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ON THE LINKING ALGEBRA OF HILBERT MODULES AND MORITA EQUIVALENCE OF LOCALLY C^*-ALGEBRAS

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Abstract. In this paper we introduce the notion of linking algebra of a Hilbert module over a locally C^*-algebra and we extend in the context of locally C^*-algebras a result of Brown, Green and Rieffel [Pacific J.,1977] which states that two C^*-algebras are strongly Morita equivalent if and only if they are isomorphic with two complementary full corners of a C^*-algebra.

1 Introduction and preliminaries

A locally C^*-algebra is a complete Hausdorff complex topological $*$-algebra A whose topology is determined by a directed family of C^*-seminorms in the sense that a net $\{a_i\}_{i\in I}$ converges to 0 if and only if the net $\{p(a_i)\}_{i\in I}$ converges to 0 for all continuous C^*-seminorm p on A. The term of locally C^*-algebra is due to Inoue [3].

Now, we recall some facts about locally C^*-algebras from [2], [3], [5] and [8].

For a given locally C^*-algebra A we denote by $S(A)$ the set of all continuous C^*-seminorms on A. For $p \in S(A)$, the quotient $*$-algebra $A/\ker p$, denoted by A_p, where $\ker p = \{a \in A; p(a) = 0\}$ is a C^*-algebra in the C^*-norm induced by p. The canonical map from A to A_p is denoted by π_p. For $p, q \in S(A)$ with $p \geq q$, there is a surjective canonical map π_{pq} from A_p onto A_q such that $\pi_{pq}(\pi_p(a)) = \pi_q(a)$ for all $a \in A$. Then $\{A_p; \pi_{pq}\}_{p,q\in S(A),p\geq q}$ is an inverse system of C^*-algebras and $\lim\limits_{\leftarrow p} A_p$ is a locally C^*-algebra which can be identified with A.

A Fréchet locally C^*-algebra is a locally C^*-algebra whose topology is determined by a countable family of C^* -seminorms.

A morphism of locally C^*-algebras is a continuous $*$-morphism from a locally C^*-algebra A to another locally C^*-algebra B. An isomorphism of locally C^*-algebras

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from A to B is a bijective map $\Phi : A \to B$ such that Φ and Φ^{-1} are morphisms of locally C^*-algebras.

Hilbert modules over locally C^*-algebras are generalizations of Hilbert C^*-modules by allowing the inner product to take values in a locally C^*-algebra rather than in a C^*-algebra. Here we recall some facts about Hilbert modules over locally C^* -algebras from [5] and [8].

A pre -Hilbert A-module is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$ which is $\mathbb{C}$ -and A-linear in its second variable and satisfies the following relations:

i. $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$;

ii. $\langle \xi, \xi \rangle \geq 0$ for every $\xi \in E$;

iii. $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$.

We say that E is a Hilbert A-module if E is complete with respect to the topology determined by the family of seminorms $\{\overline{p}_E\}_{p \in S(A)}$ where $\overline{p}_E(\xi) = \sqrt{p(\langle \xi, \xi \rangle)}, \xi \in E$.

We say that a Hilbert A -module E is full if the linear subspace $\langle E, E \rangle$ of A generated by $\{\langle \xi, \eta \rangle, \ \xi, \eta \in E\}$ is dense in A.

Let E be a Hilbert A-module. For $p \in S(A)$, $\ker \overline{p}_E = \{\xi \in E; \overline{p}_E(\xi) = 0\}$ is a closed submodule of E and the quotient linear space $E/\ker \overline{p}_E$, denoted by E_p, is a Hilbert A_p-module with $(\xi + \ker \overline{p}_E)\pi_p(a) = \xi a + \ker \overline{p}_E$ and $\langle \xi + \ker \overline{p}_E, \eta + \ker \overline{p}_E \rangle = \pi_p(\langle \xi, \eta \rangle)$. The canonical map from E onto E_p is denoted by σ_p^E. For $p, q \in S(A)$, $p \geq q$ there is a canonical surjective morphism of vector spaces σ_{pq}^E from E_p onto E_q such that $\sigma_{pq}^E(\sigma_p^E(\xi)) = \sigma_q^E(\xi)$, $\xi \in E$. Then $\{E_p; A_p; \sigma_{pq}^E, \pi_{pq}\}_{p,q \in S(A), p \geq q}$ is an inverse system of Hilbert C^*-modules in the following sense: $\sigma_{pq}^E(\xi_p a_p) = \sigma_{pq}^E(\xi_p)\pi_{pq}(a_p), \xi_p \in E_p, a_p \in A_p$; $\langle \sigma_{pq}^E(\xi_p), \sigma_{pq}^E(\eta_p) \rangle = \pi_{pq}(\langle \xi_p, \eta_p \rangle), \xi_p, \eta_p \in E_p$; $\sigma_{pp}^E(\xi_p) = \xi_p$, $\xi_p \in E_p$ and $\sigma_{qr}^E \circ \sigma_{pq}^E = \sigma_{pr}^E$ if $p \geq q \geq r$, and $\lim\limits_{\leftarrow p} E_p$ is a Hilbert A-module which can be identified with E.

Let E and F be Hilbert A-modules. The set $L_A(E, F)$ of all adjointable A-module morphisms from E into F becomes a locally convex space with topology defined by the family of seminorms $\{\widetilde{p}_{L_A(E,F)}\}_{p \in S(A)}$, where $\widetilde{p}_{L_A(E,F)}(T) = \|(\pi_p)_*(T)\|_{L_{A_p}(E_p,F_p)}$, $T \in L_A(E, F)$ and $(\pi_p)_*(T)(\xi + \ker \overline{p}_E) = T\xi + \ker \overline{p}_F$, $\xi \in E$. Moreover, $\{L_{A_p}(E_p, F_p); \ (\pi_{pq})_*\}_{p,q \in S(A), p \geq q}$, where $(\pi_{pq})_* : L_{A_p}(E_p, F_p) \to L_{A_q}(E_q, F_q)$, $(\pi_{pq})_*(T_p)(\sigma_q^E(\xi)) = \sigma_{pq}^F(T_p(\sigma_p^E(\xi)))$, is an inverse system of Banach spaces, and $\lim\limits_{\leftarrow p} L_{A_p}(E_p, F_p)$ can be identified with $L_A(E, F)$. Thus topologized, $L_A(E, E)$ becomes a locally C^*-algebra, and we write $L_A(E)$ for $L_A(E, E)$.

We say that the Hilbert A-modules E and F are unitarily equivalent if there is a unitary element U in $L_A(E, F)$ (namely, $U^*U = \mathrm{id}_E$ and $UU^* = \mathrm{id}_F$).

For $\xi \in E$ and $\eta \in F$ we consider the rank one homomorphism $\theta_{\eta,\xi}$ from E into F defined by $\theta_{\eta,\xi}(\zeta) = \eta \langle \xi, \zeta \rangle$. Clearly, $\theta_{\eta,\xi} \in L_A(E,F)$ and $\theta^*_{\eta,\xi} = \theta_{\xi,\eta}$.We denote by $K_A(E,F)$ the closed linear subspace of $L_A(E,F)$ spanned by $\{\theta_{\eta,\xi}; \xi \in E, \eta \in F\}$, and we write $K_A(E)$ for $K_A(E,E)$. Moreover, $K_A(E,F)$ may be identified with $\lim\limits_{\overleftarrow{p}} K_{A_p}(E_p, F_p)$.

Two locally C^*-algebras A and B are strongly Morita equivalent if there is a full Hilbert A -module E such that the locally C^*-algebras B and $K_A(E)$ are isomorphic [4]. In [4], we prove that the strong Morita equivalence is an equivalence relation on the set of all locally C^* -algebras. Also we prove that two Fréchet locally C^*-algebras A and B are strongly Morita equivalent if and only if they are stably isomorphic. This result extends in the context of locally C^*-algebras a well known result of Brown, Green and Rieffel [1, Theorem 1.2]. In this paper, we extend in the context of locally C^*-algebras another result of Brown, Green and Rieffel [1, Theorem 1.1] which states that two C^*-algebras A and B are strongly Morita equivalent if and only if there is a C^*-algebra C such that A and B are isomorphic with two complementary full corners in C, Theorem 9. For this, we introduce the notion of linking algebra of a Hilbert module E over a locally C^* -algebra A. Finally, using the fact that any Hilbert C^*-module E is non-degenerate as Hilbert module (that is, for any $\xi \in E$ there is $\eta \in E$ such that $\xi = \eta \langle \eta, \eta \rangle$), and taking into account that the linking algebra of E is in fact the inverse limit of the linking algebras of E_p, $p \in S(A)$, we prove that any Hilbert module E over a locally C^*-algebra A is non-degenerate as Hilbert module, Proposition 10.

2 The main results

Let A be a locally C^* -algebra. A multiplier of A is a pair (l,r), where l and r are linear maps from A to A such that $l(ab) = l(a)b$, $r(ab) = ar(b)$ and $al(b) = r(a)b$ for all a and b in A. The set $M(A)$ of all multipliers of A is an algebra with involution; addition is defined as usual, multiplication is $(l_1, r_1)(l_2, r_2) = (l_1 l_2, r_1 r_2)$ and involution is $(l,r)^* = (r^*, l^*)$, where $r^*(a) = r(a^*)^*$ and $l^*(a) = l(a^*)^*$ for all $a \in A$. For each $p \in S(A)$, the map $p_{M(A)} : M(A) \to [0,\infty)$ defined by $p_{M(A)}(l,r) = \sup\{p(l(a)); a \in A,\ p(a) \leq 1\}$ is a C^*-seminorm on $M(A)$, and $M(A)$ with the topology determined by the family of C^* -seminorms $\{p_{M(A)}\}_{p\in S(A)}$ is a locally C^*-algebra [8].

Let $p, q \in S(A)$ with $p \geq q$. Since the C^*-morphism $\pi_{pq} : A_p \to A_q$ is surjective, it extends to a unique morphism of W^*-algebras $\pi''_{pq} : A''_p \to A''_q$, where A''_p is the enveloping W^*-algebra of A_p. Moreover, $\pi''_{pq}(M(A_p)) \subseteq M(A_q)$, and $\{M(A_p); \pi''_{pq}|_{M(A_p)}\}_{p\geq q, p,q\in S(A)}$ is an inverse system of C^*-algebras. The locally C^*-algebras $M(A)$ and $\lim\limits_{\overleftarrow{p}} M(A_p)$ are isomorphic [8].

Any locally C^*-algebra A is a Hilbert A -module with the inner product defined by $\langle a,b\rangle = a^*b$, $a,b \in A$, and the locally C^*-algebras $M(A)$ and $L_A(A)$ are isomorphic [8].

Let C be a locally C^*-algebra.

Definition 1. *A locally C^* -subalgebra A of C is called a corner if there is a projection $\mathcal{P} \in M(C)$ (that is, $\mathcal{P} = \mathcal{P}^*$ and $\mathcal{P}\mathcal{P} = \mathcal{P}$) such that $A = \mathcal{P}C\mathcal{P}$.*

Remark 2. *If A is a corner in C, then $A_p = \pi_p^{''}(\mathcal{P})\ C_p\pi_p^{''}(\mathcal{P})$ and so A_p is a corner in C_p for each $p \in S(C)$.*

Definition 3. *Two corners $\mathcal{P}C\mathcal{P}$ and $\mathcal{Q}C\mathcal{Q}$ in C are complementary if $\mathcal{P} + \mathcal{Q} = 1_{M(C)}$ ($1_{M(C)}$ denotes the unit of $M(C)$).*

Remark 4. *If $\mathcal{P}C\mathcal{P}$ and $\mathcal{Q}C\mathcal{Q}$ are two complementary corners in C, then $\pi_p^{''}(\mathcal{P})C_p\pi_p^{''}(\mathcal{P})$ and $\pi_p^{''}(\mathcal{Q})C_p\pi_p^{''}(\mathcal{Q})$ are two complementary corners in C_p for each $p \in S(C)$.*

Definition 5. *A corner $\mathcal{P}C\mathcal{P}$ in C is full if $C\mathcal{P}C$ is dense in C.*

Remark 6. *If $\mathcal{P}C\mathcal{P}$ is a full corner in C, then $\pi_p^{''}(\mathcal{P})C_p\pi_p^{''}(\mathcal{P})$ is a full corner in C_p for each $p \in S(C)$.*

Proposition 7. *Let C be a locally C^*-algebra and let A be a full corner in C. Then A and C are strongly Morita equivalent.*

Proof. Let $\mathcal{P} \in M(C)$ such that $A = \mathcal{P}C\mathcal{P}$. Then $C\mathcal{P}$ is a full Hilbert A -module with the action of A on $C\mathcal{P}$ defined by $(c\mathcal{P}, a) \to ca$ and the inner product defined by $\langle c\mathcal{P}, d\mathcal{P}\rangle = \mathcal{P}c^*d\mathcal{P}$. Moreover, for each $p \in S(C)$, the Hilbert A_p -modules $(C\mathcal{P})_p$ and $C_p\pi_p^{''}(\mathcal{P})$ are unitarily equivalent. Then the locally C^*-algebras $K_A(C\mathcal{P})$ and $\lim\limits_{\leftarrow p} K_{A_p}(\ C_p\pi_p^{''}(\mathcal{P}))$ are isomorphic [8, Proposition 4.7].

For each $p \in S(A)$, since A_p is a full corner in C_p, the C^*-algebras A_p and C_p are strongly Morita equivalent [1, Theorem 1.1]. Moreover, $C_p\pi_p^{''}(\mathcal{P})$ is a full Hilbert A_p -module such that the C^*-algebras $K_{A_p}\left(C_p\pi_p^{''}(\mathcal{P})\right)$ and C_p are isomorphic. The linear map $\Phi_p : C_p \to K_{A_p}\left(C_p\pi_p^{''}(\mathcal{P})\right)$ defined by

$$\Phi_p\left(\pi_p\left(c\mathcal{P}d\right)\right) = \theta_{\pi_p(c\mathcal{P}),\pi_p(d^*\mathcal{P})}$$

is a morphism of C^*-algebras. Indeed, from

$$\begin{aligned} \left\|\Phi_p\left(\pi_p\left(c\mathcal{P}d\right)\right)\right\|_{K_{A_p}\left(C_p\pi_p^{''}(\mathcal{P})\right)} &= \left\|\theta_{\pi_p(c\mathcal{P}),\pi_p(d^*\mathcal{P})}\right\|_{K_{A_p}\left(C_p\pi_p^{''}(\mathcal{P})\right)} \\ &= \sup\{\left\|\pi_p\left(c\mathcal{P}de\mathcal{P}\right)\right\|_{C_p}; e \in C, \left\|\pi_p\left(e\mathcal{P}\right)\right\|_{C_p} \le 1\} \\ &\le \left\|\pi_p\left(c\mathcal{P}d\right)\right\|_{C_p} \end{aligned}$$

for all $c, d \in E$, we conclude that Φ_p is continuous, and since

$$\Phi_p\left(\pi_p\left(c\mathcal{P}d\right)^*\right) = \theta_{\pi_p(d^*\mathcal{P}),\pi_p(c\mathcal{P})} = \theta^*_{\pi_p(c\mathcal{P}),\pi_p(d^*\mathcal{P})} = \Phi_p\left(\pi_p\left(c\mathcal{P}d\right)\right)^*$$

and

$$\begin{aligned}\Phi_p\left(\pi_p\left(c\mathcal{P}d\right)\right)\Phi_p\left(\pi_p\left(e\mathcal{P}f\right)\right) &= \theta_{\pi_p(c\mathcal{P}),\pi_p(d^*\mathcal{P})}\theta_{\pi_p(e\mathcal{P}),\pi_p(f^*\mathcal{P})} \\ &= \theta_{\pi_p(c\mathcal{P}),\pi_p(f^*\mathcal{P}e^*d^*)} \\ &= \Phi_p\left(\pi_p\left(c\mathcal{P}de\mathcal{P}f\right)\right)\end{aligned}$$

for all $c, d, e, f \in C$, Φ_p is a morphism of C^*-algebras. Moreover, Φ_p is surjective. If $\Phi_p\left(\pi_p\left(c\mathcal{P}d\right)\right) = 0$, then $\pi_p\left(c\mathcal{P}de\mathcal{P}\right) = 0$ for all $e \in C$, in particular we have $\pi_p\left(c\mathcal{P}dd^*\mathcal{P}c^*\right) = 0$. This implies that $\pi_p\left(c\mathcal{P}d\right) = 0$ and so Φ_p is injective. Therefore Φ_p is an isomorphism of C^* -algebras from C_p onto $K_{A_p}\left(C_p\pi_p^{''}(\mathcal{P})\right)$.

It is not difficult to check that $(\Phi_p)_p$ is an inverse system of isomorphisms of C^*-algebras. Let $\Phi = \lim\limits_{\leftarrow p}\Phi_p$. Then Φ is an isomorphism of locally C^* -algebras from $\lim\limits_{\leftarrow p} C_p$ onto $\lim\limits_{\leftarrow p} K_{A_p}\left(C_p\pi_p^{''}(\mathcal{P})\right)$. From this fact and taking into account that the locally C^*-algebras $\lim\limits_{\leftarrow p} K_{A_p}\left(C_p\pi_p^{''}(\mathcal{P})\right)$ and $K_A(C\mathcal{P})$ can be identified as well as the locally C^*-algebras $\lim\limits_{\leftarrow p} C_p$ and C, we conclude that the locally C^*-algebras $K_A(C\mathcal{P})$ and C are isomorphic. Therefore the locally C^*-algebras A and C are strongly Morita equivalent. □

From Proposition 7 and taking into account that the strong Morita equivalence is an equivalence relation on the set of all locally C^*-algebras we obtain the following corollary.

Corollary 8. *Let C be a locally C^*-algebra. If A and B are two full corners in C, then A and B are strongly Morita equivalent.*

Let E be a Hilbert A-module.

The direct sum $A \oplus E$ of the Hilbert A -modules A and E is a Hilbert A -module with the action of A on $A \oplus E$ defined by

$$(A \oplus E, A) \ni (a \oplus \xi, b) \to (a \oplus \xi)\, b = ab \oplus \xi b \in A \oplus E$$

and the inner product defined by

$$(A \oplus E, A \oplus E) \ni (a \oplus \xi, b \oplus \eta) \to \langle a \oplus \xi, b \oplus \eta\rangle = a^*b + \langle \xi, \eta\rangle \in A.$$

Moreover, for each $p \in S(A)$, the Hilbert A_p-modules $(A \oplus E)_p$ and $A_p \oplus E_p$ can be identified. Then the locally C^*-algebras $L_A(A \oplus E)$ and $\lim\limits_{\leftarrow p} L_{A_p}(A_p \oplus E_p)$ can be identified.

Let $a \in A$, $\xi \in E$, $\eta \in E$ and $T \in K_A(E)$. The map

$$L_{a,\xi,\eta,T} : A \oplus E \to A \oplus E$$

defined by

$$L_{a,\xi,\eta,T}(b \oplus \zeta) = (ab + \langle \xi, \zeta \rangle) \oplus (\eta b + T(\zeta))$$

is an element in $L_A(A \oplus E)$. Moreover, $(L_{a,\xi,\eta,T})^* = L_{a^*,\eta,\xi,T^*}$. The locally C^* -subalgebra of $L_A(A \oplus E)$ generated by

$$\{L_{a,\xi,\eta,T}; a \in A, \xi \in E, \eta \in E, T \in K_A(E)\}$$

is denoted by $\mathcal{L}(E)$ and it is called the linking algebra of E.

By Lemma III 3.2 in [7], we have

$$\mathcal{L}(E) = \lim_{\substack{\leftarrow \\ p}} \overline{(\pi_p)_* (\mathcal{L}(E))},$$

where $\overline{(\pi_p)_* (\mathcal{L}(E))}$ means the closure of the vector space $(\pi_p)_* (\mathcal{L}(E))$ in $L_{A_p}(A_p \oplus E_p)$. Let $p \in S(A)$. From

$$(\pi_p)_* (L_{a,\xi,\eta,T}) = L_{\pi_p(a),\sigma_p^E(\xi),\sigma_p^E(\eta),(\pi_p)_*(T)}$$

for all $a, b \in A$, for all $\xi, \eta, \zeta \in E$, and taking into account that $\mathcal{L}(E_p)$, the linking algebra of E_p, is generated by

$$\{L_{\pi_p(a),\sigma_p^E(\xi),\sigma_p^E(\eta),(\pi_p)_*(T)}; a \in A, \xi \in E, \eta \in E, T \in K_A(E)\}$$

since $\pi_p(A) = A_p$, $\sigma_p^E(E) = E_p$, and $\overline{(\pi_p)_* (K(E))} = K(E_p)$, we conclude that

$$\mathcal{L}(E) = \lim_{\substack{\leftarrow \\ p}} \mathcal{L}(E_p).$$

Moreover, since $\mathcal{L}(E_p) = K_{A_p}(A_p \oplus E_p)$ and the locally C^* -algebras $K_A(A \oplus E)$ and $\lim\limits_{\substack{\leftarrow \\ p}} K_{A_p}(A_p \oplus E_p)$ can be identified, the linking algebra of E coincides with $K_A(A \oplus E)$.

The following theorem is a generalization of Theorem 1.1 in [1].

Theorem 9. *Two locally C^*-algebras A and B are strongly Morita equivalent if and only if there is a locally C^*-algebra C with two complementary full corners isomorphic with A respectively B.*

Proof. First we suppose that A and B are strongly Morita equivalent. Then there is a full Hilbert A -module E such that the locally C^* -algebras $K_A(E)$ and B are isomorphic. Let $C = \mathcal{L}(E)$. Then, for each $p \in S(A)$, $C_p = \mathcal{L}(E_p)$. By Theorem 1.1 in [1] we have:

1. $i_{A_p} : A_p \to C_p$ defined by $i_{A_p}(a_p) = L_{a_p,0,0,0}$ is an isometric morphism of C^*-algebras;

2. the map $\mathcal{P}_p : A_p \oplus E_p \to A_p \oplus E_p$ defined by $\mathcal{P}_p\left(a_p \oplus \xi_p\right) = a_p \oplus 0$ is a projection in $L_{A_p}\left(A_p \oplus E_p\right)$;

3. $C_p\mathcal{P}_pC_p$ is dense in C_p;

4. $\mathcal{P}_pC_p\mathcal{P}_p = i_{A_p}(A_p)$;

5. $i_{K_{A_p}(E_p)} : K_{A_p}(E_p) \to C_p$ defined by $i_{K_{A_p}(E_p)}(T_p) = L_{0,0,0,T_p}$ is an isometric morphism of C^*-algebras;

6. the map $\mathcal{Q}_p : A_p \oplus E_p \to A_p \oplus E_p$ defined by $\mathcal{Q}_p\left(a_p \oplus \xi_p\right) = 0 \oplus \xi_p$ is a projection in $L_{A_p}\left(A_p \oplus E_p\right)$;

7. $C_p\mathcal{Q}_pC_p$ is dense in C_p;

8. $\mathcal{Q}_pC_p\mathcal{Q}_p = i_{K_{A_p}(E_p)}(K_{A_p}(E_p))$.

It is not difficult to check that $(\mathcal{P}_p)_p$ and $(\mathcal{Q}_p)_p$ are elements in $\lim\limits_{\overleftarrow{p}} L_{A_p}\left(A_p \oplus E_p\right)$. Let $\mathcal{P} \in L_A\left(A \oplus E\right)$ such that $\left(\pi_p\right)_*(\mathcal{P}) = \mathcal{P}_p$ for all $p \in S(A)$ and $\mathcal{Q} \in L_A\left(A \oplus E\right)$ such that $\left(\pi_p\right)_*(\mathcal{Q}) = \mathcal{Q}_p$ for each $p \in S(A)$. Clearly, $\mathcal{P}$ and $\mathcal{Q}$ are projections in $L_A\left(A \oplus E\right)$ and $\mathcal{P} + \mathcal{Q} = \mathrm{id}_{A\oplus E}$. Also it is not difficult to check that $\left(i_{A_p}\right)_p$ is an inverse system of isometric morphisms of C^*-algebras as well as $\left(i_{K_{A_p}(E_p)}\right)_p$. Let $i_A = \lim\limits_{\overleftarrow{p}} i_{A_p}$ and $i_{K_A(E)} = \lim\limits_{\overleftarrow{p}} i_{K_{A_p}(E_p)}$. Then i_A is the embedding of A in C and $i_{K_A(E)}$ is the embedding of $K_A(E)$ in C. Moreover, we have:

$$i_A\left(A\right) = \lim_{\overleftarrow{p}} i_{A_p}(A_p) = \lim_{\overleftarrow{p}} \mathcal{P}_pC_p\mathcal{P}_p = \lim_{\overleftarrow{p}} \overline{\pi_p\left(\mathcal{P}C\mathcal{P}\right)} = \mathcal{P}C\mathcal{P};$$

$$\begin{aligned} i_{K_A(E)}\left(K_A(E)\right) &= \lim_{\overleftarrow{p}} i_{K_{A_p}(E_p)}(K_{A_p}(E_p)) = \lim_{\overleftarrow{p}} \mathcal{Q}_pC_p\mathcal{Q}_p \\ &= \lim_{\overleftarrow{p}} \overline{\pi_p\left(\mathcal{Q}C\mathcal{Q}\right)} = \mathcal{Q}C\mathcal{Q} \end{aligned}$$

Therefore the locally C^* -algebras A and $K_A(E)$ are isomorphic with two complementary corners in C. Moreover, these corners are full since

$$\overline{C\mathcal{P}C} = \lim_{\overleftarrow{p}} \overline{\pi_p\left(C\mathcal{P}C\right)} = \lim_{\overleftarrow{p}} \overline{C_p\mathcal{P}_pC_p} = \lim_{\overleftarrow{p}} C_p = C$$

and

$$\overline{C\mathcal{Q}C} = \lim_{\substack{\leftarrow \\ p}} \overline{\pi_p\left(C\mathcal{Q}C\right)} = \lim_{\substack{\leftarrow \\ p}} \overline{C_p\mathcal{Q}_pC_p} = \lim_{\substack{\leftarrow \\ p}} C_p = C.$$

Thus we showed that the locally C^*-algebras A and B are isomorphic with two complementary full corners in C.

The converse implication is proved by Corollary 8. □

It is well known that a Hilbert module E over a locally C^*-algebra A is non-degenerate as topological A -module in the sense that the linear space generated by $\{\xi a, \xi \in E, a \in A\}$ is dense in E (see, for example, [5]). Also it is well known that a Hilbert C^* -module E is non-degenerate as Hilbert module, that is, for any $\xi \in E$ there is $\eta \in E$ such that $\xi = \eta \langle \eta, \eta \rangle$. By analogy with the case of Hilbert C^*-modules, we prove that any Hilbert module E over a locally C^*-algebra A is non-degenerate as Hilbert module.

Proposition 10. *Let E be a Hilbert A -module. Then for each $\xi \in E$ there is $\eta \in E$ such that $\xi = \eta \langle \eta, \eta \rangle$.*

Proof. Let $T = L_{0,\xi,\xi,0} \in \mathcal{L}(E)$. Clearly, $T = T^*$. As in the case of Hilbert C^* -modules, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t) = t^{\frac{1}{3}}$. Then, for each $p \in S(A)$, there is $\eta_p \in E_p$ such that

$$f\left(L_{0,\sigma_p^E(\xi),\sigma_p^E(\xi),0}\right) = L_{0,\eta_p,\eta_p,0}$$

(see, for example, [9, Proposition 2.31]) and moreover, $\sigma_p^E(\xi) = \eta_p \left\langle \eta_p, \eta_p \right\rangle$. But, by functional calculus (see, for example, [8]),

$$(\pi_p)_*\left(f(T)\right) = f\left((\pi_p)_*(T)\right) = f\left(L_{0,\sigma_p^E(\xi),\sigma_p^E(\xi),0}\right)$$

for all $p \in S(A)$, and then

$$\begin{aligned} L_{0,\sigma_{pq}^E(\eta_p),\sigma_{pq}^E(\eta_p),0} &= (\pi_{pq})_*\left(L_{0,\eta_p,\eta_p,0}\right) \\ &= (\pi_{pq})_*\left(f\left(L_{0,\sigma_p^E(\xi),\sigma_p^E(\xi),0}\right)\right) \\ &= (\pi_{pq})_* \circ (\pi_p)_*\left(f\left(L_{0,\xi,\xi,0}\right)\right) \\ &= (\pi_q)_*\left(f\left(L_{0,\xi,\xi,0}\right)\right) \\ &= f\left(L_{0,\sigma_q^E(\xi),\sigma_q^E(\xi),0}\right) \\ &= L_{0,\eta_q,\eta_q,0} \end{aligned}$$

for all $p, q \in S(A)$ with $p \geq q$. This implies that for $p, q \in S(A)$ with $p \geq q$ we have

$$\left\langle \sigma_{pq}^E\left(\eta_p\right) - \eta_q, \sigma_q^E\left(\zeta\right)\right\rangle = 0$$

for all $\zeta \in E$. From this fact, and taking into account that $\sigma_q^E(E) = E_q$,we obtain $\sigma_{pq}^E(\eta_p) = \eta_q$. Therefore $(\eta_p)_p \in \lim\limits_{\leftarrow p} E_p$. Let $\eta \in E$ such that $\sigma_p^E(\eta) = \eta_p$ for all $p \in S(A)$. From

$$\begin{aligned} \sigma_p^E(\xi - \eta\langle\eta,\eta\rangle) &= \sigma_p^E(\xi) - \sigma_p^E(\eta)\left\langle\sigma_p^E(\eta), \sigma_p^E(\eta)\right\rangle \\ &= \sigma_p^E(\xi) - \eta_p\left\langle\eta_p,\eta_p\right\rangle = 0 \end{aligned}$$

for all $p \in S(A)$, *we* deduce that $\xi = \eta\langle\eta,\eta\rangle$ and the proposition is proved. □

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