IMPULSIVE EVOLUTION INCLUSIONS WITH INFINITE DELAY AND MULTIVALUED JUMPS

Mouffak Benchohra and Mohamed Ziane

Abstract. In this paper we prove the existence of a mild solution for a class of impulsive semilinear evolution differential inclusions with infinite delay and multivalued jumps in a Banach space.

1 Introduction

In this paper, we are concerned by the existence of mild solution of impulsive semilinear functional differential inclusions with infinite delay and multivalued jumps in a Banach space E. More precisely, we consider the following class of semilinear impulsive differential inclusions:

$$x'(t) \in A(t)x(t) + F(t, x_t), \quad t \in J = [0, b], \ t \neq t_k, \tag{1.1}$$

$$\Delta x\big|_{t=t_k} \in \mathcal{I}_k(x(t_k^-)), \quad k = 1, \ldots, m \tag{1.2}$$

$$x(t) = \phi(t), \quad t \in (-\infty, 0], \tag{1.3}$$

where $\{A(t) : t \in J\}$ is a family of linear operators in Banach space E generating an evolution operator, F be a Carathéodory type multifunction from $J \times \mathcal{B}$ to the collection of all nonempty compact convex subset of E, $\mathcal{B}$ is the phase space defined axiomatically (see section 2) which contains the mapping from $(-\infty, 0]$ into E, $\phi \in \mathcal{D}$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, $\mathcal{I}_k : E \to \mathcal{P}(E)$, $k = 1, \ldots, m$ are multivalued maps with closed, bounded and convex values, $x(t_k^+) = \lim_{h \to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h \to 0^+} x(t_k - h)$ represent the right and left limits of $x(t)$ at $t = t_k$. Finally $\mathcal{P}(E)$ denotes the family of nonempty subsets of E.

The theory of impulsive differential equations has become an important area of investigation in recent years, stimulated by the numerous applications to problems arising in mechanics, electrical engineering, medicine, biology, ecology, population dynamics, etc. During the last few decades there have been significant developments

---

in impulse theory, especially in the area of impulsive differential equations and inclusions with fixed moments; see the monographs of Bainov and Simeonov [7], Benchohra *et al.* [8], Lakshmikantham *et al.* [22], Samoilenko and Perestyuk [25], and the references therein. For the case where the impulses are absent (i.e. $I_k = 0$, $k = 1, \ldots, m$) and F is a single-valued or multivalued map and A is a densely defined linear operator generating a C_0-semigroup of bounded linear operators and the state space is $C([-r, 0], E)$ or E, the problem (1.1)–(1.3) has been investigated in, for instance, the monographs by Ahmed [4, 5], Hale and Verduyn Lunel [17], Hu and Papageorgiou [19], Kamenskii *et al.* [20] and Wu [26] and the papers by Benchohra and Ntouyas [9], Cardinali and Rubbioni [12], Gory *et al.* [15]. Benedetti [10] considered the existence result in the autonomous case ($A(t) \equiv A$) and finite delay. Cardinali and Rubbioni [11] considered the non autonomous case. In [24] Obukhovskii and Yao considered local and global existence results for semilinear functional differential inclusions with infinite delay and impulse characteristics in a Banach space. Recently some existence results were obtained for certain classes of functional differential equations and inclusions in Banach spaces under assumption that the linear part generates an compact semigroup (see, e.g., [1, 2, 3]).

Our goal here is to give existence results for the problem (1.1)–(1.3) without any compactness assumption. In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following sections. In Section 3, we prove existence and compactness of solutions set for problem (1.1)–(1.3).

We mention that the model with multivalued jump sizes may arise in a control problem where we want to control the jump sizes in order to achieve given objectives. To our knowledge, there are very few results for impulsive evolution inclusions with multivalued jump operators; see [3, 6, 10, 23]. The results of the present paper extend and complement those obtained in the absence of the impulse functions I_k, and for those with single-valued impulse functions I_k.

2   Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. For $\phi \in \mathcal{B}$ the norm of ϕ is defined by

$$\|\phi\|_{\mathcal{B}} = \sup\{\|\phi(\theta)\| : \theta \in (-\infty, 0]\}.$$

Let $J := [0, b]$, $b > 0$ and $(E, \|.\|)$ be a real separable Banach space. $C(J, E)$ the space of E-valued continuous functions on J with the uniform norm

$$\|x\|_{\infty} = \sup\{\|x(t)\|, \ t \in J\}.$$

$L^1(J, E)$ the space of $E-$valued Bochner integrable functions on J with the norm

$$\|f\|_{L^1} = \int_0^b \|f(t)\|dt.$$

To define the solution of problem (1.1)–(1.3), it is convenient to introduce some additional concepts and notations. Consider the following spaces

$$\mathcal{PC}(J, E) = \{y : J \to E, y_k \in C(J_k; E) \text{ there exist } y(t_k^-), y(t_k^+) \text{ with } y(t_k) = y(t_k^-)\},$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}]$, $k = 0, \ldots, m$. Let the space

$$\mathbf{\Omega} = \left\{ y \in (-\infty, b] \to E : \quad y \mid_{(-\infty, 0]} \in \mathcal{B} \text{ and } y \mid_J \in \mathcal{PC}(J, E) \right\}$$

with the semi-norm defined by

$$\|y\|_\Omega = \|y_0\|_\mathcal{B} + \sup\{\|y(s)\| : 0 \le s \le b\}, \quad y \in \mathcal{PC}.$$

In this work, we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced in [18]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into E endowed with a semi norm $\|.\|_\mathcal{B}$, and satisfies the following axioms introduced at first by Hale and Kato in [16]:

(A1) There exist a positive constant H and functions $K(.), M(.) : \mathbb{R}^+ \to \mathbb{R}^+$ with K continuous and M locally bounded, such that for any $b > 0$ if $y : (-\infty, b] \to E$, such that $y \mid_J \in \mathcal{PC}(J, E)$ and $y_0 \in \mathcal{B}$; the following conditions hold:

   (i) y_t is in $\mathcal{B}$;

   (ii) $\|y(t)\| \le H\|y_t\|_\mathcal{B}$;

   (iii) $\|y_t\|_\mathcal{B} \le K(t) \sup\{\|y(s)\| : 0 \le s \le t\} + M(t)\|y_0\|_\mathcal{B}$ and H, K and M are independent of $y(.)$.

(A2) The space $\mathcal{B}$ is complete.

In what follows we use the following notations $K_b = \sup\{K(t), t \in J\}$ and $M_b = \sup\{M(t), t \in J\}$

Definition 1. Let X and Y be two topological vector spaces. We denote by $\mathcal{P}(Y)$ the family of all non-empty subsets of Y and by

$$\mathcal{P}_k(Y) = \{C \in \mathcal{P}(Y) : \text{compact}\}, \ \mathcal{P}_b(Y) = \{C \in \mathcal{P}(Y) : \text{bounded}\},$$

$$\mathcal{P}_c(Y) = \{C \in \mathcal{P}(Y) : \text{closed}\}, \ \mathcal{P}_{cv}(Y) = \{C \in \mathcal{P}(Y) : \text{convex}\}.$$

A multifunction $G : X \to \mathcal{P}(Y)$ is said to be upper semicontinuous (u.s.c.) if $G^{-1}(V) = \{x \in X : G(x) \subseteq V\}$ is an open subset of X for every open $V \subseteq Y$. The multifunction G is called closed if its graph $\Gamma_G = \{(x, y) \in X \times Y : y \in G(x)\}$ is closed subset of the topological space $X \times Y$. The multifunction G is called quasicompact restriction to any compact subset $M \subset X$ is compact. A multifunction

$\mathcal{F} : [c, d] \subset \mathbb{R} \to \mathcal{P}_k(Y)$ is said to be strongly measurable if there exists a sequence $\mathcal{F}_n : [c, d] \to \mathcal{P}_k(Y)$, $n = 1, 2, ...$ of steps multifunctions such that

$$\lim_{n \to +\infty} h(\mathcal{F}_n(t), \mathcal{F}(t)) = 0, \quad \text{for } \mu\text{-a.e } t \in [c, d],$$

where μ denotes the Lebesgue measure on $[c, d]$ and h is the Hausdorff metric on $\mathcal{P}_k(Y)$. For details and equivalent definitions see [14, 20, 21].

Definition 2. Let $(\mathcal{A}, \geq)$ be a partially ordered set. A function $\beta : \mathcal{P}_b(E) \to \mathcal{A}$ is called a measure of noncompactness (MNC) in E if

$$\beta(\overline{\mathrm{co}}\Omega) = \beta(\Omega),$$

for every $\Omega \in \mathcal{P}_b(E)$.

Definition 3. A measure of noncompactness β is called:

   (i) monotone if $\Omega_0, \Omega_1 \in \mathcal{P}_b(E)$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$

   (ii) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in E$, $\Omega \in \mathcal{P}_b(E)$;

   (iii) regular if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

As an example of the measure of noncompactness possessing all these properties is the Hausdorff of MNC which is defined by

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon - \text{net}\}.$$

For more information about the measure of noncompactness we refer the reader to [20].

Definition 4. A multifunction $G : E \to \mathcal{P}_k(E)$ is said to be χ-condensing if for every bounded subset $\Omega \subseteq E$ the relation

$$\chi(G(\Omega)) \geq \chi(\Omega)$$

implies the relative compactness of Ω.

Definition 5. A countable set $\{f_n : n \geq 1\} \subseteq L^1(J, E)$ is said to be semicompact if

   (i) it is integrably bounded: $\|f_n(t)\| \leq \omega(t)$ for a.e. $t \in J$ and every $n \geq 1$ where $\omega \in L^1(J, \mathbb{R}^+)$

   (ii) the set $\{f_n(t) : n \geq 1\}$ is relatively compact in E for a.e. $t \in J$.

Now, let for every $t \in J$, $A(t) : E \to E$ be a linear operator such that

   (i) For all $t \in J$, $D(A(t)) = D(A) \subseteq E$ is dense and independent of t.

(ii) For each $s \in I$ and each $x \in E$ there is a unique solution $v : [s, b] \to E$ for the evolution equation

$$v'(t) = A(t)v(t), \quad t \in [s, b]$$
$$v(s) = x. \tag{2.1}$$

In this case an operator T can be defined as

$$T : \Delta = \{(t, s) : 0 \le s \le t \le b\} \to \mathcal{L}(E), \quad T(t, s)(x) = v(t),$$

where v is the unique solution of (2.1) and $\mathcal{L}(E)$ is the family of linear bounded operators on E.

Definition 6. The operator T is called the evolution operator generated by the family $\{A(t) : t \in J\}$.

1. $T(s, s) = I_E$,

2. $T(t, r)T(r, s) = T(t, s)$, for all $0 \le s \le r \le t \le b$.

3. $(t, s) \to T(t, s)$ is strongly continuous on Δ and

$$\frac{\partial T(t, s)}{\partial t} = A(t)T(t, s), \quad \frac{\partial T(t, s)}{\partial s} = -T(t, s)A(s).$$

Definition 7. The operator $G : L^1(J, E) \to C(J, E)$ defined by

$$Gf(t) = \int_0^t T(t, s)f(s)ds \tag{2.2}$$

is called the generalized Cauchy operator, where $T(., .)$ is the evolution operator generated by the family of operators $\{A(t) : t \in J\}$.

In the sequel we will need the following results.

Lemma 8. *[20] Every semicompact set in $L^1(J, E)$ is weakly compact in the space $L^1(J, E)$.*

Lemma 9 ([20, Theorem 2]). *The generalized Cauchy operator G satisfies the properties*

(G1) there exists $\zeta \ge 0$ such that

$$\|Gf(t) - Gg(t)\| \le \zeta \int_0^t \|f(s) - g(s)\|ds, \text{ for every } f, g \in L^1(J, E), \ t \in J.$$

(G2) for any compact $K \subseteq E$ and sequence $(f_n)_{n \ge 1}$, $f_n \in L^1(J, E)$ such that for all $n \ge 1$, $f_n(t) \in K$, a. e. $t \in J$, the weak convergence $f_n \rightharpoonup f_0$ in $L^1(J, E)$ implies the convergence $Gf_n \to Gf_0$ in $C(J, E)$.

Lemma 10. *[20] Let $S : L^1(J, E) \to C(J, E)$ be an operator satisfying condition (G2) and the following Lipschitz condition (weaker than (G1)).*

(G1')

$$\|Sf - Sg\|_{C(I,E)} \leq \zeta\|f - g\|_{L^1(J,E)}.$$

Then for every semicompact set $\{f_n\}_{n=1}^{+\infty} \subset L^1(J, E)$ the set $\{Sf_n\}_{n=1}^{+\infty}$ is relatively compact in $C(J, E)$. Moreover, if $(f_n)_{n\geq 1}$ converges weakly to f_0 in $L^1(J, E)$ then $Sf_n \to Sf_0$ in $C(J, E)$.

Lemma 11. *[20] Let $S : L^1(J, E) \to C(J, E)$ be an operator satisfying conditions (G1), (G2) and let the set $\{f_n\}_{n=1}^{\infty}$ be integrably bounded with the property $\chi(\{f_n(t) : n \geq 1\}) \leq \eta(t)$, for a.e. $t \in J$, where $\eta(.) \in L^1(J, \mathbb{R}^+)$ and χ is the Hausdorff MNC. Then*

$$\chi(\{Sf_n(t) : n \geq 1\}) \leq 2\zeta \int_0^t \eta(s)ds, \text{ for all } t \in J,$$

where $\zeta \geq 0$ is the constant in condition (G1).

Lemma 12. *[20] If U is a closed convex subset of a Banach space E and $R : U \to \mathcal{P}_{cv,k}(E)$ is a closed β-condensing multifunction, where β is a nonsingular MNC defined on the subsets of U. Then R has a fixed point.*

Lemma 13. *[20] Let W be a closed subset of a Banach space E and $R : W \to P_{cv,k}(E)$ be a closed multifunction which is β-condensing on every bounded subset of W, where β is a monotone measure of noncompactness. If the fixed points set $\mathcal{F}ixR$ is bounded, then it is compact.*

3   Existence Theorem

In this section we prove the existence of mild solutions for the impulsive semilinear functional differential inclusions (1.1)–(1.3). We will assume the following hypotheses.

- (A) $\{A(t) : t \in J\}$ be a family of linear (not necessarily bounded) operators, $A(t) : D(A) \subset E \to E$, $D(A)$ not depending on t and dense subset of E and $T : \Delta = \{(t, s) : 0 \leq s \leq t \leq b\} \to \mathcal{L}(E)$ be the evolution operator generated by the family $\{A(t) : t \in J\}$.

- (H1) The multifunction $F(., x)$ has a strongly measurable selection for every $x \in \mathcal{B}$.

- (H2) The multifunction $F : (t, .) \to P_{cv,k}(E)$ is upper semicontinuous for a.e. $t \in J$.

- (H3) there exists a function $\alpha \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, \psi)\| \leq \alpha(t)(1 + \|\psi\|_{\mathcal{B}}) \quad \text{for a.e. } t \in J;$$

****

Surveys in Mathematics and its Applications **7** (2012), 1 – 14
http://www.utgjiu.ro/math/sma

(H4) There exists a function $\beta \in L^1(J, \mathbb{R}^+)$ such that for all $\Omega \subset \mathbf{\Omega}$, we have

$$\chi(F(t, D)) \leq \beta(t) \sup_{-\infty \leq s \leq 0} \chi(\Omega(s)) \quad \text{for a.e. } t \in J,$$

where, $\Omega(s) = \{x(s); x \in \Omega\}$ and χ is the Hausdorff measure of noncompactness.

(H5) There exist constants $a_k > 0, k = 1, \ldots, m$ such that

$$\|I_k\| \leq a_k, \quad \text{where } I_k \in \mathcal{I}_k(x(t_k^+)).$$

Definition 14. *A function $x \in \mathbf{\Omega}$ is said to be a mild solution of system* (1.1)–(1.3) *if there exist a function $f \in L^1(J; E)$ such that $f \in F(t, x_t)$ for a.e. $t \in J$*

(i) $x(t) = T(t, 0)\phi(0) + \int_0^t T(t, s)f(s)ds + \sum_{0 < t_k < t} T(t, t_k)I_k(x(t_k))$ *, a.e. $t \in J$, $k = 1, \ldots, m$*

(ii) $x(t) = \phi(t), \quad t \in (-\infty, 0],$

with $I_k \in \mathcal{I}_k(x(t_k^+))$.

Remark 15. *Under conditions (H1)-(H3) for every piecewise continuous function $v : J \to \mathcal{B}$ the multifunction $F(t, v(t))$ admits a Bochner integrable selection (see* [20]).

We note that from assumptions (H1) and (H3) it follows that the superposition multioperator $S_F^1 : \mathbf{\Omega} \to \mathcal{P}(L^1(J, E))$ defined by

$$S_{F(.,x)}^1 = S_F^1 = \{f \in L^1(J, E) : f(t) \in F(t, x_t), \quad \text{a.e. } t \in J\}$$

is nonempty set (see [20]) and is weakly closed in the following sense.

Lemma 16. *If we consider the sequence (x^n), $(x^n) \in \mathcal{B}_b$ and $\{f_n\}_{n=1}^{+\infty} \subset L^1(J, E)$, where $f_n \in S_{F(.,x_n)}^1$ such that $x^n \to x^0$ and $f_n \to f^0$ then $f^0 \in S_F^1$.*

Now we state and prove our main result.

Theorem 17. *Under assumptions* (A) *and* (H1)–(H5), *the problem* (1.1)–(1.3) *has at least one mild solution.*

Proof. To prove the existence of a mild solution for (1.1)–(1.3) we introduce the integral multioperator $N : \mathbf{\Omega} \longrightarrow \mathcal{P}(\mathbf{\Omega})$, defined as

$$Nx = \begin{cases} y(t) \in \mathbf{\Omega} : y(t) = T(t, 0)\phi(0) + \int_0^t T(t, s)f(s)ds \\ \quad \sum_{0 < t_k < t} T(t, t_k)I_k(x(t_k)), & t \in J \\ y(t) = \phi(t), & t \in (-\infty, 0], \end{cases} \tag{3.1}$$

where S_F^1 and $I_k \in \mathcal{I}_k(x)$.

It is clear that the integral multioperator N is well defined and the set of all mild solution for the problem (1.1)–(1.3) on J is the set $\mathcal{F}ixN = \{x : x \in N(x)\}$.

We shall prove that the integral multioperator N satisfies all the hypotheses of Lemma 12. The proof will be given in several steps.

Step 1. Using in fact that the maps F and $\mathcal{I}$ has a convex values it easy to check that N has convex values.

Step 2. N has closed graph.

Let $\{x^n\}_{n=1}^{+\infty}$, $\{z^n\}_{n=1}^{+\infty}$, $x^n \to x^*$, $z^n \in N_1((x^n), n \geq 1)$ and $z^n \to z^*$. Moreover, let $\{f_n\}_{n=1}^{+\infty} \subset L^1(J;E)$ an arbitrary sequence such that $f_n \in S_F^1$ for $n \geq 1$.

Hypothesis (H3) implies that the set $\{f_n\}_{n=1}^{+\infty}$ integrably bounded and for a,e. $t \in J$ the set $\{f_n(t)\}_{n=1}^{+\infty}$ relatively compact, we can say that $\{f_n\}_{n=1}^{+\infty}$ is semicompact sequence. Consequently $\{f_n\}_{n=1}^{+\infty}$ is weakly compact in $L^1(J;E)$, so we can assume w.l.g that $f_n \rightharpoonup f^*$.

From lemma 9 we know that the generalized Cauchy operator on the interval J, $G : L^1(J;E) \to C(J;E)$, defined by

$$Gf(t) = \int_0^t T(t,s)f(s)ds, \quad t \in J \tag{3.2}$$

satisfies properties (G1) and (G2) on J.

Note that set $\{f_n\}_{n=1}^{+\infty}$ is also semicompact and sequence $(f_n)_{n=1}^{+\infty}$ weakly converges to f^* in $L^1(J;E)$. Therefore, by applying Lemma 10 for the generalized Cauchy operator G of (3.2) we have in $C(J;E)$ the convergence $Gf_n \to Gf$. By means of (3.2) and (3.1), for all $t \in J$ we can write

$$z_n(t) = T(t,0)\phi(0) + \int_0^t T(t,s)f_n(s)ds + \sum_{0<t_k<t} T(t,t_k)I_k(x^n(t_k))$$

$$= T(t,0)\phi(0) + \int_0^t T(t,s)f_n ds + \sum_{0<t_k<t} T(t,t_k)I_k(x^n(t_k))$$

$$= T(t,0)\phi(0) + Gf_n(t) + \sum_{0<t_k<t} T(t,t_k)I_k(x^n(t_k)))$$

where S_F^1, and $I_k \in \mathcal{I}_k(x)$.

By applying Lemma 9, we deduce

$$z_n \to T(.,0)\phi(0) + Gf + T(.,t)I_k(x^*(t_k))$$

in $C(J;E)$ and by using in fact that the operator S_F^1 is closed, we get $f^* \in S_F^1$. Consequently

$$z^*(t) \to T(t,0)\phi(0) + Gf + T(t,t)I_k(x^*(t_k)),$$

therefore $z^* \in N(x^*)$. Hence N is closed.

With the same technique, we obtain that N has compact values.

Step 3. We consider the measure of noncompactness defined in the following way. For every bounded subset $\Omega \subset \mathbf{\Omega}$

$$\nu_1(\Omega) = \max_{D \in \Delta(\Omega)} (\gamma_1(D), \quad \mod_C(D)) \in \mathbb{R}^2, \tag{3.3}$$

where $\Delta(\Omega)$ is the collection of all the denumerable subsets of Ω;

$$\gamma_1(D) = \sup_{t \in J} e^{-Lt} \chi(\{x(t) : x \in D\}); \tag{3.4}$$

where $\mod_C(D)$ is the modulus of equicontinuity of the set of functions D given by the formula

$$\mod_C(D) = \limsup_{\delta \to 0} \max_{x \in D} \max_{|t_1 - t_2| \le \delta} \|x(t_1) - x(t_2)\|; \tag{3.5}$$

and $L > 0$ is a positive real number chosen so that

$$q := 2M \sup_{t \in J} \int_0^t e^{-L(t-s)} \beta(s) ds < 1 \tag{3.6}$$

where $M = \sup_{(t,s) \in \Delta} \|T(t,s)\|$.

From the Arzela-Ascoli theorem, the measure ν_1 give a nonsingular and regular measure of noncompactness, (see [20]).

Let $\{y_n\}_{n=1}^{+\infty}$ be the denumerable set which achieves that maximum $\nu_1(N(\Omega))$, i,e;

$$\nu_1(N(\Omega)) = (\gamma_1(\{y_n\}_{n=1}^{+\infty}), \quad \mod_C(\{y_n\}_{n=1}^{+\infty})).$$

Then there exists a set $\{x_n\}_{n=1}^{+\infty} \subset \Omega$ such that $y_n \in \gamma_1(x_n)$, $n \ge 1$. Then

$$y_n(t) = T(t,0)\phi(0) + \int_0^t T(t,s)f(s)ds + \sum_{0 < t_k < t} T(t,t_k)I_k(x(t_k)), \tag{3.7}$$

where $f \in S_F^1$ and $I_k \in \mathcal{I}_k(x)$, so that

$$\gamma_1(\{y_n\}_{n=1}^{+\infty}) = \gamma_1(\{Gf_n\}_{n=1}^{+\infty}).$$

We give an upper estimate for $\gamma_1(\{y_n\}_{n=1}^{+\infty})$.

Fixed $t \in J$ by using condition (H4), for all $s \in [0,t]$ we have

$$
\begin{aligned}
\chi(\{f_n(s)\}_{n=1}^{+\infty}) &\le \chi(F(s, \{x_n(s)\}_{n=1}^{+\infty})) \\
&\le \chi(\{F(s, x_n(s))\}_{n=1}^{+\infty}) \\
&\le \beta(s)\chi(\{x_n(s)\}_{n=1}^{+\infty}) \\
&\le \beta(s)e^{Ls} \sup_{t \in J} e^{-Lt} \chi(\{x_n(t)\}_{n=1}^{+\infty}) \\
&= \beta(s)e^{Ls} \gamma_1(\{x_n\}_{n=1}^{+\infty}).
\end{aligned}
$$

****

Surveys in Mathematics and its Applications **7** (2012), 1 − 14
http://www.utgjiu.ro/math/sma

By using condition (H3), the set $\{f_n\}_{n=1}^{+\infty}$ is integrably bounded. In fact, for every $t \in J$, we have

$$\|f_n(t)\| \leq \|F(t, x_n(t))\|$$
$$\leq \alpha(t)(1 + \|x_n(t)\|).$$

The integrably boundedness of $\{f_n\}_{n=1}^{+\infty}$ follows from the continuity of x in J_k and the boundedness of set $\{x_n\}_{n=1}^{+\infty} \subset \Omega$. By applying Lemma 11, it follows that

$$\chi(\{gf_n(s)\}_{n=1}^{+\infty}) \leq 2M \int_0^s \beta(t)e^{Lt}(\gamma_1(\{x_n\}_{n=1}^{+\infty}))dt$$
$$= 2M\gamma_1(\{x_n\}_{n=1}^{+\infty}) \int_0^s \beta(t)e^{Lt}.$$

Thus, we get

$$\gamma_1(\{x_n\}_{n=1}^{+\infty}) \leq \gamma_1(\{y_n\}_{n=1}^{+\infty}) = \gamma_1(\{Gf_n(s)\}_{n=1}^{+\infty})$$
$$= \sup_{t \in J} e^{-Lt} 2M\gamma_1(\{x_n\}_{n=1}^{+\infty}) \int_0^s \beta(t)e^{Lt} \qquad (3.8)$$
$$\leq q\gamma_1(\{x_n\}_{n=1}^{+\infty}),$$

and hence $\gamma_1(\{x_n\}_{n=1}^{+\infty}) = 0$, then $\gamma_1(\{x_n(t)\}_{n=1}^{+\infty}) = 0$, for every $t \in J$. Consequently

$$\gamma_1(\{y_n\}_{n=1}^{+\infty}) = 0.$$

By using the last equality and hypotheses (H3) and (H4) we can prove that set $\{f_n\}_{n=1}^{+\infty}$ is semicompact. Now, by applying Lemma 9 and Lemma 10, we can conclude that set $\{Gf_n\}_{n=1}^{+\infty}$ is relatively compact in $C(J; E)$. The representation of y_n given by (3.7) yields that set $\{y_n\}_{n=1}^{+\infty}$ is also relatively compact in $C(J; E)$, therefore $\nu_1(\Omega) = (0, 0)$. Then Ω is a relatively compact set.

Step 4. A priori bounds.

We will demonstrate that the solutions set is a priori bounded. Indeed, let $x \in N$. Then there exists $f \in S_{F(.,.,x_t(.))}^1$ and $I_k \in \mathcal{I}_k(x)$ such that for every $t \in J$ we have

$$\|x(t)\| = \left\| T(t, 0)\phi(0) + \int_0^t T(t, s)f(s)ds + \sum_{0 < t_k < t} T(t, t_k)I_k(x(t_k)) \right\|$$

$$\leq M(\|\phi(0)\| + \sum_{0 < t_k < t} \|a_k\|) + M \int_0^t f(s)ds$$

$$\leq M(\|\phi(0)\| + \sum_{0 < t_k < t} \|a_k\|) + M \int_0^b \alpha(s)(1 + \|x[\phi]_s\|)ds.$$

Using condition (A1) we have

$$\|x(t)\| \le M(\|\phi(0)\| + \sum_{0 < t_k < t} \|a_k\|) + M \int_0^t \alpha(s)(1 + N_b\|\phi\|_{\mathcal{B}} + K_b \sup_{0 \le \theta \le s} \|x(\theta)\|)ds$$

$$\le M(\|\phi(0)\| + \sum_{k=1}^m \|a_k\|) + M(1 + N_b\|\phi\|_{\mathcal{B}})\|\alpha\|_{L^1(J)}$$

$$+ MK_b \int_0^t \alpha(s) \sup_{0 \le \theta \le s} \|x(\theta)\|ds.$$

Since the last expression is a nondecreasing function of t, we have that

$$\sup_{0 \le \theta \le t} \|x(\theta)\| \le \|M(\|\phi(0)\| + \sum_{k=1}^m \|a_k\|) + M(1 + N_b\|\phi\|_{\mathcal{B}})\|\alpha\|_{L^1(J)}$$

$$+ MK_b \int_0^t \alpha(s) \sup_{0 \le \theta \le s} \|x(\theta)\|ds.$$

Invoking Gronwall's inequality, we get

$$\sup_{0 \le \theta \le t} \|x(\theta)\| \le \zeta e^{MK_b\|\alpha\|_{L^1[0,t]}},$$

where

$$\zeta = M(\|\phi(0)\| + \sum_{k=1}^m \|a_k\|) + M(1 + N_b\|\phi\|_{\mathcal{B}})\|\alpha\|_{L^1(J)},$$

which completes the proof. □

4   An example

As an application of our results we consider the following impulsive partial functional differential equation of the form

$$\frac{\partial}{\partial t}z(t,x) \in a(t,x)\frac{\partial^2}{\partial x^2}z(t,x) + \int_{-\infty}^0 P(\theta)r(t, z(t+\theta, x))d\theta, \quad x \in [0, \pi],\ t \in [0, b], t \ne t_k,$$
$$(4.1)$$

$$z(t_k^+, x) - z(t_k^-, x) \in [-b_k|z(t_k^-, x), b_k|z(t_k^-, x)], \quad x \in [0, \pi],\ k = 1, \dots, m, \quad (4.2)$$

$$z(t, 0) = z(t, \pi), \quad t \in J := [0, b], \quad (4.3)$$

$$z(t, x) = \phi(t, x), \quad -\infty < t \le 0,\ x \in [0, \pi], \quad (4.4)$$

where $a(t, x)$ is continuous function and uniformly Hölder continuous in t, $b_k > 0$, $k = 1, \dots, m$, $\phi \in \mathcal{D}$,

$\mathcal{D} = \{\overline{\psi} : (-\infty, 0] \times [0, \pi] \to \mathbb{R}; \ \overline{\psi}$ is continuous everywhere except for a countable number of points at which $\overline{\psi}(s^-), \overline{\psi}(s^+)$ exist with $\overline{\psi}(s^-) = \overline{\psi}(s)\}$,
$0 = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = b, \ z(t_k^+) = \lim_{(h,x) \to (0^+, x)} z(t_k + h, x),$
$z(t_k^-) = \lim_{(h,x) \to (0^-, x)} z(t_k + h, x), \ P : (-\infty, 0] \to \mathbb{R}$ a continuous function, $r :$
$\mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cv,k}(\mathbb{R})$ a Caratheodory multivalued map.

Let

$$y(t)(x) = z(t, x), \quad x \in [0, \pi], \ t \in J = [0, b],$$

$$\mathcal{I}_k(y(t_k^-))(x) = [-b_k|z(t_k^-, x), b_k|z(t_k^-, x)], \quad x \in [0, \pi], \ k = 1, \ldots, m,$$

$$F(t, \phi)(x) = \int_{-\infty}^{0} P(\theta)r(t, z(t + \theta, x))d\theta$$

$$\phi(\theta)(x) = \phi(\theta, x), \quad -\infty < t \le 0, \ x \in [0, \pi].$$

Consider $E = L^2[0, \pi]$ and define $A(t)$ by $A(t)w = a(t, x)w''$ with domain

$D(A) = \{w \in E : \ w, w'$ are absolutely continuous, $w'' \in E, \ w(0) = w(\pi) = 0\}.$

Then $A(t)$ generates an evolution system $U(t, s)$ satisfying assumption (H1) and (H3) (see [13]). We can show that problem (4.1)–(4.4) is an abstract formulation of problem (1.1)–(1.3). Under suitable conditions, the problem (1.1)–(1.3) has at least one mild solution.

References

[1] N. Abada, R. Agarwal, M. Benchohra and H. Hammouche, Existence results for nondensely defined impulsive semilinear functional differential equations with state-dependent delay, *Asian-Eur. J. Math.* **1** (2008), no. 4, 449-468. MR2474181(2009i:34179). Zbl 1179.34070.

[2] N. Abada, M. Benchohra, H. Hammouche, Existence and controllability results for impulsive partial functional differential inclusions, *Nonlinear Anal.* **69** (2008) 2892-2909. MR2452100(2009h:34088). Zbl 1160.34068.

[3] N. Abada, M. Benchohra, H. Hammouche, Nonlinear impulsive partial functional differential inclusions with state-dependent delay and multivalued jumps, *Nonlinear Anal.* **4** (2010) 791-803. MR2680247(2011h:34132). Zbl 1207.34077.

[4] N.U. Ahmed, *Semigroup Theory with Applications to Systems and Control*, in: Pitman Research Notes in Mathematics Series, vol. 246, Longman Scientific & Technical, Harlow, John Wiley & Sons, New York, 1991. MR1100706(92e:47069). Zbl 0727.47026.

[5] N.U. Ahmed, *Dynamic Systems and Control with Applications*, World Scientific Publishing, Hackensack, NJ, 2006. MR2257896(2007m:93001). Zbl 1127.93001.

[6] N.U. Ahmed, Systems governed by impulsive differential inclusions on Hilbert spaces, *Nonlinear Anal.* **45** (2001) 693-706. MR1841203(2002e:34016). Zbl 0995.34053.

[7] D.D. Bainov, P.S. Simeonov, *Systems with Impulsive Effect*, Horwood, Chichester, 1989. MR1010418(90i:93082). Zbl 0683.34032.

[8] M. Benchohra, J. Henderson, S.K. Ntouyas, *Impulsive Differential Equations and Inclusions*, vol. 2, Hindawi Publishing Corporation, New York, 2006. MR2322133(2008f:34001). Zbl 1130.34003.

[9] M. Benchohra, S.K. Ntouyas, Nonlocal Cauchy problems for neutral functional differential and integrodifferential inclusions in Banach spaces, *J. Math. Anal. Appl.* **258** (2001) 573-590. MR1835560(2002d:34130). Zbl 0982.45008.

[10] I. Benedetti, An existence result for impulsive functional differential inclusions in Banach spaces, *Discuss. Math. Differ. Incl. Control Optim.* **24** (2004), 13-30. MR2118212(2005j:34014). Zbl 1071.34087.

[11] I. Benedetti, P. Rubbioni, Existence of solutions on compact and non-compact intervals for semilinear impulsive differential inclusions with delay. *Topol. Methods Nonlinear Anal.* **32** (2008), 227-245. MR2494056(2010f:34114). Zbl 1189.34125.

[12] T. Cardinali, P. Rubbioni, On the existence of mild solutions of semilinear evolution differential inclusions, *J. Math. Anal. Appl.* **308** (2005), 620-635. MR2150113(2006a:34172). Zbl 1083.34046.

[13] A. Freidman, *Partial Differential Equations*. Holt, Rinehat and Winston, New York, 1969. MR445088(56 #3433). Zbl 0224.35002.

[14] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Topological Fixed Point Theory and Its Applications Vol. 4, Kluwer Academic Publishers, Dordrecht, Second edition, 2006. MR2238622(2007f:58014). Zbl 1107.55001.

[15] C. Gori, V. Obukhovskii, M. Ragni and P. Rubbioni, Existence and continuous dependence results for semilinear functional differential inclusions with infinite delay, *Nonlinear Anal.* **51** (2002), 765-782. MR1921375(2003f:34149). Zbl 1018.34076.

[16] J. Hale, J. Kato, Phase space for retarded equations with infnite delay, *Funkcial. Ekvac.* **21** (1978), 11-41. MR0492721(58 #11793). Zbl 0383.34055.

[17] J.K. Hale, S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, in: Applied Mathematical Sciences, vol. 99, Springer- Verlag, New York, 1993. MR1243878(94m:34169). Zbl 0787.34002.

[18] Y. Hino, S. Murakami and T. Naito, *Functional-differential equations with infnite delay*, Lecture Notes in Mathematics, Vol. 1473, Berlin, Springer- Verlag, 1991. MR1122588(92g:34088). Zbl 0732.34051.

[19] Sh. Hu, N. Papageorgiou, *Handbook of Multivalued Analysis*, Volume I: Theory, Kluwer Academic Publishers, Dordrecht, 1997. MR1485775(98k:47001). Zbl 0887.47001.

[20] M. Kamenskii, V. Obukhovskii, P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, in: de Gruyter Series in Nonlinear Analysis and Applications, Berlin, 2001. MR1831201(2002e:47066). Zbl 0988.34001.

[21] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991. MR1135796(93c:49001). Zbl 0731.49001.

[22] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989. MR1082551(91m:34013). Zbl 0719.34002.

[23] S. Migorski, A. Ochal, Nonlinear impulsive evolution inclusions of second order, *Dynam. Systems Appl.* **16** (2007), 155-173. MR2305434(2008a:34137). Zbl 1128.34038.

[24] V. Obukhovskii, J.-C. Yao, On impulsive functional differential inclusions with Hille-Yosida operators in Banach spaces. *Nonlinear Anal.* **73** (2010), 1715-1728. MR2661354(2011m:34184). Zbl 1214.34052.

[25] A.M. Samoilenko, N.A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995. MR1355787(97i:34002). Zbl 0837.34003.

[26] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, in: Applied Mathematical Sciences, vol. 119, Springer-Verlag, New York, 1996. MR1415838(98a:35135). Zbl 0870.35116.

Mouffak Benchohra

Laboratoire de Mathématiques,

Université de Sidi Bel-Abbès,

B.P. 89, 22000, Sidi Bel-Abbès, Algérie.

e-mail: benchohra@yahoo.com

Mohamed Ziane

Départemaent des Mathématiques,

IUniversité de Tiaret,

B.P. 78, 14000, Tiaret, Algérie.

e-mail: ziane@mail.univ-tiaret.dz