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DETERMINANTAL ELLIPTIC SELBERG INTEGRALS

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Dedicated to Christian Krattenthaler on his 60th birthday

ABSTRACT. The classical Selberg integral contains a power of the Vandermonde determinant. When that power is chosen to be a square, it is easy to prove Selberg's identity by interpreting it as a determinant of one-variable integrals. We give similar proofs of summation and transformation formulas for continuous and discrete elliptic Selberg integrals. In the continuous case, the same proof was given previously by Noumi. Special cases of the resulting identities have found applications in combinatorics.

1. INTRODUCTION

In 1944, Selberg proved the integral evaluation [S2]

$$\int_{x_1,\dots,x_n=0}^{1} \prod_{1\leq j<k\leq n} |x_j-x_k|^{2c} \prod_{j=1}^{n} x_j^{a-1}(1-x_j)^{b-1}\,dx_j$$
$$=\prod_{j=1}^{n}\frac{\Gamma(a+(j-1)c)\,\Gamma(b+(j-1)c)\,\Gamma(1+jc)}{\Gamma(a+b+(n+j-2)c)\,\Gamma(1+c)}, \quad (1.1)$$

which had appeared in slightly different form in his earlier paper [S1]. Here, Γ is the classical gamma function, not the elliptic gamma function that will appear below. The integral is subject to the convergence conditions

$$\mathrm{Re}(a)>0, \quad \mathrm{Re}(b)>0, \quad \mathrm{Re}(c)>-\min\left(\frac{1}{n},\frac{\mathrm{Re}(a)}{n-1},\frac{\mathrm{Re}(b)}{n-1}\right). \quad (1.2)$$

The Selberg integral plays a fundamental role in random matrix theory and analysis on classical groups, and has been generalized in many directions [FW].

The general case of (1.1) is quite deep. It is instructive to note that in 1963 Mehta and Dyson [MD] conjectured that

$$\int_{x_1,\dots,x_n=-\infty}^{\infty} \prod_{1\leq j<k\leq n} |x_j-x_k|^{2c} \prod_{j=1}^{n} e^{-x_j^2/2}\,dx_j=(2\pi)^{n/2}\prod_{j=1}^{n}\frac{\Gamma(1+jc)}{\Gamma(1+c)}. \quad (1.3)$$

Although this was republished as a conjecture several times, no proof was found until it was recognized as a degenerate case of (1.1) in the late 1970s, see [FW].

Mehta and Dyson could prove (1.3) for $c=1/2$, 1 and 2, which are the most important cases in random matrix theory. As was pointed out by the anonymous referee, the case $c=1/2$ was proved much earlier by Hsu in a journal of "highly dubious repute" [H]. The discussion of the case $c=1$ in (1.3) is just one sentence (in their notation, $\beta=2c$): "The case $\beta=2$ is the easiest; one needs only to introduce Hermite polynomials and

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exploit their orthogonality properties". For our purpose, it is useful to explain this starting from the algebraic identity [A] (see [F] for the history of this result)

$$\det_{1\leq j,k\leq n}\left(\int f_j(x)g_k(x)\,d\mu(x)\right)$$
$$=\frac{1}{n!}\int \det_{1\leq j,k\leq n}\left(f_j(x_k)\right)\det_{1\leq j,k\leq n}\left(g_j(x_k)\right)\,d\mu(x_1)\cdots d\mu(x_n),\quad (1.4)$$

which holds for any linear functional $h\mapsto\int h(x)\,d\mu(x)$. (In the examples of interest to us, this functional is given by integrating h against a measure.) If f_j and g_j are monic polynomials of degree $j-1$, $j=1,2,\dots,n$, then the determinants on the right are column-equivalent to Vandermonde determinants, and we obtain

$$\det_{1\leq j,k\leq n}\left(\int f_j(x)g_k(x)\,d\mu(x)\right)=\frac{1}{n!}\int\prod_{1\leq j<k\leq n}(x_j-x_k)^2\,d\mu(x_1)\cdots d\mu(x_n).\qquad (1.5)$$

If we now choose $f_j=g_j$ as orthogonal with respect to $d\mu$, then the left-hand side of (1.5) reduces to the product of the squared norms of the first n monic orthogonal polynomials. The identity (1.5) is then a classical result known to Heine [I, Cor. 2.1.3]. The case of Jacobi and Hermite polynomials give the case $c=1$ of (1.1) and (1.3), respectively.

There is a less well-known but even more elementary proof of the case $c=1$ of (1.1), based on varying the parameters a and b. This proof is more relevant to the present work, so we will explain it in detail. Let I_{jk} denote the one-variable case of (1.1), after replacing (a,b) by $(a+j-1,b+n-k)$. By Euler's beta integral evaluation,

$$I_{jk}=\int_0^1 x^{a+j-2}(1-x)^{b+n-k-1}\,dx=\frac{\Gamma(a+j-1)\,\Gamma(b+n-k)}{\Gamma(a+b+n+j-k-1)}.$$

Consider the determinant $D=\det_{1\leq j,k\leq n}(I_{jk})$, where we need to assume the convergence conditions

$$\mathrm{Re}(a)>0,\quad \mathrm{Re}(b)>0.\qquad (1.6)$$

It can be identified with the left-hand side of (1.5), where $f_j(x)=x^{j-1}$, $g_j(x)=(1-x)^{n-j}$ and

$$\int f(x)\,d\mu(x)=\int_0^1 f(x)\,x^{a-1}(1-x)^{b-1}\,dx.$$

Although g_j is not monic of degree $j-1$, the sign changes resulting from replacing $(x-1)^{j-1}$ by $(1-x)^{n-j}$ cancel, so (1.5) still holds. Thus, the case $c=1$ of (1.1) can be expressed as $n!D$. This is another instance of the Vandermonde determinant. The gamma functions in the numerator can be pulled out, and the denominator can be expressed as

$$\frac{1}{\Gamma(a+b+n+j-k-1)}=\frac{p_{k-1}(j)}{\Gamma(a+b+n+j-2)},$$

where

$$p_k(x)=\prod_{j=0}^{k-1}(x+a+b+n-k-2+j)$$

is a monic polynomial of degree k, so that

$$\det_{1\leq j,k\leq n}(p_{k-1}(j))=\prod_{1\leq j<k\leq n}(k-j)=\prod_{j=1}^{n}(j-1)!\,.$$

We conclude that

$$n!D=\prod_{j=1}^{n}\frac{\Gamma(a+j-1)\,\Gamma(b+j-1)\,j!}{\Gamma(a+b+n+j-2)}.$$

Since (1.2) reduces to (1.6) when $c=1$, this proves the case $c=1$ of (1.1) for general admissible values of the other parameters.

The same method works for many variations of the $c=1$ Selberg integral; the measure of integration may be continuous or discrete, and the integrands may live at the rational, trigonometric or elliptic level. (In the most common notation, $c=1$ corresponds to $t=q$ at the trigonometric and elliptic level.) One can also prove transformation formulas, stating that two Selberg-type integrals are equal. The purpose of the present note is to illustrate this method with two examples: an elliptic Selberg integral conjectured by van Diejen and Spiridonov [DS1] and a transformation formula for discrete Selberg integrals conjectured by Warnaar [W]. Both these identities were first proved by Rains [R1, R3]; the second one was proved independently by Coskun and Gustafson [CG].

We are not claiming that our proofs are new, and the present paper should be viewed as expository. It is clear from our correspondence with Rains that he is aware of similar proofs. Moreover, Noumi [N] gave a determinantal proof of the transformation formula stated as (2.9) below. This generalizes the proof of the continuous integration formula given below and is completely parallel to our proof of the discrete transformation formula. The main motivation for writing the present note is that we have seen several recent papers where the case $t=q$ of Warnaar's identities for discrete Selberg integrals are applied [BK, BKW, FKX, KS], but the reader is referred to work on the general case [CG, R1, Ro1] for the proof. Even though it is known to some experts in the field, it seems useful to point out to the wider community that much easier proofs exist. We also hope that the same method can be used to find new results. In particular, we think of quadratic and cubic transformation formulas for $c=1$ Selberg-type integrals, which may possibly admit extensions to general c. In this direction, we mention that several quadratic transformations of elliptic Selberg integrals are given in [R4, R5]. Quadratic summations for $c=1$ discrete Selberg integrals appear in connection with tiling problems [CEKZ, Ro2].

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2. Continuous Selberg integrals

We recall some standard notation of elliptic hypergeometric functions. We fix two parameters p and q with $|p|$, $|q|<1$, which we suppress from the notation. Ruijsenaars' elliptic gamma function [Ru] is given by

$$\Gamma(z)=\prod_{j,k=1}^{n}\frac{1-p^{j+1}q^{k+1}/z}{1-p^jq^kz}.$$

It satisfies the functional equation

$$\Gamma(qz)=\theta(z)\Gamma(z) \tag{2.1}$$

and, more generally,

$$\Gamma(q^kz)=(z)_k\,\Gamma(z), \tag{2.2}$$

where the theta function and elliptic shifted factorials are given by

$$\theta(z)=\prod_{j=0}^{\infty}(1-p^jz)\left(1-\frac{p^{j+1}}{z}\right),\qquad (z)_k=\prod_{j=0}^{k-1}\theta(zq^j).$$

Repeated variables in each of these functions is a short-hand for products. For instance,

$$\Gamma(z_1,\dots,z_m)=\Gamma(z_1)\cdots\Gamma(z_m),$$

$$\Gamma(z^{\pm}w^{\pm})=\Gamma(zw)\,\Gamma(z/w)\,\Gamma(w/z)\,\Gamma(1/wz).$$

For introductions to elliptic hypergeometric series, we refer to [GR, Ro3]. We will make heavy use of elementary identities that can be found in these sources.

The elliptic Selberg integral refers to the evaluation

$$\frac{C^n}{2^nn!}\int\prod_{1\leq j<k\leq n}\frac{\Gamma(tz_j^{\pm}z_k^{\pm})}{\Gamma(z_j^{\pm}z_k^{\pm})}\prod_{j=1}^{n}\frac{\prod_{k=1}^{6}\Gamma(t_kz_j^{\pm})}{\Gamma(z_j^{\pm2})}\frac{dz_j}{2\pi\mathrm{i}\,z_j}$$

$$=\prod_{m=1}^{n}\left(\frac{\Gamma(t^m)}{\Gamma(t)}\prod_{1\leq j<k\leq 6}\Gamma(t^{m-1}t_jt_k)\right), \tag{2.3}$$

where the parameters satisfy the balancing condition

$$t^{2n-2}t_1t_2t_3t_4t_5t_6=pq \tag{2.4}$$

and

$$C=\prod_{j=1}^{\infty}(1-p^j)(1-q^j).$$

If $|t|<1$ and $|t_j|<1$ for all j, the integration is over $|z_1|=\dots=|z_n|=1$; this condition may be relaxed if the contour is deformed appropriately. The evaluation (2.3) contains the classical Selberg integral (1.1) as a limit, see [R2].

The case $p=0$ of (2.3) is due to Gustafson [G] and the case $n=1$ to Spiridonov [Sp1]. The general case was conjectured by van Diejen and Spiridonov [DS1] and first proved by Rains [R3]. Another proof follows by combining the results of [DS2, Sp2], and a third proof is given in [IN2]. For a quantum field theory interpretation of (2.3), see [SV, §12.3.2].

The parameter c in (1.1) corresponds to $\log_q t$ in (2.3). In particular, $c=1$ corresponds to $t=q$. We proceed to give a simple proof of this special case of (2.3). Let I_{jk} denote the case $n=1$ of (2.3), after the substitutions

$$(t_1,t_2,t_3,t_4)\mapsto(t_1q^{j-1},t_2q^{n-j},t_3q^{k-1},t_4q^{n-k}).$$

The balancing condition for all these integrals is

$$q^{2n-2}t_1t_2t_3t_4t_5t_6 = pq, \tag{2.5}$$

which agrees with the case $t = q$ of (2.4). By (2.2),

$$I_{jk} = \frac{C}{2}\int (t_1z^{\pm})_{j-1}\,(t_2z^{\pm})_{n-j}(t_3z^{\pm})_{k-1}\,(t_4z^{\pm})_{n-k}\frac{\prod_{j=1}^{6}\Gamma(t_jz^{\pm})}{\Gamma(z^{\pm 2})}\frac{dz}{2\pi\mathrm{i}\,z}.$$

Let $D = \det_{1\leq j,k\leq n}(I_{jk})$. Then (1.4) gives

$$D = \frac{C^n}{2^n n!}\int \Delta(t_1,t_2)\Delta(t_3,t_4)\prod_{k=1}^{n}\frac{\prod_{j=1}^{6}\Gamma(t_jz_k^{\pm})}{\Gamma(z_k^{\pm 2})}\frac{dz_k}{2\pi\mathrm{i}\,z_k},$$

where

$$\Delta(a,b) = \det_{1\leq j,k\leq n}\left((az_k^{\pm})_{j-1}\,(bz_k^{\pm})_{n-j}\right). \tag{2.6}$$

By Warnaar's determinant evaluation [W, Lemma 5.3],

$$\Delta(a,b) = b^{\binom{n}{2}}q^{\binom{n}{3}}\prod_{j=1}^{n}(q^{j-n}a/b, q^{n-j}ab)_{j-1}\prod_{1\leq j<k\leq n} z_k^{-1}\theta(z_kz_j^{\pm}). \tag{2.7}$$

Note also that

$$\prod_{1\leq j<k\leq n}\left(z_k^{-1}\theta(z_kz_j^{\pm})\right)^2 = \prod_{1\leq j<k\leq n}\frac{\Gamma(qz_j^{\pm}z_k^{\pm})}{\Gamma(z_j^{\pm}z_k^{\pm})}.$$

This gives

$$\begin{aligned} D = (t_2t_4)^{\binom{n}{2}}q^{2\binom{n}{3}}&\prod_{j=1}^{n}(q^{j-n}t_1/t_2, q^{n-j}t_1t_2, q^{j-n}t_3/t_4, q^{n-j}t_3t_4)_{j-1}\\ &\times\frac{C^n}{2^n n!}\int\prod_{1\leq j<k\leq n}\frac{\Gamma(qz_j^{\pm}z_k^{\pm})}{\Gamma(z_j^{\pm}z_k^{\pm})}\prod_{j=1}^{n}\frac{\prod_{k=1}^{6}\Gamma(t_kz_j^{\pm})}{\Gamma(z_j^{\pm 2})}\frac{dz_j}{2\pi\mathrm{i}\,z_j}, \end{aligned} \tag{2.8}$$

where we recognize the integral as the case $t = q$ of (2.3).

On the other hand, the case $n = 1$ of (2.3) (that is, Spiridonov's elliptic beta integral) gives

$$\begin{aligned} I_{jk} &= \Gamma(t_1t_2q^{n-1}, t_1t_3q^{j+k-2}, t_1t_4q^{n+j-k-1}, t_1t_5q^{j-1}, t_1t_6q^{j-1})\\ &\quad\times\Gamma(t_2t_3q^{n-j+k-1}, t_2t_4q^{2n-j-k}, t_2t_5q^{n-j}, t_2t_6q^{n-j}, t_3t_4q^{n-1})\\ &\quad\times\Gamma(t_3t_5q^{k-1}, t_3t_6q^{k-1}, t_4t_5q^{n-k}, t_4t_6q^{n-k}, t_5t_6). \end{aligned}$$

Most of the factors are independent of either j or k and can thus be pulled out of the determinant. Again using (2.2), we are left with

$$\begin{aligned} D = \Gamma(t_1t_2q^{n-1}, t_3t_4q^{n-1}, t_5t_6)^n&\prod_{m=1}^{n}\prod_{\substack{1\leq j<k\leq 6\\ (j,k)\neq(1,2),\,(3,4),\,(5,6)}}\Gamma(t_jt_kq^{m-1})\\ &\times\det_{1\leq j,k\leq n}\left((t_1t_3q^{k-1}, t_1t_4q^{n-k})_{j-1}\,(t_2t_3q^{k-1}, t_2t_4q^{n-k})_{n-j}\right). \end{aligned}$$

The final determinant is of the form (2.6), with $a = t_1\sqrt{t_3t_4q^{n-1}}$, $b = t_2\sqrt{t_3t_4q^{n-1}}$ and $z_k = q^{k-1}\sqrt{q^{1-n}t_3/t_4}$. Using (2.7) and also (2.5) to write $(q^{n-j}ab)_{j-1} = (t_5t_6)_{j-1}$, we obtain after simplification

$$D = (t_2t_4)^{\binom{n}{2}} q^{2\binom{n}{3}} \Gamma(t_1t_2q^{n-1}, t_3t_4q^{n-1})^n \prod_{m=1}^{n} \prod_{\substack{1\leq j<k\leq 6\\ (j,k)\neq(1,2),\,(3,4)}} \Gamma(t_jt_kq^{m-1}) \times \prod_{j=1}^{n} (q, q^{j-n}t_1/t_2, q^{j-n}t_3/t_4)_{j-1}.$$

Comparing this with (2.8) yields the case $t = q$ of (2.3).

Essentially the same proof works for the case $t = q$ of Rains' integral transformation [R3]

$$\begin{aligned}\int \prod_{1\leq j<k\leq n} \frac{\Gamma(tz_j^{\pm}z_k^{\pm})}{\Gamma(z_j^{\pm}z_k^{\pm})} \prod_{j=1}^{n} \frac{\prod_{k=1}^{4}\Gamma(t_kz_j^{\pm}, u_kz_j^{\pm})}{\Gamma(z_j^{\pm 2})} \frac{dz_j}{2\pi\mathrm{i}\, z_j} &= \prod_{m=1}^{n} \prod_{1\leq j<k\leq 4} \Gamma(t^{m-1}t_jt_k, t^{m-1}u_ju_k)\\ &\times \int \prod_{1\leq j<k\leq n} \frac{\Gamma(tz_j^{\pm}z_k^{\pm})}{\Gamma(z_j^{\pm}z_k^{\pm})} \prod_{j=1}^{n} \frac{\prod_{k=1}^{4}\Gamma(t_kvz_j^{\pm}, u_kv^{-1}z_j^{\pm})}{\Gamma(z_j^{\pm 2})} \frac{dz_j}{2\pi\mathrm{i}\, z_j}, \end{aligned} \tag{2.9}$$

where $v^2 = pq/t^{n-1}t_1t_2t_3t_4 = t^{n-1}u_1u_2u_3u_4/pq$. The details can be found in [N], where the integrals are interpreted as tau functions for the elliptic Painlevé equation.

3. Discrete Selberg integrals

The integral evaluation (2.3) and transformation (2.9) have analogues for finite sums, which were conjectured by Warnaar [W] prior to the discovery of the continuous versions. Warnaar's summation can be obtained from the integral evaluation (2.3) through residue calculus [DS1] (presumably, a similar argument applies to the transformations). The conjectured summation was proved in [Ro1], see also [IN1], and the more general transformation in [CG, R1].

As one would expect, the case $t = q$ of Warnaar's identities also admits a simple determinantal proof. We will focus on the transformation, which can be written as [S]

$$\begin{aligned}
&\sum_{0\le x_1<x_2<\dots<x_n\le N}\;\prod_{1\le j<k\le n}\big(q^{x_j}\theta(q^{x_k-x_j})\theta(aq^{x_j+x_k})\big)^2\\
&\qquad\times\prod_{j=1}^n\left(\frac{\theta(aq^{2x_j})}{\theta(a)}\frac{(a,b,c,d,e,f,g,q^{-N})_{x_j}}{(q,aq/b,aq/c,aq/d,aq/e,aq/f,aq/g,aq^{N+1})_{x_j}}\,q^{x_j}\right)\\
&=\left(\frac{a}{\lambda}\right)^{(N+1-n)n}\frac{(aq)_N^n}{(\lambda q)_N^n}\prod_{j=1}^n\frac{(b,c,d)_{j-1}\,(\lambda q/e,\lambda q/f,\lambda q/g)_{N+1-j}}{(\lambda b/a,\lambda c/a,\lambda d/a)_{j-1}\,(aq/e,aq/f,aq/g)_{N+1-j}}\\
&\qquad\times\sum_{0\le x_1<x_2<\dots<x_n\le N}\;\prod_{1\le j<k\le n}\big(q^{x_j}\theta(q^{x_k-x_j})\,\theta(\lambda q^{x_j+x_k})\big)^2\\
&\qquad\times\prod_{j=1}^n\left(\frac{\theta(\lambda q^{2x_j})}{\theta(\lambda)}\frac{(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,g,q^{-N})_{x_j}}{(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,\lambda q/g,\lambda q^{N+1})_{x_j}}\,q^{x_j}\right),
\end{aligned}\tag{3.1}$$

where $bcdefg = q^{4+N-2n}a^3$ and $\lambda = a^2q^{2-n}/bcd$. When $aq = cd$, the factor $(\lambda b/a)_{x_n} = (q^{1-n})_{x_n}$ on the right-hand side vanishes unless $x_n \le n-1$, so the sum reduces to the term with $(x_1,\dots,x_n) = (0,1,\dots,n-1)$. After a change of variables, this gives the case $t = q$ of Warnaar's discrete elliptic Selberg integral, namely,

$$\begin{aligned}
&\sum_{0\le x_1<x_2<\dots<x_n\le N}\;\prod_{1\le j<k\le n}\big(q^{x_j}\theta(q^{x_k-x_j})\theta(aq^{x_j+x_k})\big)^2\\
&\qquad\times\prod_{j=1}^n\left(\frac{\theta(aq^{2x_j})}{\theta(a)}\frac{(a,b,c,d,e,q^{-N})_{x_j}}{(q,aq/b,aq/c,aq/d,aq/e,aq^{N+1})_{x_j}}\,q^{x_j}\right)\\
&=b^{n(N+1-n)}q^{\frac13 n(n-1)(3N+1-2n)}(aq)_N^n\\
&\qquad\times\prod_{j=1}^n\frac{(q,b,c,d,e,q^{-N})_{j-1}\,(aq^{2-j}/bc,aq^{2-j}/bd,aq^{2-j}/be)_{N+1-n}}{(aq/b,aq/c,aq/d,aq/e)_{N+1-j}},
\end{aligned}\tag{3.2}$$

which holds for $bcde = q^{N+3-2n}a^2$.

We will give a simple proof of (3.1), which is parallel to the continuous case. We need the case $n = 1$, which is the one-variable elliptic Bailey transformation. It first appeared (rather implicitly and with some restrictions on the parameters) in the work of Date et al. on Baxter's elliptic solid-on-solid model [D] and was proved in general by Frenkel and Turaev [FT], see [GR, Ro3] for more elementary proofs.

Let S_{jk} denote the case $n = 1$ of (3.1), after the substitutions

$$(b,c,e,f)\mapsto(bq^{j-1},cq^{n-j},eq^{k-1},fq^{n-k}).$$

Some elementary manipulation gives

$$\begin{aligned}
S_{jk}=&\sum_{x=0}^N\frac{\theta(aq^{2x})}{\theta(a)}\frac{(a,b,c,d,e,f,g,q^{-N})_x}{(q,aq/b,aq/c,aq/d,aq/e,aq/f,aq/g,aq^{N+1})_x}\,q^{(2n-1)x}\\
&\times\frac{(bq^x,bq^{-x}/a)_{j-1}\,(cq^x,cq^{-x}/a)_{n-j}\,(eq^x,eq^{-x}/a)_{k-1}\,(fq^x,fq^{-x}/a)_{n-k}}{(b,b/a)_{j-1}\,(c,c/a)_{n-j}\,(e,e/a)_{k-1}\,(f,f/a)_{n-k}}.
\end{aligned}$$

Let $D=\det_{1\leq j,k\leq n}(S_{jk})$. We expand D using the Cauchy–Binet identity

$$\det_{1\leq j,k\leq n}\left(\sum_{x=0}^N a_{jx}b_{kx}\right)=\sum_{0\leq x_1<x_2<\dots<x_n\leq N}\det_{1\leq j,k\leq n}(a_{j,x_k})\det_{1\leq j,k\leq n}(b_{j,x_k}).$$

This is a special case of (1.4), where symmetry is used to restrict the range of summation. It follows that

$$\begin{aligned}D&=\frac{1}{\prod_{j=1}^n(b,b/a,c,c/a,e,e/a,f,f/a)_{j-1}}\\&\quad\times\sum_{0\leq x_1<x_2<\dots<x_n\leq N}\prod_{k=1}^n\left(\frac{\theta(aq^{2x_k})}{\theta(a)}\frac{(a,b,c,d,e,f,g,q^{-N})_{x_k}\,q^{x_k(2n-1)}}{(q,aq/b,aq/c,aq/d,aq/e,aq/f,aq/g,aq^{N+1})_{x_k}}\right)\\&\qquad\qquad\times\tilde\Delta(b,c)\tilde\Delta(e,f),\end{aligned}$$

where

$$\tilde\Delta(b,c)=\det_{1\leq j,k\leq n}\left((bq^{x_k},bq^{-x_k}/a)_{j-1}\,(cq^{x_k},cq^{-x_k}/a)_{n-j}\right).$$

This determinant is of the form (2.6), with (a,b,z_k) replaced by $(b/\sqrt{a},c/\sqrt{a},\sqrt{a}q^{x_k})$. Using (2.7) and simplifying, we find that D equals the left-hand side of (3.1) times

$$\left(\frac{cf}{a^2}\right)^{\binom n2}q^{2\binom n3}\prod_{j=1}^n\frac{(q^{j-n}b/c,q^{n-j}bc/a,q^{j-n}e/f,q^{n-j}ef/a)_{j-1}}{(b,b/a,c,c/a,e,e/a,f,f/a)_{j-1}}.\tag{3.3}$$

Repeating the same computation but starting from the alternative expression

$$\begin{aligned}S_{jk}&=\left(\frac{a}{\lambda}\right)^N\frac{(aq,\lambda q^{2-k}/e,\lambda q^{1-n+k}/f,\lambda q/g)_N}{(\lambda q,aq^{2-k}/e,aq^{1-n+k}/f,aq/g)_N}\\&\quad\times\sum_{x=0}^N\frac{\theta(\lambda q^{2x})}{\theta(\lambda)}\frac{(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,g,q^{-N})_x}{(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,\lambda q/g,\lambda q^{N+1})_x}\,q^{x(2n-1)}\\&\quad\times\frac{(\lambda bq^x/a,bq^{-x}/a)_{j-1}\,(\lambda cq^x/a,cq^{-x}/a)_{n-j}\,(eq^x,eq^{-x}/\lambda)_{k-1}\,(fq^x,fq^{-x}/\lambda)_{n-k}}{(\lambda b/a,b/a)_{j-1}\,(\lambda c/a,c/a)_{n-j}\,(e,e/\lambda)_{k-1}\,(f,f/\lambda)_{n-k}},\end{aligned}$$

we obtain after simplification the same prefactor (3.3) times the right-hand side of (3.1). This completes the proof of (3.1).

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