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VARIOUS NOTIONS OF AMENABILITY FOR NOT NECESSARILY LOCALLY COMPACT GROUPOIDS

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Abstract. We start with a groupoid G endowed with a family $\mathcal{W}$ of subsets mimicking the properties of a neighborhood basis of the unit space (of a topological groupoid with paracompact unit space). Using the family $\mathcal{W}$ we endow each G-space with a uniform structure. The uniformities of the G-spaces allow us to define various notions of amenability for the G-equivariant maps. As in [1], the amenability of the groupoid G is defined as the amenability of its range map. If the groupoid G is a group, all notions of amenability that we introduce coincide with the classical notion of amenability for topological (not necessarily locally-compact) groups.

1 Introduction

There are several notions of amenability for groupoids. An extensive study of amenability both for measured groupoids and topological locally compact groupoids can be found in [1]. The topological amenability defined in [15] implies the measure-wise amenability (the amenability with respect to all quasi-invariant measures). Moreover J. Renault proved that for locally compact topological groupoid endowed with a (continuous) Haar system the Borel amenability (a notion introduced in [16]) is equivalent to topological amenability. The definition of Borel amenability makes sense for arbitrary Borel groupoids and, in particular, for topological groups. However, in the case of a non locally compact topological group, it is strictly stronger than the classical definition. The notions of amenability that we propose here coincide with the classical notion of amenability for topological groups (which is the existence of a left invariant mean on the space of all right uniformly continuous bounded functions on the group).

Our definition requires two kinds of information about the groupoid G:

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- a family $\mathcal{W}$ of subsets mimicking the properties of a neighborhood basis of the unit space $G^{(0)}$ (of a topological groupoid with paracompact unit space).
- a family of subsets Γ_G of G such that $G \in \Gamma_G$.

For instance if G is a topological space, possible choices for Γ_G are

- $\Gamma_G = \{A \subset G : \ A \text{ open}\}$
- $\Gamma_G = \{A \subset G : \ A \text{ Borel}\}$
- $\Gamma_G = \{A \subset G : \ A\ \mu\text{-measurable}\}$, where μ is a fixed probability measure on G
- $\Gamma_G = \{A \subset G : \ A \text{ universally measurable}\}$

We use the same definition, notation and terminology concerning groupoids as in [4]. Let us state some conventions and facts about measure theory (see [2, Chapter 3]). By a Borel space $(X, \mathcal{B}(X))$ we mean a space X, together with a σ-algebra $B(X)$ of subsets of X, called Borel sets. A subspace of a Borel space $(X, \mathcal{B}(X))$ is a subset $S \subset X$ endowed with the relative Borel structure, namely the σ-algebra of all subsets of S of the form $S \cap E$, where E is a Borel subset of X. $(X, \mathcal{B}(X))$ is called countably separated if there is a sequence $(E_n)_n$ of sets in $\mathcal{B}(X)$ separating the points of X: i.e., for every pair of distinct points of X there is $n \in N$ such that E_n contains one point but not both. A function from one Borel space into another is called Borel function if the inverse image of every Borel set is Borel. A one-one onto function Borel in both directions is called Borel isomorphism. The Borel sets of a topological space are taken to be the σ-algebra generated by the open sets. The Borel space $(X, \mathcal{B}(X))$ is called standard if it is Borel isomorphic to a Borel subset of a complete separable metric space. $(X, \mathcal{B}(X))$ is called analytic if it is countably separated and if it is the image of a Borel function from a standard space. By a measure μ on a Borel space $(X, \mathcal{B}(X))$ we always mean a map $\mu : B(X) \to \overline{\mathbf{R}}$ which satisfies the following conditions:

1. μ is positive ($\mu(A) \geq 0$ for all $A \in B(X)$)
2. $\mu(\emptyset) = 0$
3. μ is countable additive (i.e. $\mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \sum\limits_{n=1}^{\infty} \mu(A_n)$ for all sequences $\{A_n\}_n$ of mutually disjoint sets $A_n \in B(X)$)

Let $(X, \mathcal{B}(X))$ be a Borel space. By a finite measure on X we mean a measure μ with $\mu(X) < \infty$ and by a probability measure a measure with value 1 on X. We denote by ε_x the unit point mass at $x \in X$, i.e. the probability measure on $(X, \mathcal{B}(X))$ such $\varepsilon_x(A) = 1$ if $x \in A$ and $\varepsilon_x(A) = 0$ if $x \notin A$ for any $A \in \mathcal{B}(X)$.

The measure μ is σ-finite if there is a sequence $\{A_n\}_n$ with $A_n \in \mathcal{B}(X)$ for all n, such that $\bigcup_{n=1}^{\infty} A_n = X$ and $\mu(A_n) < \infty$ for all n. A subset of X or a function on X is called μ-measurable (for a σ-finite measure μ) if it is measurable with respect to the completion of μ which is again denoted μ. The complement of a $\mu-$null set (a set A is $\mu-$null if $\mu(A) = 0$) is called $\mu-$conull.

If $(X, \mathcal{B}(X))$ is analytic and μ is a σ-finite measure on $(X, \mathcal{B}(X))$, then there is a Borel subset X_0 of X such that $\mu(X - X_0) = 0$ and such that X_0 is a standard space in its relative Borel structure. Analytic subsets of a countably separated space are universally measurable (i.e. μ-measurable for all finite measures μ).

The measures μ and λ on a Borel space $(X, \mathcal{B}(X))$ are called equivalent measures (and we write $\mu \sim \nu$) if they have the same null sets (i.e. $\mu(A) = 0$ iff $\nu(A) = 0$). Every measure class $[\mu] = \{\nu : \nu \sim \mu\}$ of a σ-finite measure $\mu \neq 0$ contains a probability measure. If $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ are Borel space, $p : X \to Y$ a Borel function and μ a finite measure on $(X, \mathcal{B}(X))$, then by $p_*(\mu)$ we denote the finite measure on $(Y, \mathcal{B}(Y))$ defined by $p_*(\mu)(A) = \mu\left(p^{-1}(A)\right)$ for all $A \in \mathcal{B}(Y)$, and we call it the image of μ by p. It is also possible to define the image $p_*(C)$ of a measure class C of a σ-finite measure as the class of $p_*(\mu)$, where μ is a probability measure in the class C. A pseudo-image by p of a σ-finite measure μ is a measure in $p_*([\mu])$.

We shall not mention explicitly the Borel sets when they result from the context (for instance, in the case of a topological space we shall always consider the σ-algebra generated by the open sets).

2 A neighborhood basis of the unit space of a topological groupoid with paracompact unit space

We record some basic observations about the connection between the topology near the unit space of a topological groupoid and the topology of the fibres. If G is a topological groupoid whose unit space is a T_1-space (the points are closed in $G^{(0)}$), then the topology of the r-fibres, as well as the topology of the d-fibres, is determined by a neighborhood basis $\{W\}_{W \in \mathcal{W}}$ of $G^{(0)}$. Indeed for each $u \in G^0$ and each $x \in G^u$ (respectively, $x \in G_u$), $\{xW\}_{W \in \mathcal{W}}$ (respectively, $\{Wx\}_{W \in \mathcal{W}}$) is a is a neighborhood basis (local basis) for x with respect to the topology induced by G on G^u (respectively, G_u). In order to prove that $\{xW\}_{W \in \mathcal{W}}$ is a neighborhood basis for x with respect to the topology of $G^{r(x)}$, let us notice that:

- Since the map $y \mapsto xy \left[: G^{d(x)} \to G^{r(x)}\right]$ is a homeomorphism, it follows that if D is an open subset of G then $x\left(D \cap G^{d(x)}\right) = xD$ is an open subset of $G^{r(x)}$.
- If D is an open subset of G containing x, then there is an open neighborhood U of x and an open neighborhood V of $d(x)$ such that $UV \subset D$. Thus $xV \subset D \cap G^{r(x)}$.

Since $G^{(0)}$ is a T_1-space, $G\setminus G^{d(x)}$ is open and $V\cup\left(G\setminus G^{d(x)}\right)$ is a neighborhood of $G^{(0)}$. Moreover $x\left(V\cup\left(G\setminus G^{d(x)}\right)\right)\subset D\cap G^{r(x)}$.

Similarly, we can prove that $\{Wx\}_{W\in\mathcal{W}}$ is a neighborhood basis for x with respect to the topology of $G_{d(x)}$.

If G is a topological group, then for each neighborhood V_1 of the identity e there is a neighborhood V_2 of e such that $V_2V_2\subset V_1$. For a topological paracompact groupoid a similar result was proved by Ramsay [14]. However non-Hausdorff groupoids occur in many important examples of foliations such as Reeb foliations. Let us show that the result is also true for (not necessarily Hausdorff) topological groupoids but having paracompact unit space.

Definition 1. *A topological space X is called regular if for any point $x\in X$ and neighborhood V of x, there is a closed neighborhood F of x that is a subset of V.*

Definition 2. *A paracompact space is a topological regular space in which every open cover has an open refinement that is locally finite.*

Proposition 3. *Let G be a topological groupoid whose unit space $G^{(0)}$ is paracompact. Then for each neighborhood W_0 of $G^{(0)}$ there is a symmetric neighborhood W_1 of $G^{(0)}$ such that $W_1W_1\subset W_0$.*

Proof. Let $u\in G^{(0)}$. Since G is topological and $uuu=u$, it follows that there is an open symmetrical set $U_u\subset W_0$ such that $U_uU_uU_u\subset W_0\cap W_0^{-1}$. Since $G^{(0)}$ is paracompact and regular, $\left\{U_u\cap G^{(0)}\right\}_{u\in G^{(0)}}$ has a closed locally finite refinement $\{K\}_{K\in\mathcal{I}}$ [10, Lemma 29/p. 157]. For each $K\in\mathcal{I}$, there is $U_K\in\{U_u\}_{u\in G^{(0)}}$ such that $K\subset U_K\cap G^{(0)}$, and consequently, $U_KKU_K\subset W_0\cap W_0^{-1}$. Let

$$W_K=U_K\cup\left(W_0\cap W_0^{-1}\setminus\left(r^{-1}(K)\cup d^{-1}(K)\right)\right)=(W_K)^{-1}\subset W_0\cap W_0^{-1}$$

and

$$W_1=\bigcap_{K\in\mathcal{I}}W_K.$$

Let $(x,y)\in(W_1\times W_1)\cap G^{(2)}$. There is $K\in\mathcal{I}$ such that $d(x)=r(y)\in K$, and consequently, $x,y\in U_K$. Hence $xy=xd(x)y\in W_0\cap W_0^{-1}$. Therefore $W_1W_1\subset W_0\cap W_0^{-1}$.

Let $u\in G^{(0)}$ and let us prove that u is in the interior of W_1 (with respect to the topology on G). Let $V_u\subset G^{(0)}$ be a neighborhood of u (with the respect to the topology induced on $G^{(0)}$) that intersects only finitely many of the sets $K\in\mathcal{I}$. Let $J_u\subset\mathcal{I}$ be the collection of the sets K that intersect V_u and let $D_u=\bigcap\limits_{K\in J_u}W_K$. Then D_u is a neighborhood of u with respect to the topology on G. Let us notice that

$$\left(\bigcap_{K\in\mathcal{I}\setminus J_u}W_K\right)\cap r^{-1}(V_u)\cap d^{-1}(V_u)=W_0\cap W_0^{-1}\cap r^{-1}(V_u)\cap d^{-1}(V_u),$$

is also a neighborhood of u with respect to the topology on G. Therefore

$$D_u \cap \left(\bigcap_{K \in \mathcal{I} \setminus J_u} W_K \right) \cap r^{-1}(V_u) \cap d^{-1}(V_u) \subset \left(\bigcap_{K \in \mathcal{I}_u} W_K \right) \cap r^{-1}(V_u) \cap d^{-1}(V_u)$$

is also a neighborhood of u (with respect to the topology on G) contained in W_1. $\square$

In this paper we work with a collection of subsets of a groupoid mimicking the properties of a neighborhood basis of the unit space (of a topological groupoid with paracompact unit space). Let us consider that $\{W\}_{W \in \mathcal{W}}$ is a family of subsets of G satisfying

- $G^{(0)} \subset W \subset G$ for all $W \in \mathcal{W}$;
- If W_1, $W_2 \in \mathcal{W}$, then there is $W_3 \subset W_1 \cap W_2$ such that $W_3 \in \mathcal{W}$;
- For every $W_1 \in \mathcal{W}$ there is $W_2 \in \mathcal{W}$ such that $W_2 W_2 \subset W_1$.

Then there is a topology $\tau_{\mathcal{W}}^r$ (respectively, $\tau_{\mathcal{W}}^d$) on G such that for all $x \in G$, $\mathcal{V}^r(x)$ (respectively, $\mathcal{V}^d(x)$) is a neighborhood basis (local basis) for x, where

$$\mathcal{V}^r(x) = \{V \subset G : \text{ there is } W \in \mathcal{W} \text{ such that } xW \subset V\}.$$

respectively,

$$\mathcal{V}^d(x) = \{V \subset G : \text{ there is } W \in \mathcal{W} \text{ such that } Wx \subset V\}.$$

Indeed, it is enough to prove that all $V \in \mathcal{V}^r(x)$ (respectively, $V \in \mathcal{V}^d(x)$) there is $U \in \mathcal{V}^r(x)$ (respectively, $U \in \mathcal{V}^d(x)$) such that $V \in \mathcal{V}^r(y)$ (respectively, $V \in \mathcal{V}^d(y)$) for all $y \in U$. Since $V \in \mathcal{V}^r(x)$ (respectively, $V \in \mathcal{V}^d(x)$), it follows that there is $W_1 \in \mathcal{W}$ such that $xW_1 \subset V$ (respectively, $W_1 x \subset V$). There is $W_2 \in \mathcal{W}$ such that $W_2 W_2 \subset W_1$. If we take $U = xW_2$ (respectively, $U = W_2 x$), then for all $y \in U$ there is $z_y \in W_2 \cap G^{d(x)}$ (respectively, $z_y \in W_2 \cap G_{r(x)}$) such that $y = xz_y$ (respectively, $y = z_y x$) and

$$yW_2 = xz_y W_2 \subset xW_2 W_2 \subset xW_1,$$

respectively,

$$W_2 y = W_2 z_y x \subset W_2 W_2 x \subset W_1 x.$$

Definition 4. *Let G be a groupoid and $\{W\}_{W \in \mathcal{W}}$ be a collection of subsets of G. Let us consider the following conditions:*

1. *$G^{(0)} \subset W \subset G$ for all $W \in \mathcal{W}$.*
2. *If W_1, $W_2 \in \mathcal{W}$, then there is $W_3 \subset W_1 \cap W_2$ such that $W_3 \in \mathcal{W}$.*

3. *$W = W^{-1}$for all $W \in \mathcal{W}$.*

4. *For every $W_1 \in \mathcal{W}$ there is $W_2 \in \mathcal{W}$ such that $W_2W_2 \subset W_1$.*

5. *For every $W_1 \in \mathcal{W}$ and $x \in G$ there is $W_2 \in \mathcal{W}$ such that $W_2 \cap G_{d(x)}^{d(x)} \subset x^{-1}W_1x$ (or equivalently, $xW_2x^{-1} \subset W_1$).*

Let us notice that if $\{W\}_{W\in\mathcal{W}}$ satisfies conditions $1, 2, 4$ and 5 from Definition 4, then the multiplication on G is continuous with respect to $\tau_{\mathcal{W}}^r$ (respectively, $\tau_{\mathcal{W}}^d$). Indeed by 4, for all $W \in \mathcal{W}$, there is $W_1 \in \mathcal{W}$ such that $W_1W_1 \subset W$ and by 5, for all $y \in G$ there is $W_y \in \mathcal{W}$ such that $W_y \cap G_{r(y)}^{r(y)} \subset yW_1y^{-1}$. If $x \in G_{r(y)}$, then

$$xW_yyW_1 = xyy^{-1}W_yyW_1 \subset xyW_1W_1 \subset xyW,$$

Similarly, the multiplication on G is continuous with respect to $\tau_{\mathcal{W}}^d$. However the inversion is not necessary continuous with respect to $\tau_{\mathcal{W}}^r$ or $\tau_{\mathcal{W}}^d$.

If for every $x \notin G^{(0)}$ there is $W \in \mathcal{W}$ such that $x \notin W$, then topology induced by $\tau_{\mathcal{W}}^r$ on r-fibres (respectively, by $\tau_{\mathcal{W}}^d$ on d-fibres) is Hausdorff.

In the following we use a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4 to define various uniform structures. Let us first recall basic terminology from uniform spaces.

A uniform space $(S,\mathcal{U})$ is a set S equipped with a nonempty family $\mathcal{U}$ of subsets of the Cartesian product $S \times S$ ($\mathcal{U}$ is called the uniform structure or uniformity of S and its elements entourages) that satisfy the following conditions:

1. if U is in $\mathcal{U}$, then U contains the diagonal $\Delta = \{(s, s) : s \in S\}$.

2. if U is in $\mathcal{U}$ and V is a subset of $S \times S$ which contains U, then $V \in \mathcal{U}$.

3. if U and V are in $\mathcal{U}$, then $U \cap V \in \mathcal{U}$

4. if U is in $\mathcal{U}$, then there exists V in $\mathcal{U}$ such that, whenever (s_1, s_2) and (s_2, s_3) are in V, then $(s_1, s_3) \in U$.

5. if $U \in \mathcal{U}$, then $U^{-1} = \{(t, s) : (s, t) \in U\}$ is also in $\mathcal{U}$

One usually writes $U[s] = \{t : (s, t) \in U\}$ for $U \in \mathcal{U}$ and $s \in S$. Then there is a topology (associated to the uniformity $\mathcal{U}$) on S such that for all $s \in S$

$$\{U[s] : U \in \mathcal{U}\}$$

is a neighborhood basis for s.

A fundamental system of entourages of a uniformity $\mathcal{U}$ is any set $\mathcal{B}$ of entourages of $\mathcal{U}$ such that every entourage of $\mathcal{U}$ contains a set belonging to $\mathcal{B}$. Thus a fundamental systems of entourages $\mathcal{B}$ is enough to specify the uniformity $\mathcal{U}$ unambiguously: $\mathcal{U}$

is the set of subsets of $S \times S$ that contain a set of $\mathcal{B}$. Every uniform space has a fundamental system of entourages consisting of symmetric entourages.

If $\mathcal{W}$ satisfies conditions $1-4$ from Definition 4, then $\{(r,d)(W)\}_{W\in\mathcal{W}}$ is a fundamental system of symmetric entourages of a uniformity on $G^{(0)}$. Thus induces a topology on $G^{(0)}$.

Let us consider the trivial groupoid $G = X \times X$ on a set X. $\mathcal{W}$ satisfies conditions $1-4$ from Definition 4 if and only if $\mathcal{W}$ is a fundamental system of symmetric entourages of a uniformity on X. Condition 5 is automatically satisfied.

Let us reformulate the definition of uniform continuity [3, Definition 3.1/p. 39] in the setting of a groupoid endowed with a family of subsets satisfying the conditions from Definition 4.

Definition 5. *Let G be a groupoid, $\mathcal{W}$ be a family of subsets of G satisfying conditions $1-5$ from Definition 4, $A \subset G$ and E be a Banach space. The function $h : A \to E$ is said to be left uniformly continuous on fibres if and only if for each $\varepsilon > 0$ there is $W_\epsilon \in \mathcal{W}$ such that:*

$$\|h(x) - h(xy)\| < \varepsilon \ \textit{ for all } y \in W_\varepsilon \textit{ and } x \in A \cap G_{r(y)} \textit{ such that } xy \in A$$

The function $h : A \to E$ is said to be right uniformly continuous on fibres if and only if for each $\varepsilon > 0$ there is $W_\epsilon \in \mathcal{W}$ such that:

$$\|h(x) - h(yx)\| < \varepsilon \ \textit{ for all } \ y \in W_\varepsilon \textit{ and } x \in A \cap G^{d(y)} \textit{ such that } yx \in A.$$

If $h : A \to E$ is left (respectively, right) uniformly continuous on fibres, then h is continuous with respect to the topology induced by $\tau^r_{\mathcal{W}}$ (respectively, $\tau^d_{\mathcal{W}}$) on A.

If $f, g : G \to \mathbb{C}$ are left uniformly, respectively right uniformly on fibres, then $|f|$, $\overline{f}$, $f+g$ are left uniformly, respectively right uniformly continuous on fibres. If $f, g : G \to \mathbb{C}$ are left uniformly, respectively right uniformly continuous on fibres bounded functions, then fg is a left uniformly, respectively right uniformly continuous on fibres bounded function.

3 Invariant systems of means for equivariant maps

In order to fix notation let us recall the notions of groupoid action and semi-direct product.

Definition 6. *Let G be a groupoid. A set S is said to be a (left) G-space if G acts on S (to the left).*

We say G acts (to the left) on S if there is a map $\rho : S \to G^{(0)}$ (called a momentum map) and a map $(x, s) \mapsto x \cdot s$ from

$$G_d *_\rho S = \{(x, s) : d(x) = \rho(s)\}$$

to S, called (left) action, such that:

1. *$\rho(x\cdot s)=r(x)$ for all $(\gamma,x)\in G*_{\rho}S$.*

2. *$\rho(s)\cdot s=s$ for all $s\in S$.*

3. *If $(x,y)\in G^{(2)}$ and $(y,s)\in G*_{\rho}S$, then $(xy)\cdot s=x\cdot(y\cdot s)$.*

In the same manner, we define a right action of G on S, using a map $\sigma:S\to G^{(0)}$ and a map $(s,x)\mapsto s\cdot x$ from

$$S_{\sigma}*_{r}G=\{(s,x):\sigma(s)=r(x)\}$$

to S.

The simplest example of a left (or right) G-space is the case when the groupoid G acts upon itself by either left (or right) translation (multiplication). Also $G^{(0)}$ can be seen as a left, respectively, right G-space under the action $(x,u)\mapsto r(x)$, respectively, $(u,x)\mapsto d(x)$ from

$$G_{d}*_{id}G^{(0)}=\{(x,u):d(x)=u\}\text{, respectively, }G_{id}^{(0)}*_{r}G=\{(u,x):r(x)=u\}.$$

to $G^{(0)}$.

Let us notice that if S is a left G-space, then

$$S_{\rho}*_{r}G=\{(s,x):\rho(s)=r(x)\}$$

has a groupoid structure (called semi-direct product) with the following operations

$$\begin{aligned}(s,x)\left(x^{-1}\cdot s,y\right)&=(s,xy)\\ (s,x)^{-1}&=\left(x^{-1}\cdot s,x^{-1}\right).\end{aligned}$$

When $S_{\rho}*_{r}G$ is viewed as a groupoid, it will be denoted by $S\rtimes G$. The unit space of $S\rtimes G$ will be identified with S. If S is a left G-space, then the momentum map of the action of G on S will be denoted ρ_{S}.

In the following we shall assume that the groupoid G is endowed with a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4. Obviously every groupoid can be endowed with such a family (for instance, we can take $\mathcal{W}=\left\{G^{(0)}\right\}$). However the topology $\tau_{\mathcal{W}}^{r}$ (respectively, $\tau_{\mathcal{W}}^{d}$) induced on the r-fibres (respectively, d-fibres) by $\mathcal{W}=\left\{G^{(0)}\right\}$ is the discrete topology.

According to a result of Ramsay, Mackey's groupoids [11] may be assume to have locally compact topologies. More precisely, a Mackey's groupoid G [11] has an inessential reduction G_0 which has a locally compact metric topology in which it is a topological groupoid [13]. Thus G_0 can be endowed with a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4 and such that the topology $\tau_{\mathcal{W}}^{r}$ (respectively, $\tau_{\mathcal{W}}^{d}$) coincides on the r-fibres (respectively, d-fibres) with the topology coming from G_0.

Proposition 7. *Let G be a groupoid endowed with a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4. If S is a left G-space then the family*

$$\{S_{\rho_S} *_r W : \ W \in \mathcal{W}\}$$

satisfies conditions $1-5$ from Definition 4 with respect to the groupoid $S \rtimes G$. Hence

$$\{(r,d)\,(S_{\rho_S} *_r W) : \ W \in \mathcal{W}\}$$

is a fundamental system of symmetric entourages of a uniformity on S.

Proof. Let us check condition 5, for instance. Let $W_1 \in \mathcal{W}$ and $(s,x) \in S \rtimes G$. Then there is $W_2 \in \mathcal{W}$ such that $W_2 \cap G_{d(x)}^{d(x)} \subset x^{-1}W_1 x$. Let $(s', y) \in (S_{\rho_S} *_r W_2) \cap (S \rtimes G)_{d(s,x)}^{d(s,x)}$. Then $s' = x^{-1} \cdot s$, $\ x^{-1} \cdot s = y^{-1} \cdot \left(x^{-1} \cdot s\right)$ and $y \in W_2 \cap G_{d(x)}^{d(x)}$. Since $W_2 \cap G_{d(x)}^{d(x)} \subset x^{-1}W_1 x$, it follows that there is $z \in W_1$ such that $y = x^{=1}zx$. Moreover since $z = xyx^{-1}$ and $x^{-1} \cdot s = y^{-1} \cdot \left(x^{-1} \cdot s\right)$, it follows that

$$z^{-1} \cdot s = \left(xy^{-1}x^{-1}\right) \cdot s = x \cdot \left(y^{-1} \cdot \left(x^{-1} \cdot s\right)\right) = x \cdot \left(x^{-1} \cdot s\right) = s.$$

Consequently,

$$\begin{aligned} \left(s', y\right) &= \left(x^{-1} \cdot s, x^{-1}zx\right) = \left(x^{-1} \cdot s, x^{-1}\right)(s,z)\left(z^{-1} \cdot s, x\right) \\ &= \left(x^{-1} \cdot s, x^{-1}\right)(s,z)(s,x) \in \left(x^{-1} \cdot s, x^{-1}\right)(S_{\rho_S} *_r W_1)(s,x). \end{aligned}$$

□

Definition 8. *Let G be a groupoid endowed with a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4 and let S be a left G-space. Then any subset A of S is a uniform space with respect to the uniformity induced from S. The family*

$$\{(A \times A) \cap (r,d)\,(S_{\rho_S} *_r W) : \ W \in \mathcal{W}\}$$

is a fundamental system of symmetric entourages of the uniformity on A. Let us denote by $RCUB\,(A)$ the space of uniformly continuous bounded functions $f : A \to \mathbb{C}$ with respect to above uniformity on A. A function $f \in RCUB\,(A)$ will be called right uniformly continuous bounded function on A (with respect to the action of G on S).

If $W \in \mathcal{W}$, then

$$(r,d)\,(S_{\rho_S} *_r W) = \left\{\left(s, x^{-1} \cdot s\right) : s \in S, \ x \in W, \ \rho_S(s) = r(x)\right\}.$$

Thus a function $f : A \to \mathbb{C}$ belongs to $RCUB\,(A)$ if and only if f is bounded and for each $\varepsilon > 0$ there is $W_\varepsilon \in \mathcal{W}$ such that

$\left|f(s) - f\left(x^{-1} \cdot s\right)\right| < \varepsilon$ for all $s \in A$ and all $x \in W_\varepsilon \cap G^{\rho_S(s)}$ satisfying $x^{-1} \cdot s \in A$.

Remark 9. *Let G be a groupoid endowed with a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4 and let S be a left G-space. Then the family*

$$\{S_{\rho_S} *_r W : \ W \in \mathcal{W}\}$$

satisfies conditions $1-5$ from Definition 4 with respect to the groupoid $S \rtimes G$. Thus the family

$$\{S_{\rho_S} *_r W : \ W \in \mathcal{W}\}$$

defines a uniformity on $S \rtimes G$ viewed as a left $S \rtimes G$-space. On the other hand the space

$$S_{\rho_S} *_r G = \{(s,x) : r(x) = \rho_S(t)\} = S \rtimes G$$

can be seen as a left G-space under the action

$$x \cdot (s,y) = (x \cdot s, xy)$$

with momentum map $(s,x) \mapsto r(x)$. The uniformity defined by the action of G on $S \rtimes G$ coincides with the uniformity defined by the action of $S \rtimes G$ on itself by multiplication.

Proposition 10. *Let G be a groupoid endowed with a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4 and let $A \subset G$. If G is seen a left G-space (G acting on G by multiplication), then a function $f \in RCUB(A)$ if and only if f is bounded and f is right uniformly continuous on A in the sense of Definition 5.*

Proof. As we have remarked $f \in RCUB(A)$ if and only if f is bounded and for each $\varepsilon > 0$ there is $W_\varepsilon \in \mathcal{W}$ such that

$$\left|f(y) - f\left(x^{-1}y\right)\right| < \varepsilon \text{ for all } y \in A \text{ and all } x \in W_\varepsilon \cap G^{r(y)} \text{ satisfying } x^{-1}y \in A.$$

This is means that f is right uniformly continuous on A in the sense of Definition 5. □

Definition 11. *Let G be a groupoid and let T and S be two left G-spaces. A map $\pi : T \to S$ is said to be G-equivariant if the following conditions are satisfied*

1. *$\rho_S(\pi(t)) = \rho_T(t)$ for all $t \in T$*
2. *$\pi(x \cdot t) = x \cdot \pi(t)$ for all $(x,t) \in G_d *_{\rho_T} T$.*

If $s \in S$ and $x \in G^{\rho_S(s)}$, then for each function $f : \pi^{-1}(\{s\}) \to \mathbb{C}$ the left translate of f by x with respect to π is the function $f^{\pi}_{(s,x)} : \pi^{-1}\left(\left\{x^{-1} \cdot s\right\}\right) \to \mathbb{C}$ defined by

$$f^{\pi}_{(s,x)}(t) = f(x \cdot t) \ \text{ for all } t \in \pi^{-1}\left(\left\{x^{-1} \cdot s\right\}\right).$$

(Obviously, the equivariance of π guarantees the fact that $f^{\pi}_{(s,x)}$ is correctly defined.)

In particular, for the equivariant map $\rho_T : T \to G^{(0)}$ (where $G^{(0)}$ is considered a left G-space under the action $x \cdot d(x) = r(x)$) we use the notation f_x for $f^{\rho_T}_{(r(x),x)}$.

Proposition 12. *Let G be a groupoid endowed with a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4 and let T and S be two left G-spaces. Let $\pi : T \to S$ be G-equivariant map and $s \in S$. If $f \in RCUB\left(\pi^{-1}(\{s\})\right)$, then $f^{\pi}_{(s,x)} \in RCUB\left(\pi^{-1}\left(\left\{x^{-1}\cdot s\right\}\right)\right)$ for all $x \in G^{\rho_S(s)}$.*

Proof. Let $\varepsilon > 0$. Since $f \in RCUB\left(\pi^{-1}(\{s\})\right)$, it follows that there is $W_\varepsilon \in \mathcal{W}$ such that

$$\left|f(t) - f\left(y^{-1}\cdot t\right)\right| < \varepsilon$$

for all $t \in \pi^{-1}(\{s\})$ and all $y \in W_\varepsilon \cap G^{\rho_T(t)}$ satisfying $y^{-1} \cdot s = s$. Let $W'_\varepsilon \in \mathcal{W}$ be such that $W'_\varepsilon \cap G^{d(x)}_{d(x)} \subset x^{-1}W_\varepsilon x$ (or equivalently, $xW'_\varepsilon x^{-1} \subset W_\varepsilon$). Since $x \cdot t \in \pi^{-1}(\{s\})$ for all $t \in \pi^{-1}\left(\left\{x^{-1}\cdot s\right\}\right)$, it follows that

$$\left|f(x\cdot t) - f\left(x \cdot \left(y^{-1}\cdot t\right)\right)\right| = \left|f(x\cdot t) - f\left(\left(xy^{-1}x^{-1}\right)\cdot(x\cdot t)\right)\right| < \varepsilon$$

for all $t \in \pi^{-1}\left(\left\{x^{-1}\cdot s\right\}\right)$ and all $y \in W'_\varepsilon \cap G^{\rho_T(t)}$ satisfying $y^{-1}\cdot\left(x^{-1}\cdot s\right) = x^{-1}\cdot s$. $\square$

Definition 13. *Let G be a groupoid endowed with a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4. Let T and S be two left G-spaces, and let $\pi : T \to S$ be G-equivariant map. A G-invariant π-system of means (with respect to $\mathcal{W}$) (or a G-invariant system of means for π) is a family $\{m^s, \in S\}$ of states m^s on $RCUB\left(\pi^{-1}(\{s\})\right)$ such that for all $\varphi \in RCUB\left(\pi^{-1}(\{s\})\right)$*

$$m^s(\varphi) = m^{x^{-1}\cdot s}\left(\varphi^{\pi}_{(s,x)}\right) \text{ for all } (s,x) \in S_{\rho_s} *_r G.$$

In the preceding definition by a state m^s on $RCUB\left(\pi^{-1}(\{s\})\right)$ we mean a linear map m^s: $RCUB\left(\pi^{-1}(\{s\})\right) \to \mathbb{C}$ that is positive ($m^s(f) \geq 0$ for $f \geq 0$) and such that $m^s(1) = 1$. Thus m^s is continuous with respect to sup-norm.

If the groupoid G is a group then a G-invariant system of means for the map $G \to \{1\}$ (where 1 is the unity of G) is in fact a left invariant mean on $RUCB(G)$. Thus the existence of a G-invariant system of means for the map $G \to \{1\}$ is equivalent in this case to the amenability of the group G seen as a topological group with the topology defined by $\mathcal{W}$ (as neighborhood basis of the unity).

If G is a principal groupoid (seen as the graph of an equivalence relation $G \subset X \times X$) then a G-invariant system of means (with respect to $\mathcal{W}$) for the G-equivariant map $r : G \to X$ (r is the first projection) in the sense of the preceding definition is in fact a family $\{m^x, \in X\}$ of states m^x on $l^\infty([x])$ (the space of bounded function

$\varphi : [x] \to \mathbb{C}$ on the class $[x]$ of x) such that $m^x = m^y$ for all $y \in [x]$. Usually we write $m^{[x]} = m^y$ for all $y \in [x]$.

For each equivalence relation $G \subset X \times X$, the map $r : G \to X$ admits G-invariant system of means. Indeed, let σ be a section of the canonical quotient map $p : X \to X/G$ (this means $p \circ \sigma = id_X$) and let us define

$$m^x(\varphi) = \varphi(x, \sigma(p(x))),\ x \in X,\ \varphi \in l^\infty(\{x\} \times [x]) = l^\infty([x]).$$

Then $\{m^x, \in X\}$ G-invariant system of means for the map $r : G \to X$.

Proposition 14. *Let G be a groupoid endowed with a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4. Let T and S be two left G-spaces, and let $\{m^s, \in S\}$ be a G-invariant system of means for the G-equivariant map $\pi : T \to S$. If $A \subset S$ and $\varphi \in RCUB\left(\pi^{-1}(A)\right)$, then the map $m(\varphi) : A \to \mathbb{C}$ defined by*

$$m(\varphi)(s) = m^s\left(\varphi|_{\pi^{-1}(\{s\})}\right) \text{ for all } s \in A$$

is a right uniformly continuous bounded function on A (i.e. $m(\varphi) \in RCUB(A)$).

Proof. Since $\varphi \in RCUB\left(\pi^{-1}(A)\right)$, it follows that there is $W_\varepsilon \in \mathcal{W}$ such that

$$\left|\varphi(t) - \varphi\left(y^{-1} \cdot t\right)\right| < \varepsilon$$

for all $t \in \pi^{-1}(A)$ and all $y \in W_\varepsilon \cap G^{\rho_T(t)}$ satisfying $y^{-1} \cdot t \in \pi^{-1}(A)$. Therefore for all $s \in A$ and all $x \in W_\varepsilon \cap G^{\rho_S(s)}$ such that $x^{-1} \cdot s \in A$, we have

$$\begin{aligned}
\left|m(\varphi)(s) - m(\varphi)\left(x^{-1} \cdot s\right)\right| &= \left|m^{x^{-1}\cdot s}\left(\varphi^\pi_{(s,x)}|_{\pi^{-1}(\{x^{-1}\cdot s\})}\right) - m^{x^{-1}\cdot s}\left(\varphi|_{\pi^{-1}(\{x^{-1}\cdot s\})}\right)\right| \\
&= \left|m^{x^{-1}\cdot s}\left(\left(\varphi^\pi_{(s,x)} - \varphi\right)|_{\pi^{-1}(\{x^{-1}\cdot s\})}\right)\right| \\
&\le \sup_{t \in \pi^{-1}(\{x^{-1}\cdot s\})} |\varphi(x \cdot t) - \varphi(t)| \\
&< \varepsilon.
\end{aligned}$$

□

4 Amenable equivariant maps

Definition 13 does not use any information on the additional structure of the G-spaces. If we assume that these spaces are endowed with additional structures (such as topologies or σ-algebras) then the map

$$s \mapsto m^s$$

should be compatible with those structures. We use an approach similar to that in [9].

Definition 15. *If S is a set endowed with a family of subsets $\Gamma_S \subset \mathcal{P}(S)$ such that $S \in \Gamma_S$, then (S, Γ_S) will be called "measurable" space.*

If A is a subset of a "measurable" space (S, Γ_S), then A can be seen as a "measurable" space endowed with $\Gamma_A = \{A \cap X : X \in \Gamma_S\}$.

Let (T, Γ_T) and (S, Γ_S) be two "measurable" spaces.

- A function $f : T \to S$ is said to be (Γ_T, Γ_S)-"measurable" if $f^{-1}(A) \in \Gamma_T$ for all $A \in \Gamma_S$.

- $T \times S$ will be always endowed with a family of subsets $\Gamma_{T\times S}$ with the property that for all $X \in \Gamma_T$ and $Y \in \Gamma_S$ we have $X \times Y \in \Gamma_{T\times S}$.

If (T, Γ_T) is a "measurable" space, then a function $f : T \to \mathbb{C}$ is said to be Γ_T-"measurable" if $f^{-1}(A) \in \Gamma_T$ for all open sets $A \subset \mathbb{C}$ ($\mathbb{C}$ is endowed with the usual topology).

For instance if T is a topological space, possible choices for Γ_T are

- $\Gamma_T = \{A \subset T : A \text{ open}\}$
- $\Gamma_T = \{A \subset T : A \text{ Borel}\}$
- $\Gamma_T = \{A \subset T : A\ \mu\text{-measurable}\}$, where μ is a fixed probability measure on T
- $\Gamma_T = \{A \subset T : A \text{ universally measurable}\}$

Definition 16. *Let G be a groupoid endowed with a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4. Let (T, Γ_T) and (S, Γ_S) be two left "measurable" G-spaces, and let $\pi : T \to S$ be a G-equivariant map. An invariant π-system of means $\{m^s, \in S\}$ (with respect to $\mathcal{W}$) is said to be (Γ_T, Γ_S)-"measurable" if for all bounded Γ_T-measurable maps $\varphi : T \to \mathbb{C}$ with the property that $\varphi|_{\pi^{-1}(\{s\})} \in RCUB\left(\pi^{-1}(\{s\})\right)$ for all $s \in S$, the map*

$$s \mapsto m^s\left(\varphi|_{\pi^{-1}(\{s\})}\right) [: S \to \mathbb{C}]$$

is Γ_S-"measurable".

The G-equivariant map $\pi : T \to S$ is said to be a (Γ_T, Γ_S)-amenable map (with respect to $\mathcal{W}$) if there is a (Γ_T, Γ_S)-"measurable" invariant π-system of means (with respect to $\mathcal{W}$).

Proposition 17. *Let G be a groupoid endowed with a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4 and let (S_1, Γ_{S_1}), (S_2, Γ_{S_2}) and $(S_3, , \Gamma_{S_3})$ be three left G-spaces. If $\pi_1 : S_1 \to S_2$, respectively $\pi_2 : S_2 \to S_3$, is a G-equivariant $(\Gamma_{S_1}, \Gamma_{S_2})$-amenable, respectively $(\Gamma_{S_2}, \Gamma_{S_3})$-amenable map, then $\pi_2 \circ \pi_1 : S_1 \to S_3$ is a G-equivariant $(\Gamma_{S_1}, \Gamma_{S_3})$-amenable map.*

Proof. Let $\{m_1^s,\, s \in S_2\}$, respectively $\{m_2^s,\, s \in S_3\}$, be an invariant π_1-system, respectively π_2-system, of means.

For each $s \in S_3$ and each $\varphi \in RCUB\left((\pi_2 \circ \pi_1)^{-1}(\{s\})\right)$ let us define $\tilde{m}_s(\varphi) : \pi_2^{-1}(\{s\}) \to \mathbb{C}$ by $\tilde{m}_s(\varphi)(t) = m_1^t\left(\varphi|_{\pi_1^{-1}(\{t\})}\right)$ for all $t \in \pi_2^{=1}(\{s\})$. According to Proposition 14 $\tilde{m}_s(\varphi) \in RCUB\left(\pi_2^{-1}(\{s\})\right)$. Thus for each $s \in S_3$ and each $\varphi \in RCUB\left((\pi_2 \circ \pi_1)^{=1}(\{s\})\right)$ we can define

$$m^s(\varphi) = m_2^s(\tilde{m}_s(\varphi)).$$

Obviously, m^s is a state on $RCUB\left((\pi_2 \circ \pi_1)^{=1}(\{s\})\right)$. For each $(s, x) \in S_{3\rho_{S_3}} *_r G$ and $\varphi \in RCUB\left((\pi_2 \circ \pi_1)^{=1}(\{s\})\right)$ the left translate of φ by x with respect to $\pi_2 \circ \pi_1$ is the function $\varphi_{(s,x)}^{\pi_2\circ\pi_1} : (\pi_2 \circ \pi_1)^{-1}\left(\{x^{=1} \cdot s\}\right) \to \mathbb{C}$ defined by

$$\varphi_{(s,x)}^{\pi_2\circ\pi_1}(t) = \varphi(x \cdot t) \text{ for all } t \in (\pi_2 \circ \pi_1)^{-1}\left(\{x^{=1} \cdot s\}\right).$$

Let us denote by $\tilde{m}_{(s,x)}(\varphi)$ the left translate of $\tilde{m}_s(\varphi)$ by x with respect to π_2, i.e. the function $\tilde{m}_{(s,x)}(\varphi) : \pi_2^{-1}\left(\{x^{=1} \cdot s\}\right) \to \mathbb{C}$ defined by

$$\begin{aligned}
\tilde{m}_{(s,x)}(\varphi)(t) &= \tilde{m}_s(\varphi)(x \cdot t) \\
&= m_1^{x\cdot t}\left(\varphi|_{\pi_1^{-1}(\{x\cdot t\})}\right) \\
&= m_1^{x^{=1}\cdot(x\cdot t)}\left(\left.\varphi_{(s,x)}^{\pi_2\circ\pi_1}\right|_{\pi_1^{-1}(\{t\})}\right) \\
&= m_1^{t}\left(\left.\varphi_{(s,x)}^{\pi_2\circ\pi_1}\right|_{\pi_1^{-1}(\{t\})}\right) \\
&= \tilde{m}_{x.^{=1}\cdot s}\left(\varphi_{(s,x)}^{\pi_2\circ\pi_1}\right)(t) \text{ for all } t \in \pi_2^{-1}\left(\{x.^{=1} \cdot s\}\right).
\end{aligned}$$

Hence for each $(s, x) \in S_{3\rho_{S_3}} *_r G$ and $\varphi \in RCUB\left((\pi_2 \circ \pi_1)^{=1}(\{s\})\right)$ we have

$$\begin{aligned}
m^s(\varphi) &= m_2^s(\tilde{m}_s(\varphi)) = m_2^{x^{=1}\cdot s}\left(\tilde{m}_{(s,x)}(\varphi)\right) \\
&= m_2^{x^{=1}\cdot s}\left(\tilde{m}_{x^{=1}\cdot s}\left(\varphi_{(s,x)}^{\pi_2\circ\pi_1}\right)\right) = m^{x^{=1}\cdot s}\left(\varphi_{(s,x)}^{\pi_2\circ\pi_1}\right).
\end{aligned}$$

Let $\varphi : S_1 \to \mathbb{C}$ be a bounded Γ_{S_1}-"measurable" map with the property that $\varphi|_{(\pi_2\circ\pi_1)^{=1}(\{s\})} \in RCUB\left((\pi_2 \circ \pi_1)^{=1}(\{s\})\right)$ for all $s \in S_3$. Let us define $\tilde{m}(\varphi) : S_2 \to \mathbb{C}$ by $\tilde{m}(\varphi)(s) = m_1^s\left(\varphi|_{\pi_1^{-1}(\{s\})}\right)$ for all $s \in S_2$. Since $\{m_1^s, \in S_2\}$ is $(\Gamma_{S_1}, \Gamma_{S_2})$-"measurable", if follows that $\tilde{m}(\varphi)$ is Γ_{S_2}-"measurable". On the other hand for all $s \in S_3$

$$m^s\left(\varphi|_{(\pi_2\circ\pi_1)^{=1}(\{s\})}\right) = m_2^s\left(\tilde{m}(\varphi)|_{\pi_2^{=1}(\{s\})}\right).$$

and $\{m_2^s, \in S_3\}$ is $(\Gamma_{S_2}, \Gamma_{S_3})$-"measurable". Thus

$$s \mapsto m^s\left(\varphi|_{(\pi_2\circ\pi_1)^{-1}(\{s\})}\right) [: S \to \mathbb{C}]$$

is Γ_{S_3}-"measurable". □

Proposition 18. *Let G be a groupoid endowed with a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4 and let (S_1, Γ_{S_1}), (S_2, Γ_{S_2}) and $(S_3, , \Gamma_{S_3})$ be three left G-spaces. If $\pi_1 : S_1 \to S_2$ and $\pi_2 : S_2 \to S_3$ are G-equivariant such that $\pi_2 \circ \pi_1$ is $(\Gamma_{S_1}, \Gamma_{S_3})$-amenable and if π_1 is $(\Gamma_{S_1}, \Gamma_{S_2})$-"measurable", then π_2 is $(\Gamma_{S_2}, \Gamma_{S_3})$-amenable.*

Proof. Let $\{m^s,\ s \in S_2\}$ be an invariant $\pi_2\circ\pi_1$-system of means $(\Gamma_{S_1}, \Gamma_{S_3})$-measurable. For each $s \in S_3$ and each $\varphi \in RCUB\left(\pi_2^{-1}(\{s\})\right)$ let us define

$$m_2^s(\varphi) = m^s\left(\varphi\circ\pi_1|_{(\pi_2\circ\pi_1)^{-1}(\{s\})}\right).$$

Obviously, m^s is a state on $RCUB\left((\pi_2\circ\pi_1)^{-1}(\{s\})\right)$. Since for each $(s,x) \in S_{3\rho_{S_3}} *_r G$ and $\varphi \in RCUB\left(\pi_2^{-1}(\{s\})\right)$ the equivariance of π_1 implies

$$\varphi^{\pi_2}_{(s,x)}\circ\pi_1\Big|_{(\pi_2\circ\pi_1)^{-1}(\{x^{-1}\cdot s\})} = \left(\varphi\circ\pi_1|_{(\pi_2\circ\pi_1)^{-1}(\{s\})}\right)^{\pi_2\circ\pi_1}_{(s,x)},$$

it follows that

$$\begin{aligned} m_2^s(\varphi) &= m^s\left(\varphi\circ\pi_1|_{(\pi_2\circ\pi_1)^{-1}(\{s\})}\right) \\ &= m^{x^{-1}\cdot s}\left(\left(\varphi\circ\pi_1|_{(\pi_2\circ\pi_1)^{-1}(\{s\})}\right)^{\pi_2\circ\pi_1}_{(s,x)}\right) \\ &= m^{x^{-1}\cdot s}\left(\varphi^{\pi_2}_{(s,x)}\circ\pi_1\Big|_{(\pi_2\circ\pi_1)^{-1}(\{x^{-1}\cdot s\})}\right) \\ &= m_2^{x^{-1}\cdot s}\left(\varphi^{\pi_2}_{(s,x)}\right). \end{aligned}$$

Let $\varphi : S_2 \to \mathbb{C}$ be a bounded Γ_{S_2}-"measurable" map with the property that $\varphi|_{\pi_2^{-1}(\{s\})} \in RCUB\left(\pi_2^{-1}(\{s\})\right)$ for all $s \in S$. Let us define $m_2(\varphi) : S_2 \to \mathbb{C}$ by $m_2(\varphi)(s) = m_2\left(\varphi|_{\pi_2(\{s\})}\right) = m^s\left(\varphi\circ\pi_1|_{(\pi_2\circ\pi_1)^{-1}(\{s\})}\right)$ for all $s \in S_2$. Since $\{m^s, \in S_2\}$ is $(\Gamma_{S_1}, \Gamma_{S_3})$-"measurable" and $\varphi\circ\pi_1$ is Γ_{S_1}-"measurable", if follows that $m_2(\varphi)$ is Γ_{S_3}-"measurable". Thus

$$s \mapsto m_2\left(\varphi|_{\pi_2(\{s\})}\right) [: S \to \mathbb{C}]$$

is Γ_{S_3}-"measurable". □

In [1] the amenability of a measure groupoid (G, λ, μ) (a groupoid G endowed with a Haar system λ and a quasi invariant measure μ) was defined as the amenability of the range map with respect to (λ, μ) ([1, Definition 3.2.8/p. 71]). We shall define the amenability of a groupoid G in a similar way.

Definition 19. *Let G be a groupoid endowed with a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4. In addition let us assume that (G, Γ_G) and $\left(G^{(0)}, \Gamma_{G^{(0)}}\right)$ are "measurable" spaces. The groupoid G is said to be $(\Gamma_G, \Gamma_{G^{(0)}})$-amenable (with respect to $\mathcal{W}$) if the G-equivariant map $r: G \to G^{(0)}$ is $(\Gamma_G, \Gamma_{G^{(0)}})$-amenable (with respect to $\mathcal{W}$), where G acts on G by multiplication and on $G^{(0)}$ by $x \cdot d(x) = r(x)$.*

If G is a Borel groupoid, $\Gamma_G = \{A \subset G : A \text{ Borel}\}$ and

$$\Gamma_{G^{(0)}} = \left\{A \subset G^{(0)} : A \ \mu\text{-measurable}\right\}$$

(where μ is σ-finite measure on $G^{(0)}$) and $r : G \to G^{(0)}$ is $(\Gamma_G, \Gamma_{G^{(0)}})$-amenable (with respect to $\mathcal{W}$), then G is said to be μ-amenable (with respect to $\mathcal{W}$).

Remark 20. *For a principal groupoid G (seen as the graph of an equivalence relation $G \subset X \times X$) the various notions of amenability introduced in Definition 19 do not depend on the family $\mathcal{W}$ satisfying conditions $1-5$ from Definition 4. Indeed, a G-invariant system of means (with respect to $\mathcal{W}$) for the G-equivariant map $r: G \to X$ (r is the first projection) is in fact a family $\{m^x, \in X\}$ of states m^x on $l^\infty([x])$ (the space of bounded function $\varphi : [x] \to \mathbb{C}$ on the class $[x]$ of x) such that $m^x = m^y$ for all $y \in [x]$.*

The notion of topological amenability of a locally compact topological groupoid endowed with a continuous Haar system (introduced in [15, Definition II.3.6] and extensively studied in [1]) as well as the notion of Borel amenability (introduced in [16, Definition 2.1] for a Borel groupoid) does not coincide with the notion of amenability in the sense of Definition 19 even when $\Gamma_G = \{A \subset G : A \text{ open}\}$ (or $\Gamma_G = \{A \subset G : A \text{ Borel}\}$ in the Borel case) and $\Gamma_{G^{(0)}} = \left\{A \cap G^{(0)} : A \in \Gamma_G\right\}$. J. Renault remarked in [16] (see also [17]) that the unitary group $\mathcal{U}(H)$ of an infinite-dimensional Hilbert space H, endowed with the weak operator topology, is amenable in the classical sense (and consequently, in the sense of Definition 19 in which the amenability for topological groups coincides with the classical notion). However it is not Borel amenable in sense of [16, Definition 2.1].

On the other hand for let us consider a countable Borel equivalence relation $G \subset X \times X$, where X is a Polish space. Suppose that μ is a Borel probability measure on X that is quasi-invariant, i.e., the saturation of a μ-null Borel set is also μ-null. The equivalence relation G is μ-amenable in the sense of [5] (tor equivalently, in the sense of [18]) if it admits a family $\left\{m^{[x]}, \in X\right\}$ of states $m^{[x]}$on $l^\infty([x])$ such that for all Borel bounded functions $\varphi : G \to \mathbb{C}$, the map

$$x \mapsto m^{[x]}(y \mapsto \varphi(x, y)) \,[: X \to \mathbb{C}]$$

is μ-measurable. In the setting of Definition 19, G is μ-amenable (with respect to $\mathcal{W} = \{X\}$) or equivalently the first projection $r : G \to X$ is (Γ_G, Γ_X)-amenable with respect to $\mathcal{W} = \{X\}$, where $\Gamma_G = \{A \subset G : \ A \text{ Borel}\}$ and

$$\Gamma_X = \{A \subset X : \ A \ \mu\text{-measurable}\} .$$

In [5] it is shown that the existence of such a family $\left\{m^{[x]}, \in X\right\}$ is equivalent to the existence of a μ-conull Borel set B of X such that $G|_B$ is hyperfinite (this means $G|_B$ is of the form $\bigcup\limits_{n\in\mathbb{N}} R_n$, where $R_0 \subset R_1 \subset ...$ is an increasing sequence of finite Borel equivalence relations). This notion of μ-amenability also coincides to the amenability in the sense of [1, Definition 3.2.8/p. 71] of the measure groupoid (G, λ, μ) (where λ is the Haar system consisting in counting measures) .

If we consider $\Gamma_G = \{A \subset G : \ A \text{ Borel}\}$ and $\Gamma_X = \{A \subset X : \ A \text{ Borel}\}$, then according [9, Theorem 5.8] the first projection $r : G \to X$ is (Γ_G, Γ_X)-amenable if and only if G is smooth (this means there is a Borel set $B \subset X$ which contains exactly one point of every class of the equivalence relation G).

Definition 21. *Let G be a groupoid endowed with a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4 and let S be a left G-space. In addition let us assume that (G, Γ_G) and (S, Γ_S) are "measurable" spaces. A left G-space S is said to be (Γ_G, Γ_S)-amenable (with respect to $\mathcal{W}$) if the groupoid $S \rtimes G$ is $\left(\Gamma_{S_{\rho_S} *_r G}, \Gamma_{G^{(0)}}\right)$-amenable (with respect to $\{S_{\rho_S} *_r W : \ W \in \mathcal{W}\}$).*

Definition 22. *Let G be a groupoid endowed with a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4. Let (T, Γ_T), (S, Γ_S) be two left "measurable" G-spaces, (Z, Γ_Z) be a "measurable" space and let $\pi : T \to S$ be a G-equivariant map. An invariant π-system of means $\{m^s, \in S\}$ (with respect to $\mathcal{W}$) is said to be $(\Gamma_Z, \Gamma_T, \Gamma_S)$-"measurable" if for all bounded $\Gamma_{Z_{\rho_Z} * \rho_T T}$-"measurable" maps φ : $Z_{\rho_Z} * \rho_T T \to \mathbb{C}$ with the property that $t \mapsto \varphi(z, t)|_{\pi^{-1}(\{s\})} \in RCUB\left(\pi^{-1}(\{s\})\right)$ for all $(z, s) \in Z \times S$, the map*

$$(s, z) \mapsto m^s\left(t \mapsto \varphi(z, t)|_{\pi^{-1}(\{s\})}\right) [: Z \times S \to \mathbb{C}]$$

is $\Gamma_{Z\times S}$-measurable.

The G-equivariant map $\pi : T \to S$ is said to be a $(\Gamma_Z, \Gamma_T, \Gamma_S)$-amenable map (with respect to $\mathcal{W}$) if there is a $(\Gamma_Z, \Gamma_T, \Gamma_S)$-"measurable" invariant π-system of means (with respect to $\mathcal{W}$).

If (G, Γ_G) and $\left(G^{(0)}, \Gamma_{G^{(0)}}\right)$ are "measurable" spaces, then G is said to be $(\Gamma_Z, \Gamma_G, \Gamma_{G^{(0)}})$-amenable (with respect to $\mathcal{W}$) if G-equivariant map $r : G \to G^{(0)}$ is $(\Gamma_Z, \Gamma_G, \Gamma_{G^{(0)}})$-amenable (with respect to $\mathcal{W}$), where G acts on G by multiplication and on $G^{(0)}$ by $x \cdot d(x) = r(x)$.

Let us consider again a countable Borel equivalence relation $G \subset X \times X$, where X is standard Borel space. G is measure-amenable in the sense of [7, Definition 2.7] if there is a family $\left\{m^{[x]}, x \in X\right\}$ of states $m^{[x]}$ on $l^{\infty}([x])$ such that for standard Borel space Z and for all Borel bounded functions $\varphi : X \times Z \to \mathbb{C}$, the map

$$(x, z) \mapsto m^{[x]}(y \mapsto \varphi(y, z)) \,[: X \to \mathbb{C}]$$

is universally measurable. In the setting of Definition 22, $\left\{m^{[x]}, \in X\right\}$ is $(\Gamma_Z, \Gamma_G, \Gamma_X)$-"measurable" (or equivalently the first projection $r : G \to X$ is $(\Gamma_Z, \Gamma_G, \Gamma_X)$-amenable) for all standard Borel space Z, where

$$\Gamma_G = \{A \subset G : \ A \text{ Borel}\}, \ \Gamma_X = \{A \subset X : \ A \text{ universally measurable}\}$$

and

$$\Gamma_Z = \{A \subset Z : \ A \text{ universally measurable}\}.$$

Obviously, measure-amenability implies μ-amenability for all quasi-invariant probability measures μ (so called measurewise amenability of G). Under the Continuum Hypothesis (CH) the converse is true: If the countable Borel equivalence relation G is measurewise amenable, then G measure-amenable (see, [7, Theorem 2.8]).

Proposition 23. *Let G be a groupoid endowed with a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4 and let S be a left G-space. In addition let us assume that (G, Γ_G), $\left(G^{(0)}, \Gamma_{G^{(0)}}\right)$ and (S, Γ_S) are "measurable" spaces such that $\rho_S : S \to G^{(0)}$ is $(\Gamma_S, \Gamma_{G^{(0)}})$-"measurable". If G is $(\Gamma_S, \Gamma_G, \Gamma_{G^{(0)}})$-amenable, then S is (Γ_S, Γ_G)-amenable (with respect to $\mathcal{W}$).*

Proof. Let $\left\{m_G^u, \, u \in G^{(0)}\right\}$ be a $(\Gamma_S, \Gamma_G, \Gamma_{G^{(0)}})$-"measurable" invariant r-system of means for $r : G \to G^{(0)}$. For each $s \in S$ and $\varphi \in RCUB(\{(s, x) : r(x) = \rho_S(s)\})$ let us define

$$m^s(\varphi) = m_G^{\rho_S(s)}(x \mapsto \varphi(s, x)),$$

Then $\{m^s, \in S\}$ is an invariant r-system of means (Γ_G, Γ_S)-"measurable" for the range map r of $S \rtimes G$. $\square$

Remark 24. *Let G be a groupoid endowed with a family $\mathcal{W}$ of subsets satisfying conditions $1-5$ from Definition 4. In addition let us assume that (G, Γ_G) and $\left(G^{(0)}, \Gamma_{G^{(0)}}\right)$ are "measurable" spaces. Let us endow the group bundle G' with $\Gamma_{G'} = \{A \cap G' : A \in \Gamma_G\}$ and $\mathcal{W}_{G'} = \{W \cap G' : W \in \mathcal{W}\}$. If G is $(\Gamma_G, \Gamma_{G^{(0)}})$-amenable (with respect to $\mathcal{W}$) and $G' \in \Gamma_G$, then G' is $(\Gamma_{G'}, \Gamma_{G^{(0)}})$-amenable (with respect to $\mathcal{W}_{G'}$). Indeed, let $\left\{m_G^u, \, u \in G^{(0)}\right\}$ be a $(\Gamma_G, \Gamma_{G^{(0)}})$-"measurable" invariant r-system of means for $r : G \to G^{(0)}$ and for each $u \in G^{(0)}$ and each $\varphi \in RCUB(G_u^u)$ let us define $m_{G'}^u(\varphi) = m_G^u(\tilde{\varphi})$, where $\tilde{\varphi}|_{G_u^u} = \varphi$ and $\tilde{\varphi}(x) = 1$ for all $x \in G^u \setminus G_u^u$. Then $\left\{m_{G'}^u, \, u \in G^{(0)}\right\}$ is a $(\Gamma_{G'}, \Gamma_{G^{(0)}})$-"measurable" invariant r-system of means* *for* $r: G' \to G^{(0)}$. *In particular, each isotropy group* G_u^u *is amenable (as a topological group with the topology defined by* $\{W \cap G_u^u : W \in \mathcal{W}\}$ *seen as neighborhood basis of the unity).*

In fact the existence of an invariant r*-system of means for* $r: G \to G^{(0)}$ *(with respect to* $\mathcal{W}$*) is equivalent to the amenability of all isotropy groups* G_u^u *(*G_u^u *endowed with the topology defined by* $\{W \cap G_u^u : W \in \mathcal{W}\}$*).*

If the principal groupoid R *associated to* G *is endowed with*

$$\mathcal{W}_R = \{(r,d)(W) : W \in \mathcal{W}\}$$

(or $\mathcal{W}_R = \left\{diag\left(G^{(0)}\right)\right\}$ *or any other* $\mathcal{W}$ *satisfying conditions* $1-5$ *from Definition 4) and* Γ_R *is such that* $(r,d): G \to R$ *is* (Γ_G, Γ_R)*-"measurable", then if* G *is* $(\Gamma_G, \Gamma_{G^{(0)}})$*-amenable (with respect to* $\mathcal{W}$*), then* R *is* $(\Gamma_R, \Gamma_{G^{(0)}})$*-amenable (with respect to* $\mathcal{W}_R$*). Indeed, the application* $r: R \to G^{(0)}$ *can be seen as a* R*-equivariant map as well as a* G*-equivariant map. Since the composition of the maps*

$$G \overset{(r,d)}{\to} R \overset{r}{\to} G^{(0)}$$

is $(\Gamma_G, \Gamma_{G^{(0)}})$*-amenable, it follows that* $r: R \to G^{(0)}$ *is* $(\Gamma_G, \Gamma_{G^{(0)}})$*-amenable.*

5 Measured groupoids

An analytic (respectively, standard) Borel groupoid is a groupoid G such that $G^{(2)}$ is a Borel set in the product structure on $G \times G$, and the functions $(x,y) \mapsto xy$ $\left[: G^{(2)} \to G\right]$ and $x \mapsto x^{-1}$ $[: G \to G]$ are Borel functions.

An analytic Borel groupoid G is said to be Borel amenable in the sense of [16, Definition 2.1] or [7] if for each $u \in G^{(0)}$ there exists a sequence $(m_n^u)_n$ of finite positive measures m_n^u of mass not greater than one on G^usuch that:

1. For all $n \in \mathbb{N}$ and for all bounded Borel functions $f: G \to \mathbb{R}$ the application $u \mapsto \int f(x)\, dm_n^u(x)$ is Borel.
2. $\|m_n^u\|_1 \to 1$ for all $u \in G^{(0)}$.
3. $\left\|x m_n^{d(x)} - m_n^{r(x)}\right\|_1 \to 1$ for all $x \in G$.

Proposition 25. *Let* G *be a principal analytic Borel groupoid (seen as the graph of an equivalence relation* $G \subset G^{(0)} \times G^{(0)}$*). If* G *is Borel amenable in the sense of [16, Definition 2.1], then for every* σ*-finite measure* μ *on* $G^{(0)}$ *and every family* $\mathcal{W}$ *satisfying conditions* $1-5$ *from Definition 4, the groupoid* G *is* μ*-amenable with respect to* $\mathcal{W}$ *(in the sense of Definition 19).*

Proof. Let μ be a σ-finite measure μ on $G^{(0)}$ and $\mathcal{W}$ be a family satisfying conditions $1-5$ from Definition 4. Let LIM be a medial limit as defined by Mokobodski in [12]. For each Borel bounded function φ on G and each $u \in G^{(0)}$ let us define

$$m^u(\varphi) = LIM\left(m_n^u(f)\right)$$

Then $u \mapsto m^u(\varphi)$ is μ-measurable. In addition for all φ bounded and all $x \in G$ we have

$$m^{d(x)}(\varphi_x) = LIM\left(m_n^{d(x)}(\varphi(xy))\right) = LIM\left(m_n^{r(x)}(y \mapsto \varphi(y))\right) = m^{r(x)}(\varphi).$$

Thus we obtain a family $\left\{m^u, \in G^{(0)}\right\}$ with the property that each m^u can be seen as a state on the space of Borel bounded functions $\varphi : [u] \to \mathbb{C}$ (defined on the orbit $[u]$ of u) and such that $m^u = m^v$ for all $v \in [u]$. If for each $u \in G^{(0)}$ we extend $m^u = m^{[u]}$ to $\tilde{m}^{[u]} : l^\infty([u]) \to \mathbb{C}$ ($l^\infty([u])$ is the space of bounded function $\varphi : [u] \to \mathbb{C}$) using Hahn-Banach Extension Theorem, we obtain a G-invariant r-system $\left\{\tilde{m}^{[u]}, u \in G^{(0)}\right\}$ of means (with respect to $\mathcal{W} = \left\{G^{(0)}\right\}$ and consequently, with respect to all $\mathcal{W}$ satisfying conditions $1-5$ from Definition 4) with the property that for all Borel bounded functions $\varphi : G \to \mathbb{C}$, the map

$$u \mapsto m^{[u]}(v \mapsto \varphi(u, v)) \left[: G^{(0)} \to \mathbb{C}\right]$$

is μ-measurable. Therefore G is μ-amenable in the sense of Definition 19. $\square$

A Borel Haar system $\lambda = \left\{\lambda^u, u \in G^{(0)}\right\}$ for an analytic Borel groupoid G is a family of non-zero measures λ^u on the fibers G^u such that

1. For all non-negative Borel functions f on G, the map $u \mapsto \int f(x)\, d\lambda^u(x)$ is Borel.
2. For all $x \in G$ and all non-negative Borel functions f on G we have
$$\int f(y)\, d\lambda^{r(x)}(y) = \int f(xy)\, d\lambda^{d(x)}(y).$$
3. G is the union of an increasing sequence $(A_n)_{n\in\mathbb{N}}$ of Borel subsets such that for all $n \in \mathbb{N}$, the functions $u \mapsto \lambda^u(A_n)$ are bounded on $G^{(0)}$.

A measure μ on $G^{(0)}$ is said to be quasi-invariant with respect to the Haar system $\lambda = \left\{\lambda^u, u \in G^{(0)}\right\}$ if the measure $\nu = \int \lambda^u d\mu(u)$ is equivalent to its inverse ν^{-1}, where $\nu^{-1}(A) = \nu\left(A^{-1}\right)$ for all Borel subsets $A \subset G$. In this case the triple (G, λ, μ) is called measured groupoid [1].

The measured groupoid (G, λ, μ) in the sense of [1] is amenable in the sense of [1, Definition 3.2.8/p. 71] if and only if there is a family $\{m^u, u \in G^{(0)}\}$ of states (or means) m^u on $L^\infty(G, \lambda^u)$ such that:

1. $u \mapsto m^u(\varphi)$ is μ-measurable for all $\varphi \in L^\infty(G,\nu)$.

2. $m^{d(x)}(y \mapsto \varphi(xy)) = m^{r(x)}(\varphi)$ for all $\varphi \in L^\infty\left(G, \lambda^{r(x)}\right)$ ν-a.e.

where $\nu = \int \lambda^u d\mu(u)$ ([1, 3.2]).

Mackey [11] took into consideration the analytic Borel groupoids that can be endowed with a measure class C (a generalization of the class obtained in the group action case by product of quasi-invariant measures μ on X with measures in the Haar measure class on G) such that each measure in the class C has a decomposition satisfying a kind of quasi-invariance condition. Hahn proved in [6, Theorem 3.9/p. 17] that if (G,C) is a measure groupoid in the sense of [11], then there is an $r_*(C)$-conull Borel set $U \subset G^{(0)}$ such that the groupoid $G|_U$ admits a Borel Haar system $\lambda = \{\lambda^u, u \in U\}$. Moreover if $\mu \in r_*(C)$, then $(G|_U, \lambda, \mu)$ is a measured groupoid in the sense of [1] and $\int \lambda^u d\mu(u) \in C$. Conversely, if (G,λ,μ) is a measured groupoid in the sense of [1], then $\left(G, \left[\int \lambda^u d\mu(u)\right]\right)$ is a Mackey measure groupoid.

If $R = (r,d)(G)$ is the principal groupoid associated to a Mackey measure groupoid (G,C), then $(R, (r,d)_*(C))$ is a Mackey measure groupoid and $r_*(C) = r_*((r,d)_*(C))$, where R is endowed with the Borel structure coming from $G^{(0)} \times G^{(0)}$ ([6, Example 2.8/p. 8]). Consequently, there is an $r_*(C)$-conull Borel set $U \subset G^{(0)}$ such that $R|_U$ can be endowed with a Borel Haar system α.

Hahn also proved in [6, Theorem 4.4/p. 23] that the Haar system $\lambda = \{\lambda^u, u \in U\}$ associated to Mackey measure groupoid (G,C) admits an (r,d)-decomposition

$$\{\nu_{u,v}, (u,v) \in R \cap (U_0 \times U_0)\}$$

(where R is the principal groupoid associated to G and U_0 is an $r_*(C)$-conull Borel subset of $G^{(0)}$). In particular, $\{\nu_{u,u}, u \in U_0\}$ is a Borel Haar system for $G'|_{U_0}$ (G', the isotropy group bundle of G, being endowed with the Borel structure coming from G).

According [1, Corollary 5.3.33/p. 127] a measured groupoid (G,λ,μ) in the sense of [1] is amenable in the sense of [1, Definition 3.2.8/p. 71] if and only if (R,α,μ) is amenable and μ-a.a isotropy groups of G are amenable. Equivalently, (G,λ,μ) is amenable in the sense of [1, Definition 3.2.8/p. 71] if and only if $(R|_U, \alpha, \mu)$ and $\left(G'|_{U_0}, \{\nu_{u,u}, u \in U_0\}, \mu\right)$ are amenable in the sense of [1].

On the other hand a Mackey measure groupoid (G,C) has an inessential reduction $G_0 = G|_{U_0}$ (this means that $U_0 \subset G^{(0)}$ is a $r_*(C)$-conull Borel set) which has a locally compact metric topology in which it is a topological groupoid [13, Theorem 4.1/p. 330]. Then the family $\mathcal{W}$ of the symmetric neighborhoods of the unit space of G_0 satisfying conditions $1-5$ from Definition 4. Thus we can study the amenability of G_0 with respect of $\mathcal{W}$.

Proposition 26. *Let (G,λ,μ) be a measured groupoid in the sense of [1] and let $G_0 = G|_{U_0}$ be an inessential reduction of G which has a locally compact metric* *topology in which it is a topological groupoid. If G_0 is μ-amenable with respect to the family $\mathcal{W}$ of the symmetric neighborhoods of the unit space of G_0, then (G, λ, μ) is amenable in the sense of [1, Definition 3.2.8/p. 71].*

Proof. If G_0 is μ-amenable with respect to $\mathcal{W}$, then each isotropy group G_u^u $(u \in U_0)$ is amenable (as a topological group with the topology defined by $\{W \cap G_u^u : W \in \mathcal{W}\}$ seen as neighborhood basis of the unity). Indeed, let $\{m_G^u, u \in U_0\}$ be an invariant r-system of means for $r : G_0 \to U_0$ and for each $u \in U_0$ and each $\varphi \in RCUB\left(G_u^u\right)$ let us define $m_{G'}^u(\varphi) = m_G^u(\tilde{\varphi})$, where $\tilde{\varphi}|_{G_u^u} = \varphi$ and $\tilde{\varphi}(x) = 1$ for all $x \in G^u \setminus G_u^u$. Then $m_{G'}^u$ is a left invariant mean on $RUCB\left(G_u^u\right)$.

If $R = (r, d)(G)$ is the principal groupoid associated to a Mackey measure groupoid (G, C), where $C = \left[\int \lambda^u d\mu(u)\right]$, then $(R, (r, d)_*(C))$ is a Mackey measure groupoid and $r_*(C) = r_*((r, d)_*(C))$, where R is endowed with the Borel structure coming from $G^{(0)} \times G^{(0)}$ ([6, Example 2.8/p. 8]). Consequently, there is an $r_*(C)$-conull Borel set $U_1 \subset U_0$ such that $R|_{U_1}$ can be endowed with a Borel Haar system α. Thus $\left(R|_{U_1}, \alpha, \mu\right)$ is a measured groupoid in the sense of [1].

On the other hand since $(r, d) : G_0 \to R|_{U_0}$ is Borel, and the composition of the maps

$$G_0 \overset{(r,d)}{\to} R|_{U_0} \overset{r}{\to} U_0$$

is μ-amenable, it follows that $r : R|_{U_0} \to U_0$ is μ-amenable and therefore $\left(R|_{U_1}, \alpha, \mu\right)$ is amenable in the sense of [1, Definition 3.2.8/p. 71]. Thus (G, λ, μ) is amenable in the sense of [1, Definition 3.2.8/p. 71] (because $\left(R|_{U_1}, \alpha, \mu\right)$ is amenable and μ-a.a isotropy groups of G are amenable [1, Corollary 5.3.33/p. 127]). $\square$

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