alpha\circ\varphi\circ\alpha}$ if and only if (by Lemma 4) $M_{f\circ\alpha}$ commutes with $C_{\alpha\circ\varphi\circ\alpha}$ if and only if (by Theorem 3) $f\circ\alpha(z) = \sum_{n=0}^{\infty} a_{nq} z^{nq}$ if and only if $f(z) = \sum_{n=0}^{\infty} a_{nq}\left(\frac{b-z}{1-\bar{b}z}\right)^{nq}$. □

Proposition 6. *Let φ be a self-map of $\mathbb{D}$, and let $f \in H^\infty$. Also suppose that φ is neither an elliptic disc automorphism nor the identity mapping, and φ has an interior fixed point. If M_f commutes with C_φ, then f is a constant.*

Proof. Let $a \in \mathbb{D}$ and $\varphi(a) = a$. Since $f\circ\varphi = f$, we have $f(\varphi^{[n]}(z)) = f(z)$ for each $z \in \mathbb{D}$ and all $n \in \mathbb{N}$. From this, we have $f(z) = f(a)$ for all $z \in \mathbb{D}$, because $\{\varphi^{[n]}(z)\}$ converges to a as $n \to \infty$ for every $z \in \mathbb{D}$. □

3 Some properties of the commutant of composition operators on weighted Bergman spaces

In this section, we consider the commutant of composition operator C_φ on A^p_α for $1 < p < \infty$ and $-1 < \alpha < \infty$, where φ is an analytic self-map of $\mathbb{D}$ which is neither an elliptic disc automorphism nor the identity and a constant. Also we assume that $\varphi(a) = a$ for some $a \in \mathbb{D}$.

Lemma 7. *There exists a point z_0 in $\mathbb{D}$ such that the iterates of φ at z_0 are distinct.*

Proof. See [10]. □

Lemma 8. *Let z_0 satisfy the properties of Lemma 7. Then the linear span of reproducing kernels, $\{K_{\varphi^{[n]}(z_0)} : n \geq 0\}$ is dense in A^p_α for $1 < p < \infty$.*

Proof. Let A be the linear span of $\{K_{\varphi^{[n]}(z_0)} : n \geq 0\}$. Suppose that x^* is a bounded linear function on A^p_α for $1 < p < \infty$. If $\frac{1}{p} + \frac{1}{q} = 1$, then there is $g \in A^q_\alpha$ such that $x^* = F_g$ and F_g define by

$$F_g(f) = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z)$$

for each $f \in A^p_\alpha$ (see, e.g., [8]). Hence

$$\begin{aligned} A^\perp &= \{F_g \ : \ F_g(K_{\varphi^{[n]}(z_0)}) = 0 \ \ (\forall n)\} \\ &= \{F_g \ : \ g(\varphi^{[n]}(z_0)) = 0 \ \ (\forall n)\}. \end{aligned}$$

By the Denjoy-Wolff Theorem, the sequence $\{\varphi^{[n]}(z_0)\}_{n=0}^\infty$ has a limit point in $\mathbb{D}$. Then $A^\perp = \{0\}$ and ${}^\perp A^\perp = \overline{A} = A^p_\alpha$, so the proof is complete. □

Proposition 9. *C^*_φ is cyclic.*

Proof. Since $C^*_\varphi(K_{\varphi^{[n]}(z_0)}) = K_{\varphi^{[n+1]}(z_0)}$, by Lemmas 7 and 8, the proof is complete. □

Remark 10. *If the Denjoy-Wolff point of φ is in the boundary of $\mathbb{D}$, then Lemma 8 is not true in general. For example, if $\varphi(z) = az+b$, where $a, b \neq 0$ and $|a|+|b| = 1$, then the sequence $\{\varphi^{[n]}(0)\}_{n=0}^\infty$ has distinct elements and each Blaschke product with zeros $\{\varphi^{[n]}(0)\}_{n=0}^\infty$ is in $A^\perp$. So A is not dense in A^p_α.*

By Lemma 8, we can answer to some questions about the commutant of C_φ.

Theorem 11. *Let S be a bounded operator such that $SC_\varphi = C_\varphi S$ and $S^*K_{z_0} = \sum_{j=0}^m a_j K_{\varphi^{[j]}(z_0)}$ for some z_0 in $\mathbb{D}$ for which $\{\varphi^{[n]}(z_0)\}_{n=0}^\infty$ are distinct. Then S is a polynomial in C_φ.*

Proof. Let $p(z)=\sum_{j=0}^{m} a_j z^j$, we show that $p(C_\varphi^*)=S^*$. By an easy computation, we have $p(C_\varphi^*)K_{z_0}=S^*K_{z_0}$. Let $\epsilon>0$ and $f\in A_\alpha^p$. Since the linear span of $\{K_{\varphi^{[n]}(z_0)}: n\geq 0\}$ is dense in A_α^p, there is $g=\sum_{k=0}^{n} g_k K_{\varphi^{[k]}(z_0)}$ such that

$$\|f-g\|_{p.\alpha}<\epsilon/(1+\|p(C_\varphi^*)-S^*\|).$$

Since $C_{\varphi^{[k]}}^*K_{z_0}=K_{\varphi^{[k]}(z_0)}$, we have

$$\begin{aligned}\|(p(C_\varphi^*)-S^*)f\|_{p.\alpha} &\leq \|(p(C_\varphi^*)-S^*)(f-g)\|_{p.\alpha}+\|(p(C_\varphi^*)-S^*)(g)\|_{p.\alpha}\\ &\leq \epsilon+\|\sum_{k=0}^{n} g_k C_{\varphi^{[k]}}^*(p(C_\varphi^*)-S^*)K_{z_0}\|_{p.\alpha}\\ &= \epsilon.\end{aligned}$$

Hence $p(C_\varphi^*)=S^*$ and so the proof is complete. □

Corollary 12. *Let iterates of φ at zero be distinct, and let S be a bounded operator such that $SC_\varphi=C_\varphi S$ and $S^*(1)=\lambda I$. Then S is a multiple of the identity.*

Proof. Since $K_0=1$, by Theorem 11, we have $S^*=\lambda I$. □

Theorem 13. *Let S be a bounded operator such that $SC_\varphi=C_\varphi S$. Then there is a dense subset on which S can be approximated by polynomials in C_φ.*

Proof. Assume φ and z_0 are as in the Lemma 7 and $S^*K_{z_0}=f$. Since the linear span of $\{K_{\varphi^{[n]}(z_0)}: n\geq 0\}$ is dense in A_α^p, there exists $f_j=\sum_{k=0}^{m_j} a_{j,k}K_{\varphi^{[k]}(z_0)}$ such that $\|f-f_j\|_{p.\alpha}\to 0$ as $j\to\infty$. If $p_j=\sum_{k=0}^{m_j} a_{j,k}z^k$, then we show that $p_j(C_\varphi^*)$ approximate S^* on the linear span of $\{K_{\varphi^{[n]}(z_0)}: n\geq 0\}$. Let $g=\sum_{n=0}^{m} g_n K_{\varphi^{[n]}(z_0)}$. Since $C_{\varphi^{[n]}}^*K_{z_0}=K_{\varphi^{[n]}(z_0)}$ and $S^*C_{\varphi^{[n]}}^*=C_{\varphi^{[n]}}^*S^*$, by an easy computation, we have $S^*g=\sum_{n=0}^{m} g_n C_{\varphi^{[n]}}^* f$ and

$$p_j(C_\varphi^*)g=\sum_{k=0}^{m_j}\sum_{n=0}^{m} a_{j,k}g_n K_{\varphi^{[k+n]}(z_0)}=\sum_{n=0}^{m} g_n C_{\varphi^{[n]}}^* f_j.$$

Since $\{\varphi^{[n]}(0)\}$ converges to the Denjoy-Wolff point in the disc as $n\to\infty$, by using similar arguments as the proof of [9, Theorem 2.3], we have

$$\|C_{\varphi_n}^*\|\leq\left(\frac{1+|\varphi^{[n]}(0)|}{1-|\varphi^{[n]}(0)|}\right)^{\frac{2+\alpha}{p}}\leq b,$$

where b is independent of n on A_α^p and so we have

$$\begin{aligned}\|(S^*-p_j(C_\varphi^*))g\|_{p.\alpha} &\leq \|\sum_{n=0}^{m} g_n C_{\varphi^{[n]}}^*(f-f_j)\|_{p.\alpha}\\ &\leq b\|f-f_j\|_{p.\alpha}\sum_{n=0}^{m}|g_n|,\end{aligned}$$

which converges to zero as $j \to 0$. □

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