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COMBINATORICS OF (q,y)-LAGUERRE POLYNOMIALS AND THEIR MOMENTS

QIONGQIONG PAN AND JIANG ZENG

To Christian Krattenthaler on the occasion of his 60th birthday

Abstract. We consider a (q,y)-analogue of Laguerre polynomials $L_n^{(\alpha)}(x;y\,|\,q)$ for integral $\alpha\geq -1$, which turns out to be a rescaled version of Al-Salam–Chihara polynomials. A combinatorial interpretation for the (q,y)-Laguerre polynomials is given using a colored version of Foata and Strehl's Laguerre configurations with suitable statistics. When $\alpha\geq 0$, the corresponding moments are described using certain classical statistics on permutations, and the linearization coefficients are proved to be a polynomial in y and q with nonnegative integral coefficients.

1. Introduction

The monic Laguerre polynomials $L_n^{(\alpha)}(x)$ are defined by the generating function

$$(1+t)^{-\alpha-1}\exp\left(\frac{xt}{t+1}\right)=\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)\frac{t^n}{n!}. \tag{1.1}$$

They are the multiple of the usual (general) Laguerre polynomials [16, pp. 241–242] by $(-1)^n n!$. We have the explicit formula

$$L_n^{(\alpha)}(x)=\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}\binom{n+\alpha}{n-k}x^k \tag{1.2}$$

and the three-term recurrence relation

$$L_{n+1}^{(\alpha)}(x)=(x-(2n+\alpha+1))L_n^{(\alpha)}(x)-n(n+\alpha)L_{n-1}^{(\alpha)}(x). \tag{1.3}$$

The Laguerre polynomials $L_n^{(\alpha)}(x)$ are orthogonal with respect to the moments $\mathcal{L}(x^n)=(\alpha+1)_n$, where $(x)_n=x(x+1)\cdots(x+n-1)$ $(n\geq 1)$ is the shifted factorial with $(x)_0=1$, and $\mathcal{L}$ is the linear functional defined by

$$\mathcal{L}(f)=\frac{1}{\Gamma(\alpha+1)}\int_0^{\infty}f(x)x^{\alpha}e^{-x}dx. \tag{1.4}$$

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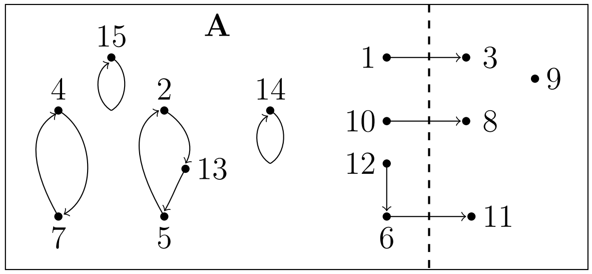

FIGURE 1. A Laguerre configuration (A, f) on $[15]$ with $A = [15] \setminus \{3, 8, 9, 11\}$.

The linearization formula [23] reads as follows:

$$\mathcal{L}(L_{n_1}^{(\alpha)}(x)L_{n_2}^{(\alpha)}(x)L_{n_3}^{(\alpha)}(x)) = \sum_{s\geq 0} \frac{n_1!\, n_2!\, n_3!\, 2^{n_1+n_2+n_3-2s}\,(\alpha+1)_s}{(s-n_1)!\,(s-n_2)!\,(s-n_3)!\,(n_1+n_2+n_3-2s)!}. \quad (1.5)$$

A combinatorial model for Laguerre polynomials with parameter α was first given by Foata and Strehl [7]. Recall that a *Laguerre configuration* on $[n] := \{1, \dots, n\}$ is a pair (A, f), where $A \subset [n]$ and f is an injection from A to $[n]$. A Laguerre configuration can be depicted by a digraph on $[n]$ by drawing an edge $i \to j$ if and only if $f(i) = j$. Clearly, such a graph has two types of connected components called *cycles* and *paths*, see Figure 1. Let $\mathcal{LC}_{n,k}$ be the set of Laguerre configurations (A, f) on $[n]$ with $|A| = n-k$. Then Foata and Strehl's interpretation [7] reads

$$\sum_{(A,f)\in\mathcal{LC}_{n,k}} (\alpha+1)^{\mathsf{cyc}(f)} = \frac{n!}{k!}\binom{n+\alpha}{n-k}, \quad (1.6)$$

where $\mathsf{cyc}(f)$ is the number of cycles of f.

Note that one can derive (1.6) from any of the three formulas (1.1)–(1.3), see [1, 7]. The aim of this paper is to study combinatorial aspects of more general (q, y)-Laguerre polynomials $L_n^{(\alpha)}(x; y \,|\, q)$ $(n \geq 0)$ defined by the three term-recurrence relation

$$\begin{aligned} L_{n+1}^{(\alpha)}(x; y \,|\, q) &= (x - (y[n+\alpha+1]_q + [n]_q))\, L_n^{(\alpha)}(x; y \,|\, q) \\ &\quad - y[n]_q[n+\alpha]_q\, L_{n-1}^{(\alpha)}(x; y \,|\, q), \qquad \alpha \geq -1,\ n \geq 1, \quad (1.7) \end{aligned}$$

with $L_0^{(\alpha)}(x; y \,|\, q) = 1$, $L_{-1}^{(\alpha)}(x; y \,|\, q) = 0$. Here and throughout this paper, we use the standard q-notations: $[n]_q = \frac{1-q^n}{1-q}$ for $n \geq 0$, the q-analogue of n-factorial $n!_q = \prod_{i=1}^n [i]_q$, and the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{n!_q}{k!_q\,(n-k)!_q} \quad \text{for} \quad 0 \leq k \leq n.$$

Clearly we have $L_n^{(\alpha)}(x; 1 \,|\, 1) = L_n^{(\alpha)}(x)$. Kasraoui et al. [17] gave a combinatorial interpretation for the linearization coefficients of the polynomials $L_n^{(0)}(x; y \,|\, q)$ and pointed out that a combinatorial model for $L_n^{(0)}(x; y \,|\, q)$ can be derived from Simion and Stanton's model for octabasic q-Laguerre polynomials in [22]. For $k \in \mathbb{Z}$, let

$$\mathbb{N}_k := \{n \in \mathbb{Z} : n \geq k\}$$

and $\mathbb{N} := \mathbb{N}_1$. Recently, using the theory of q-Riordan matrices, Cheon, Jung and Kim [3] derived a combinatorial model for the q-Laguerre polynomials $L_n^{(\alpha)}(x; q \,|\, q)$ when $\alpha \in \mathbb{N}_0$. It is then natural to search for a combinatorial structure unifying the above two special cases, as was alluded to at the end of [3]. Our first goal is to give such a combinatorial model for $L_n^{(\alpha)}(x; y \,|\, q)$ with variable y and integer $\alpha \in \mathbb{N}_{-1}$ by using a q-analogue of Foata and Strehl's Laguerre configurations. Moreover, for $\alpha \in \mathbb{N}_0$, the (q, y)-Laguerre polynomials $L_n^{(\alpha)}(x; y \,|\, q)$ are orthogonal polynomials. It is our second goal to give a combinatorial interpretation for the moments of (q, y)-Laguerre polynomials and prove that the linearization coefficients are polynomials in y and q with nonnegative integral coefficients. We achieve this by making use of the combinatorial theory of continued fractions.

By (1.7), the first few values of $L_n^{(\alpha)}(x; y \,|\, q)$ are

$$\begin{aligned}
L_1^{(\alpha)}(x; y \,|\, q) &= x - y[\alpha+1]_q, \\
L_2^{(\alpha)}(x; y \,|\, q) &= x^2 - (y[\alpha+1]_q + y[\alpha+2]_q + 1)\, x + [\alpha+1]_q[\alpha+2]_q y^2, \\
L_3^{(\alpha)}(x; y \,|\, q) &= x^3 - \big(y([\alpha+1]_q + [\alpha+2]_q + [\alpha+3]_q) + 2 + q\big)x^2 \\
&\quad + \big(y^2([\alpha+1]_q[\alpha+2]_q + [\alpha+2]_q[\alpha+3]_q + [\alpha+1]_q[\alpha+3]_q) \\
&\quad + y([\alpha+3]_q + [2]_q[\alpha+1]_q) + [2]_q\big)\, x - y^3[\alpha+1]_q[\alpha+2]_q[\alpha+3]_q.
\end{aligned}$$

For convenience, we introduce the signless (q, y)-Laguerre polynomials

$$\mathsf{L}_n^{(\alpha)}(x; y \,|\, q) := (-1)^n L_n^{(\alpha)}(-x; y \,|\, q) = \sum_{k=0}^{n} \ell_{n,k}^{(\alpha)}(y; q) x^k. \tag{1.8}$$

For $\alpha \in \mathbb{N}_{-1}$, we observe that $\ell_{n,k}^{(\alpha)}(y; q)$ is a polynomial in y, q with nonnegative integral coefficients, which is far from obvious from the explicit Formula (2.8). For $\alpha \in \mathbb{N}_{-1}$, Formula (1.6) implies that $\ell_{n,k}^{(\alpha)}(1; 1)$ is equal to the number of Laguerre configurations in $\mathcal{LC}_{n,k}$ such that each cycle carries a *color* $\in [1+\alpha]$. In particular, the number of Laguerre configurations in $\mathcal{LC}_{n,k}$ without cycles (i.e., consisting of only k paths) is equal to the *Lah numbers* [18]:

$$\ell_{n,k}^{(-1)}(1; 1) = \frac{n!}{k!} \binom{n-1}{k-1}.$$

Remark 1. Two different q-analogues of Lah numbers were defined and studied by Garsia and Remmel [9] and Lindsay et al. [19], respectively. Moreover an elliptic analogue of Garsia and Remmel's q-Lah numbers was constructed by Schlosser and Yoo [21].

The organization of this paper is as follows. In Section 2 we identify the (q,y)-Laguerre polynomials as a rescaled version of Al-Salam–Chihara polynomials and derive several expansion formulas for (q,y)-Laguerre polynomials. In Section 3 we present a combinatorial interpretation for the (q,y)-Laguerre polynomials in terms of α-Laguerre configurations, which are in essence the product structure of "cycles" and "paths". In Section 4 we give a combinatorial interpretation for the moments of (q,y)-Laguerre polynomials and prove that the linearization coefficients are polynomials in y and q with nonnegative integral coefficients. As the Laguerre polynomials play an important role in the theory of rook polynomials, we translate our α-Laguerre configurations in terms of rook placements in Section 5 and set up the connection between our α-Laguerre configurations and the matching model of *complete bipartite graphs* $K_{n,n+\alpha}$ (see Godsil and Gutman [12]).

2. A detour to Al-Salam–Chihara polynomials

The q-Pochhammer symbol or q-shifted factorial $(a;q)_n$ is defined by

$$(a;q)_n = \begin{cases} \prod_{i=0}^{n-1}(1-aq^i) & \text{for } n\in\mathbb{Z}^+\cup\{\infty\}, \\ 1 & \text{for } n=0. \end{cases}$$

The Al-Salam–Chihara polynomials $Q_n(x) := Q_n(x;a,b\,|\,q)$ are defined by the generating function (see [16, Chapter 14])

$$\sum_{n=0}^{\infty} Q_n(x;a,b\,|\,q)\frac{t^n}{(q;q)_n} = \frac{(at,bt;q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}, \tag{2.1}$$

with $(a,b;q)_\infty = (a;q)_\infty(b;q)_\infty$, and they satisfy the recurrence relation (op. cit.)

$$\begin{cases} Q_{-1}(x)=0, \quad Q_0(x)=1, \\ Q_{n+1}(x) = (2x-(a+b)q^n)Q_n(x) - (1-q^n)(1-abq^{n-1})Q_{n-1}(x), \quad n\geq 0. \end{cases} \tag{2.2}$$

We have the explicit formula

$$Q_n(x;a,b\,|\,q) = \frac{(ab;q)_n}{a^n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k\,(au;q)_k\,(au^{-1};q)_k}{(ab;q)_k(q;q)_k}q^k, \tag{2.3}$$

where $x=\frac{u+u^{-1}}{2}$ or $x=\cos\theta$ if $u=e^{i\theta}$.

Comparing (1.7) with (2.2) and using (1.8), we see that our polynomials $L_n^{(\alpha)}(x;y\,|\,q)$ are a rescaled version of the Al-Salam–Chihara polynomials:

$$L_n^{(\alpha)}(x;y\,|\,q) = \left(\frac{\sqrt{y}}{1-q}\right)^n Q_n\left(\frac{(1-q)x+y+1}{2\sqrt{y}};\frac{1}{\sqrt{y}},\sqrt{y}q^{\alpha+1}\,\middle|\,q\right). \tag{2.4}$$

The Al-Salam–Chihara polynomials (see [16, pp. 455–456] and [14]) are orthogonal with respect to the linear functional $\hat{\mathcal{L}}_q$ defined by

$$\hat{\mathcal{L}}_q(f) = \frac{(q, ab; q)_\infty}{2\pi} \int_{-1}^{+1} \frac{f(x)dx}{\sqrt{1-x^2}} \prod_{k=0}^{\infty} \frac{1 - 2(2x^2-1)q^k + q^{2k}}{[1 - 2xaq^k + a^2q^{2k}][1 - 2xbq^k + b^2q^{2k}]}. \tag{2.5}$$

Hence, for $\alpha \in \mathbb{N}_0$, the polynomials $L_n^{(\alpha)}(x; y \,|\, q)$ are orthogonal with respect to the linear functional $\mathcal{L}_q$ given by

$$\begin{aligned}\mathcal{L}_q(f) = {}& \frac{(q, q^{\alpha+1}; q)_\infty}{2\pi} \frac{1-q}{2\sqrt{y}} \int_{B_-}^{B_+} \frac{f(x)dx}{\sqrt{1-v(x)^2}} \\ & \times \prod_{k=0}^{\infty} \frac{[1 - 2(2v(x)^2-1)q^k + q^{2k}]}{[1 - 2v(x)q^k/\sqrt{y} + q^{2k}/y][1 - 2v(x)q^{k+\alpha+1}\sqrt{y} + q^{2k+2\alpha+2}y]},\end{aligned} \tag{2.6}$$

where $B_\pm = \frac{(1\pm\sqrt{y})^2}{1-q}$ and

$$v(x) = \frac{1}{2\sqrt{y}}((q-1)x + (y+1)). \tag{2.7}$$

Now, by (2.4), we may derive an explicit formula from (2.3), namely

$$L_n^{(\alpha)}(x; y \,|\, q) = \sum_{k=0}^{n} \frac{n!_q}{k!_q} \begin{bmatrix} n+\alpha \\ k+\alpha \end{bmatrix}_q q^{k(k-n)} y^{n-k} \prod_{j=0}^{k-1} \left(x + (1 - yq^{-j})[j]_q\right), \tag{2.8}$$

and, from (2.1), the generating function

$$\begin{aligned}\mathcal{L}^{(\alpha)}(x; y; t \,|\, q) &:= \sum_{n\geq 0} L_n^{(\alpha)}(x; y \,|\, q) \frac{t^n}{n!_q} \\ &= \frac{(t; q)_\infty \, (ytq^{\alpha+1}; q)_\infty}{\prod_{k=0}^{\infty} [1 - ((1-q)x + y + 1)tq^k + yt^2q^{2k}]},\end{aligned} \tag{2.9}$$

which can be written as

$$\mathcal{L}^{(\alpha)}(x; y; t \,|\, q) = \mathcal{L}^{(\alpha)}(0; y; t \,|\, q) \cdot \mathcal{L}^{(-1)}(x; y; t \,|\, q). \tag{2.10}$$

Define the "vertical generating function"

$$\mathcal{L}_k^{(\alpha)}(y; t \,|\, q) := [x^k]\mathcal{L}^{(\alpha)}(x; y; t \,|\, q) = \sum_{n\geq k} \ell_{n,k}^{(\alpha)}(y, q) \frac{t^n}{n!_q}, \tag{2.11}$$

and the q-derivative operator $\mathcal{D}_q$ for $f(t) \in \mathbf{R}[[t]]$ by

$$\mathcal{D}_q(f(t)) = \frac{f(t) - f(qt)}{(1-q)t},$$

where $\mathbf{R} = \mathbb{C}[[x, y, q, \ldots]]$. Thus $\mathcal{D}_q(1) = 0$ and $\mathcal{D}_q(t^n) = [n]_q t^{n-1}$ for $n > 0$.

It follows from (2.9) that

$$\mathcal{D}_q\mathcal{L}^{(-1)}(x;y;t\,|\,q)=\frac{x}{(1-t)(1-yt)}\mathcal{L}^{(-1)}(x;y;t\,|\,q), \tag{2.12}$$

which in particular gives

$$\begin{aligned}\mathcal{D}_q\mathcal{L}_1^{(-1)}(y;t\,|\,q)&=[x]\mathcal{D}_q\mathcal{L}^{(-1)}(x;y;t\,|\,q)\\&=\frac{1}{(1-t)(1-ty)}\\&=\sum_{n\ge 0}n!_q\,[n+1]_y\,\frac{t^n}{n!_q}.\end{aligned} \tag{2.13}$$

So we can rewrite (2.12) as

$$\mathcal{D}_q\mathcal{L}^{(-1)}(x;y;t\,|\,q)=x\cdot\mathcal{D}_q\mathcal{L}_1^{(-1)}(y;t\,|\,q)\cdot\mathcal{L}^{(-1)}(x;y;t\,|\,q), \tag{2.14}$$

which is equivalent to the following result.

Proposition 2. *For $n\in\mathbb{N}$, we have*

$$L_{n+1}^{(-1)}(x;y\,|\,q)=x\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q k!_q\,[k+1]_y\,L_{n-k}^{(-1)}(x;y\,|\,q). \tag{2.15}$$

Now, applying the q-binomial formula (see [10, Chapter 1])

$$\sum_{n\ge 0}\frac{(a;q)_n}{(q;q)_n}z^n=\frac{(az;q)_\infty}{(z;q)_\infty}$$

with $a=q^{\alpha+1}$ and $z=yt$, we have

$$\mathcal{L}^{(\alpha)}(0;y;t\,|\,q)=\frac{(ytq^{\alpha+1};q)_\infty}{(yt;q)_\infty}=\sum_{n\ge 0}\left(\prod_{k=1}^{n}[\alpha+k]_q\right)\frac{(yt)^n}{n!_q}. \tag{2.16}$$

Substitution of the latter into (2.10) gives the following result.

Proposition 3. *For $n\in\mathbb{N}$, we have*

$$L_n^{(\alpha)}(x;y\,|\,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\left(\prod_{j=1}^{k}[\alpha+j]_q\right)y^kL_{n-k}^{(-1)}(x;y\,|\,q). \tag{2.17}$$

Remark 4. (1) More generally we can prove the following connection formula for $\alpha\ge\beta\ge -1$:

$$L_n^{(\alpha)}(x;y\,|\,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\left(\prod_{j=0}^{k-1}[\alpha-\beta+j]_q\right)(yq^{\beta+1})^kL_{n-k}^{(\beta)}(x;y\,|\,q). \tag{2.18}$$

(2) For $q \to 1$, Identity (2.9) reduces to

$$\sum_{n\geq 0} L_n^{(\alpha)}(x; y \,|\, 1)\frac{t^n}{n!} = (1-yt)^{-(\alpha+1)}\left(1 - \frac{(1-y)t}{1-yt}\right)^{-x/(1-y)}.$$

Comparing with the generating function of the Meixner polynomials (see [16, Equation (1.9.11)])

$$\sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} M_n(x; \beta, c)t^n = (1-t)^{-x-\beta}(1-t/c)^x,$$

we derive

$$L_n^{(\alpha)}(x; y \,|\, 1) = y^n(\alpha+1)_n M_n\left(\frac{-x}{1-y}; \alpha+1, y\right).$$

Hence the (q, y)-Laguerre polynomials $L_n^{(\alpha)}(x; y \,|\, q)$ are a q-analogue of rescaled Meixner polynomials.

3. Combinatorial interpretation of (q, y)-Laguerre polynomials

The reader is referred to [1, 6, 13] for the general combinatorial theory of exponential generating functions for labeled structures. For our purpose we need only a q-version of this theory for special labeled structures. A *labeled structure* on a (finite) set $A \subset \mathbb{N}$ is a graph with vertex set A. Consider a family of labeled $\mathcal{F}$-structures $\mathcal{F} = \bigcup_{n=0}^{\infty} \mathcal{F}_n$, where $\mathcal{F}_n$ consists of the $\mathcal{F}$-structures on $[n]$. If $A = \{a_1, \ldots, a_n\} \subset \mathbb{N}$, where $a_1 < \cdots < a_n$, an $\mathcal{F}$-structure on A is obtained by replacing i by a_i for $i = 1, \ldots, n$ in the elements of $\mathcal{F}_n$. Let $\mathcal{F}[A]$ denote the set of $\mathcal{F}$-structures on A and associate a weight $u(f)$ to each object $f \in \mathcal{F}$. For the set of weighted $\mathcal{F}$-structures $\mathcal{F}_u$ (where the valuation u may involve the parameter q), the q-generating function is defined as

$$\mathcal{F}_u(t) = \sum_{f\in\mathcal{F}} u(f)\frac{t^{|f|}}{|f|!_q},$$

where $|f| = n$ if $f \in \mathcal{F}[n]$. If $\mathcal{F}_u$ and $\mathcal{G}_v$ are two weighted structures, we denote by $(\mathcal{F} \cdot \mathcal{G})_w[n]$ the set of pairs $(f, g) \in \mathcal{F}[S] \times \mathcal{G}[T]$ with weight

$$w(f, g) = u(f) \cdot v(g) \cdot q^{\mathrm{inv}(S,T)},$$

where (S, T) is an ordered bipartition of $[n]$ and $\mathrm{inv}(S, T)$ is the number of pairs $(i, j) \in S \times T$ such that $i > j$. Recall (see [13, p. 98]) that

$$\sum_{(S,T)} q^{\mathrm{inv}(S,T)} = \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

where the sum is over all ordered bipartitions (S, T) of $[n]$ with $|S| = k$. It is folklore and immediately checked that

$$(\mathcal{F} \cdot \mathcal{G})_w(t) = \mathcal{F}_u(t) \cdot \mathcal{G}_v(t). \tag{3.1}$$

We need some further definitions.

(a) For a permutation σ of a set $A \subset \mathbb{N}$, let the word $\hat{\sigma}$ denote its linear representation in the usual sense, i.e., $\hat{\sigma} = \sigma(i_1)\ldots\sigma(i_n)$ if $A = \{i_1, \ldots, i_n\}$ with $i_1 < \cdots < i_n$.

(b) A list of (nonnegative) integers, taken as a word over $\mathbb{N}$, is *strict* if no element occurs more than once. For a strict list ρ let $\mathsf{rl}(\rho)$ be the number of elements that come after the maximum element.

(c) For a set λ of k non-empty and disjoint strict lists of integers, order these lists according to their minimum element (increasing). This gives a list of k words $(\lambda_1, \ldots, \lambda_k)$, which will be identified with λ. Then $\underline{\lambda} = \lambda_1 \ldots \lambda_k$ denotes the concatenation of these lists.

Two particular structures will be used to interpret the (q, y)-Laguerre polynomials.

(d) The structures $\mathcal{S}^{(\alpha)}$ consist of permutations σ, where each cycle carries a color $\in \{0, 1, 2, \ldots, \alpha\}$. Write σ as a product of *unicolored permutations*, $\sigma = \sigma_0 \cdot \sigma_1 \cdots \sigma_\alpha$, where σ_i is the product of cycles with color i. Now consider the concatenation

$$\underline{\sigma} = \hat{\sigma}_0 \cdot \hat{\sigma}_1 \cdots \hat{\sigma}_\alpha$$

and the word with letters from $\{0, 1\}$ given by

$$\underline{\underline{\sigma}} = 0^{|\hat{\sigma}_0|}10^{|\hat{\sigma}_1|}1 \cdots 10^{|\hat{\sigma}_\alpha|}.$$

Define the valuation u on $\mathcal{S}^{(\alpha)}$ by

$$u(\sigma) = y^{|\underline{\sigma}|} q^{\mathsf{inv}(\underline{\sigma}) + \mathsf{inv}(\underline{\underline{\sigma}})}.$$

(e) The structures $\mathcal{L}in^{(k)}$ consist of sets $\lambda = (\lambda_1, \ldots, \lambda_k)$ of k nonempty and disjoint strict lists (cf. (c)). Define the valuation v on $\mathcal{L}in^{(k)}$ by

$$v(\lambda) = y^{\mathsf{rl}(\lambda)} q^{\mathsf{inv}(\underline{\lambda}) - \mathsf{rl}(\lambda)},$$

where $\mathsf{rl}(\lambda) = \sum_{i=1}^k \mathsf{rl}(\lambda_i)$.

Let $\mathcal{LC}_{n,k}^{(\alpha)} := \mathcal{S}^{(\alpha)} \cdot \mathcal{L}in^{(k)}[n]$. For any α-Laguerre configuration $(\sigma, \lambda) \in \mathcal{S}^{(\alpha)}[A] \times \mathcal{L}in^{(k)}[B]$ with $A \cap B = \emptyset$ and $A \cup B = [n]$, in order to invoke the folklore statement (3.1), one should use as valuation

$$\begin{aligned} w(\sigma, \lambda) &= u(\lambda) \cdot v(\lambda) \cdot q^{\mathsf{inv}(A,B)} \\ &= y^{|\underline{\sigma}|} q^{\mathsf{inv}(\underline{\sigma}) + \mathsf{inv}(\underline{\underline{\sigma}})} y^{\mathsf{rl}(\lambda)} q^{\mathsf{inv}(\underline{\lambda}) - \mathsf{rl}(\lambda)} q^{\mathsf{inv}(A,B)} \\ &= y^{|\underline{\sigma}| + \mathsf{rl}(\lambda)} q^{\mathsf{inv}(\underline{\sigma}) + \mathsf{inv}(\underline{\underline{\sigma}}) + \mathsf{inv}(\underline{\lambda}) - \mathsf{rl}(\lambda)} q^{\mathsf{inv}(A,B)} \\ &= y^{|\underline{\sigma}| + \mathsf{rl}(\lambda)} q^{\mathsf{inv}(\underline{\sigma}.\underline{\lambda}) - \mathsf{rl}(\lambda) + \mathsf{inv}(\underline{\underline{\sigma}})}. \end{aligned} \tag{3.2}$$

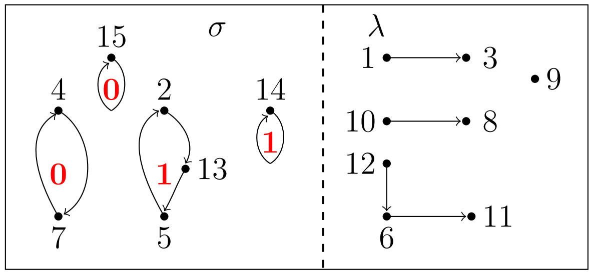

FIGURE 2. A 1-Laguerre configuration $(\sigma, \lambda) \in \mathcal{LC}^{(1)}_{15,4}$, which is the Laguerre configuration in Figure 1 of which each cycle gets a color 0 or 1.

The essential point is $\mathsf{inv}(\underline{\sigma}) + \mathsf{inv}(\underline{\lambda}) + \mathsf{inv}(A, B) = \mathsf{inv}(\underline{\sigma}.\underline{\lambda})$. This describes the weighted configurations $(\mathcal{LC}^{(\alpha)}_{n,k})_w$. An element of $(\mathcal{LC}^{(\alpha)}_{n,k})_w$ is called an α-Laguerre configuration on $[n]$, see Figure 2.

Lemma 5. *For $\alpha \in \mathbb{N}$, we have*

$$\mathcal{S}^{(\alpha)}_u(t) = \mathcal{L}^{(\alpha)}(0; y; t \,|\, q).$$

Proof. Let $\mathsf{P}(n, \alpha)$ be the set of words of length $n+\alpha$ with n 0's and α 1's, i.e., lattice paths from $(0, 0)$ to (n, α). For $\sigma \in \mathcal{S}^{(\alpha)}[n]$, the word $\underline{\sigma}$ can be seen as the linear representation of an (ordinary) permutation $\tilde{\sigma} \in \mathcal{S}^{(0)}[n]$, whereas $\underline{\underline{\sigma}} \in \mathsf{P}(n, \alpha)$. The mapping

$$\begin{aligned} \mathcal{S}^{(\alpha)}[n] &\to \mathcal{S}^{(0)}[n] \times \mathsf{P}(n, \alpha) \\ \sigma &\mapsto (\tilde{\sigma}, \underline{\underline{\sigma}}) \end{aligned}$$

is a bijection, and from summing both contributions separately, one obtains

$$\begin{aligned} \sum_{\sigma \in \mathcal{S}^{(\alpha)}[n]} q^{\mathsf{inv}(\underline{\sigma}) + \mathsf{inv}(\underline{\underline{\sigma}})} &= \sum_{\sigma \in \mathcal{S}^{(0)}[n]} q^{\mathsf{inv}(\underline{\sigma})} \sum_{\underline{\underline{\sigma}} \in P(n,\alpha)} q^{\mathsf{inv}(\underline{\underline{\sigma}})} \\ &= n!_q \begin{bmatrix} n+\alpha \\ \alpha \end{bmatrix}_q, \end{aligned}$$

which is $\prod_{i=1}^{n} [\alpha + i]_q$. So we get

$$\mathcal{S}^{(\alpha)}_u(t) = \sum_{n \geq 0} \left(\prod_{i=1}^{n} [\alpha + i]_q \right) (yt)^n.$$

The result then follows from (2.16). $\square$

Lemma 6. *For integers $k \geq 1$, we have*

$$\mathcal{L}in^{(k)}_v(t) = \mathcal{L}^{(-1)}_k(y; t \,|\, q).$$

Proof. We proceed by induction on $k \geq 1$.

- The case $k = 1$. For a single list $\lambda = \underline{\lambda} \in \mathcal{L}in^{(1)}[n+1]$, let j_λ be the position of the maximum element, let $\lambda' = \underline{\lambda}' \in \mathcal{L}in^{(1)}[n]$ be the list obtained by deleting this maximum element. Then
$$\mathcal{L}in^{(1)}[n+1] \to \mathcal{L}in^{(1)}[n] \times [n+1]$$
$$\lambda \mapsto (\lambda', j_\lambda)$$
is a bijection such that $\mathsf{inv}(\underline{\lambda}) = \mathsf{inv}(\underline{\lambda}') + \mathsf{rl}(\lambda)$. Furthermore, we have
$$\sum_{\lambda \in \mathcal{L}in^{(1)}[n+1]} y^{\mathsf{rl}(\lambda)} q^{\mathsf{inv}(\underline{\lambda}) - \mathsf{rl}(\lambda)} = \sum_{\lambda' \in \mathcal{L}in^{(1)}[n]} q^{\mathsf{inv}(\underline{\lambda}')} \sum_{j \in [n+1]} y^{n+1-j},$$
and thus
$$\sum_{\lambda \in \mathcal{L}in^{(1)}[n+1]} v(\lambda) = n!_q\, [n+1]_y,$$
which, in view of (2.13), gives
$$\mathcal{D}_q \mathcal{L}in_v^{(1)}(t) = \mathcal{D}_q \mathcal{L}_1^{(-1)}(y; t \,|\, q),$$
and by q-integration
$$\mathcal{L}in_v^{(1)}(t) = \mathcal{L}_1^{(-1)}(y; t \,|\, q)$$
because the series on both sides have a zero constant term.
- The case $k > 1$. Assuming that $\mathcal{L}in_v^{(k)}(t) = \mathcal{L}_k^{(-1)}(y; t \,|\, q)$ has already been proved for $k \geq 1$, the goal is to show
$$\mathcal{L}in_v^{(k+1)}(t) = \mathcal{L}_{k+1}^{(-1)}(y; t \,|\, q).$$
Comparing the coefficients of x^{k+1} on both sides of Equation (2.14), we obtain
$$\mathcal{D}_q \mathcal{L}_{k+1}^{(-1)}(y; t \,|\, q) = \mathcal{D}_q \mathcal{L}_1^{(-1)}(y; t \,|\, q) \cdot \mathcal{L}_k^{(-1)}(y; t \,|\, q).$$
If we can show that similarly
$$\mathcal{D}_q \mathcal{L}in_v^{(k+1)}(t) = \mathcal{D}_q \mathcal{L}in_v^{(1)}(t) \cdot \mathcal{L}in_v^{(k)}(t), \tag{3.3}$$
then we would be done. Again, the final integration step poses no problem because in both $\mathcal{L}in_v^{(k+1)}(t)$ and $\mathcal{L}_{k+1}^{(-1)}(y; t \,|\, q)$ the first $k+1$ coefficients vanish. Recall that a configuration $\lambda \in \mathcal{L}in^{(k+1)}[n]$ consists of a list of $k+1$ disjoint strict lists, written as a list $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_k)$, with $\lambda_i \in \mathcal{L}in^{(1)}[A_i]$, where
$$\biguplus_{i=0}^{k} A_i = [n] \quad \text{and} \quad \min A_{i-1} < \min A_i, \quad 1 \leq i \leq k.$$
We have a bijection
$$\mathcal{L}in^{(k+1)}[A] \to \mathcal{L}in^{(1)}[A_0] \times \mathcal{L}in^{(k)}[A']$$
$$\lambda \mapsto (\lambda_0, \lambda'),$$

where $\lambda' = (\lambda_1, \dots, \lambda_k)$ and $A' = \bigcup_{i=1}^{k} A_i$, which also satisfies the requirement for applying the folklore statement (3.1):

$$v(\lambda) = v(\lambda_0) \cdot v(\lambda') \cdot q^{\mathsf{inv}(A_0, A')}.$$

All this holds only if for the bipartition $A = A_0 \uplus A'$ it is guaranteed that $\min A_0 < \min A'$. This is where the derivative $\mathcal{D}_q$ comes into play. Differentiation for a collection of structures means that the minimum element of the underlying set of a structure is tagged and no longer counted in the w-valuation of the base set. In the present situation, this implies that only structures are considered where tagging the minimum element of λ means the same as tagging the minimum element of λ_0. This shows that (3.3) holds. □

Theorem 7. *For integers $\alpha \geq -1$, we have*

$$\ell_{n,k}^{(\alpha)}(y; q) = \sum_{(\sigma;\lambda) \in \mathcal{LC}_{n,k}^{(\alpha)}} y^{|\underline{\sigma}| + \mathsf{rl}(\lambda)} q^{\mathsf{inv}(\underline{\sigma}.\underline{\lambda}) - \mathsf{rl}(\lambda) + \mathsf{inv}(\underline{\underline{\sigma}})}.$$

Proof. From (2.10) we infer

$$\mathcal{L}_k^{(\alpha)}(y; t \,|\, q) = \mathcal{L}^{(\alpha)}(0; y; t \,|\, q) \mathcal{L}_k^{(-1)}(y; t \,|\, q),$$

and the result follows from Lemmas 1 and 2. □

Here we give an example to illustrate the α-Laguerre configurations.

Example 8. Consider the 1-Laguerre configuration $(\sigma; \lambda) \in \mathcal{LC}_{15,4}^{(1)}$ in Figure 2. We have

$$\begin{aligned} &\sigma = \sigma_0 \cdot \sigma_1 \quad \text{with} \quad \sigma_0 = (15)(7\,4),\ \sigma_1 = (14)(13\,5\,2); \\ &\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \quad \text{with} \quad \lambda_1 = 1\,3,\ \lambda_2 = 12\,6\,11,\ \lambda_3 = 10\,8,\ \lambda_4 = 9. \end{aligned}$$

Thus,

$$\begin{aligned} \underline{\sigma} &= \hat{\sigma}_0 \cdot \hat{\sigma}_1 = 7\,4\,15 \cdot 13\,2\,5\,14, \\ \underline{\underline{\sigma}} &= 0^3\,1\,0^4; \\ \underline{\lambda} &= 1\,3 \cdot 12\,6\,11 \cdot 10\,8 \cdot 9. \end{aligned}$$

We have $|\underline{\sigma}| = 7$, $\mathsf{rl}(\lambda) = 3$, $\mathsf{inv}(\underline{\underline{\sigma}}) = 4$, and $\mathsf{inv}(\underline{\sigma} \cdot \underline{\lambda}) = 52$.

Remark 9. Our model of α-Laguerre configurations is simpler than the model in [3]. Actually, the α-Laguerre configurations are essentially the Laguerre configurations of which each cycle has a color in $\{0, \dots, \alpha\}$. A linear order of paths and colored cycles is needed only for the valuation w in (3.2).

4. Moments of (q,y)-Laguerre polynomials

For $\alpha \in \mathbb{N}_0$, by (2.6) the moments of the (q,y)-Laguerre polynomials are defined by

$$\mu_n^{(\alpha)}(q,y) := \mathcal{L}_q(x^n). \tag{4.1}$$

According to the theory of orthogonal polynomials (see [4]) and the three-term recurrence relation (1.7), we have the orthogonality relation

$$\mathcal{L}_q(L_n^{(\alpha)}(x;y\,|\,q)L_m^{(\alpha)}(x;y\,|\,q)) = y^n n!_q\left(\prod_{j=1}^{n}[\alpha+j]_q\right)\delta_{n\,m}. \tag{4.2}$$

Moreover, we have the following continued fraction expansion:

$$\sum_{n\geq 0}\mu_n^{(\alpha)}(q,y)t^n = \cfrac{1}{1-b_0t-\cfrac{\lambda_1t^2}{1-b_1t-\cfrac{\lambda_2t^2}{\ddots}}}, \tag{4.3}$$

where $b_n = y[n+\alpha+1]_q+[n]_q$ and $\lambda_n = y[n]_q[n+\alpha]_q$.

Let $\mathfrak{S}_n$ be the set of permutations of $\{1,2,\dots,n\}$. For $\sigma\in\mathfrak{S}_n$, we define three statistics, namely:

- the number of *weak excedances*, $\mathsf{wex}(\sigma)$, given by
$$\mathsf{wex}(\sigma) = |\{i\in[n] : \sigma(i)\geq i\}|;$$
- the number of *records* (or *left-to-right maxima*), $\mathsf{rec}(\sigma)$, given by
$$\mathsf{rec}(\sigma) = |\{i\in[n] : \sigma(i)>\sigma(j) \text{ for all } j<i\}|;$$
- the *number of crossings*, $\mathsf{cros}(\sigma)$, given by
$$\mathsf{cros}(\sigma) = |\{(i,j)\in[n]\times[n] : i<j\leq\sigma(i)<\sigma(j) \text{ or } \sigma(j)<\sigma(i)<j<i\}|.$$

Theorem 10. *Let $\beta = [\alpha+1]_q$. Then*

$$\mu_n^{(\alpha)}(y,q) = \sum_{\sigma\in\mathfrak{S}_n}\beta^{\mathsf{rec}(\sigma)}y^{\mathsf{wex}(\sigma)}q^{\mathsf{cros}(\sigma)}. \tag{4.4}$$

The first values of the moments are as follows:

$$\begin{aligned}
\mu_1^{(\alpha)}(y,q) &= y\beta,\\
\mu_2^{(\alpha)}(y,q) &= y\beta+y^2\beta^2,\\
\mu_3^{(\alpha)}(y,q) &= y\beta+\beta(1+(2+q)\beta)y^2+y^3\beta^3.
\end{aligned}$$

Due to the contraction formula [13, p. 292], we can rewrite (4.3) as

$$\sum_{n\geq 0} \mu_n^{(\alpha)}(q,y)t^n = \cfrac{1}{1-\cfrac{\gamma_1 t}{1-\cfrac{\gamma_2 t}{\ddots}}}, \tag{4.5}$$

where $\gamma_{2n} = [n]_q$ and $\gamma_{2n+1} = y[n+\alpha]_q = y([n]_q + [\alpha+1]_q q^n)$ for $n \geq 0$.

Recall that a *Dyck path* of length $2n$ is a sequence of points $(\omega_0, \dots, \omega_{2n})$ in $\mathbb{N}_0 \times \mathbb{N}_0$ satisfying $\omega_0 = (0,0)$, $\omega_{2n} = (2n, 0)$ and $\omega_{i+1} - \omega_i = (1,1)$ or $(1,-1)$ for $i = 0, \dots, 2n-1$. Clearly we can also identify a Dyck path with its sequence of steps (or *Dyck word*) $s = s_1 \dots s_{2n}$ on the alphabet $\{\mathsf{u}, \mathsf{d}\}$, and we use $|s|_{\mathsf{u}}$ and $|s|_{\mathsf{d}}$ to denote the number of u's and d's, respectively, in s. So, for a Dyck word s, we have $|s|_{\mathsf{u}} = |s|_{\mathsf{d}} = n$ and $|s_1 \dots s_k|_u \geq |s_1 \dots s_k|_{\mathsf{d}}$ for $k \in [2n]$. The height h_k of step s_k is defined to be $h_1 = 0$ and

$$h_k = |s_1 \dots s_{k-1}|_u - |s_1 \dots s_{k-1}|_d \quad \text{for} \quad k = 2, \dots, 2n.$$

A *Laguerre history* of length $2n$ is a pair (s, ξ), where s is a Dyck word of length $2n$ and $\xi = (\xi_1, \dots, \xi_{2n})$ is a sequence of integers such that $\xi_i = 1$ if $s_i = \mathsf{u}$ and $1 \leq \xi_i \leq \lceil h_i/2 \rceil$ if $s_i = \mathsf{d}$. Let $\mathcal{LH}_n$ be the set of Laguerre histories of length $2n$. We essentially use Biane's bijection [2] to construct a bijection Φ from $\mathfrak{S}_n$ to $\mathcal{LH}_n$.

Proof of Theorem 10. We identify a permutation $\sigma \in \mathfrak{S}_n$ with the bipartite graph $\mathcal{G}$ on $\{1, \dots, n; 1', \dots, n'\}$ with an edge (i, j') if and only if $\sigma(i) = j$. We display the vertices on two rows called top row and bottom row as follows:

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ 1' & 2' & \cdots & n' \end{pmatrix},$$

and we read the graph column by column from left to right and from top to bottom. In other words, the order of vertices is $v_1 = 1, v_2 = 1', \dots, v_{2n-1} = n, v_{2n} = n'$.

For $k = 1, \dots, 2n$, the k-th restriction of $\mathcal{G}$ is the graph $\mathcal{G}_k$ on $\{v_1, v_2, \dots, v_k\}$ with edge (v_i, v_j) in $\mathcal{G}_k$ if and only if $i, j \in [k]$, so isolated vertices may exist in $\mathcal{G}_k$.

For $i = 1, \dots, n$, the Dyck path $s = s_1 \dots s_{2n}$ is defined as follows:

- if $\sigma^{-1}(i) > i < \sigma(i)$ (i.e., i is a cycle valley), then $s_{2i-1}s_{2i} = \mathsf{uu}$;
- if $\sigma^{-1}(i) < i < \sigma(i)$ (i.e., i is a cycle double ascent), then $s_{2i-1}s_{2i} = \mathsf{ud}$;
- if $\sigma^{-1}(i) > i > \sigma(i)$ (i.e., i is a cycle double descent), then $s_{2i-1}s_{2i} = \mathsf{du}$;
- if $\sigma^{-1}(i) < i > \sigma(i)$ (i.e., i is a cycle peak), then $s_{2i-1}s_{2i} = \mathsf{dd}$;
- if $\sigma^{-1}(i) = i = \sigma(i)$ (i.e., i is a fixed point), then $s_{2i-1}s_{2i} = \mathsf{ud}$.

It is easy to see that

- s is a Dyck path;
- the height h_i is the number of isolated vertices in $\mathcal{G}_{i-1}$ for $i \in [2n]$ with $\mathcal{G}_{i-1} = \emptyset$; thus h_{2i-1} (respectively h_{2i}) is even (respectively odd) for $i = 1, \dots, n$ and there are $\lceil h_i/2 \rceil$ isolated vertices in the top row.

Next, the sequence $\xi = (\xi_1, \dots, \xi_{2n})$ is defined as follows:

- $s_i = \mathsf{u}$ then $\xi_i = 1$;
- $s_i = \mathsf{d}$, then
  - if $\sigma(i) < i$ (i.e., i is a cycle double descent or cycle peak), then $h_{2i-1} > 0$; let $\xi_i = m$ if $\sigma(2i)$ is the m-th isolated vertex in the bottom row of $\mathcal{G}_{2i-2}$ from right-to-left ($1 \le m \le \lceil h_{2i}/2 \rceil$); clearly the value i will contribute $m-1$ crossings $l < k < i < j$ such that $l = \sigma(i),\ k = \sigma(j)$;
  - if $\sigma^{(-1)}(i) \le i$ (i.e., i is a cycle double ascent, cycle peak or fixed point), then $h_{2i} > 0$; let $\xi_i = m$ if $\sigma^{(-1)}(i)$ is the m-th isolated vertex in the top row of $\mathcal{G}_{2i-2}$ from right-to-left, so $1 \le m \le \lceil h_{2i}/2 \rceil$; clearly the value i will contribute $m-1$ crossings $l < k < i < j$ such that $l = \sigma^{-1)}(i),\ k = \sigma(j)$, and i is a record if and only if $m = \lceil h_{2i}/2 \rceil$.

Let $\Phi(\sigma) = (s, \xi)$. Then

$$\begin{aligned}\mathsf{wex}(\sigma) &= |\{i \in [n] : s_{2i} = \mathsf{d}\}|,\\ \mathsf{rec}(\sigma) &= |\{i \in [n] : s_{2i} = \mathsf{d},\, \xi_{2i} = \lceil h_{2i}/2 \rceil\}|,\\ \mathsf{cros}(\sigma) &= \sum_{i: s_i = \mathsf{d}} (\xi_i - 1).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{\sigma \in \mathfrak{S}_n} \beta^{\mathsf{rec}(\sigma)} y^{\mathsf{wex}(\sigma)} q^{\mathsf{cros}(\sigma)} &= \sum_{(s,\xi) \in \mathcal{LH}_n} \prod_{i: s_i = \mathsf{d}} q^{\xi_i - 1} \prod_{i: s_{2i} = \mathsf{d}} y \beta^{\chi(\xi_{2i} = \lceil h_{2i}/2 \rceil)}\\ &= \sum_{s \in \mathsf{Dyck}_n} \prod_{i: s_i = \mathsf{d}} w(s_i), \end{aligned} \tag{4.6}$$

where Dyck_n denotes the set of Dyck paths of semilength n, and the weight of each down step $s_i = \mathsf{d}$ is defined by

$$w(s_i) = \begin{cases} 1 + q + \cdots + q^{k-1}, & \text{if } h_i = 2k,\\ y(1 + q + \cdots + q^{k-1} + \beta q^k), & \text{if } \ h_i = 2k+1. \end{cases}$$

A folklore theorem [5] implies that the generating function of (4.6) has the continued fraction expansion (4.5), and we are done. □

Example 11. If $\sigma = 4\,1\,2\,7\,9\,6\,5\,8\,3 \in \mathfrak{S}_9$, then the Laguerre history $\Phi(\sigma) = (s, \xi)$ is given by

$$\begin{pmatrix} s \\ \xi \end{pmatrix} = \begin{pmatrix} \mathsf{u\,u} & \mathsf{d\,u} & \mathsf{d\,u} & \mathsf{u\,d} & \mathsf{u\,u} & \mathsf{u\,d} & \mathsf{d\,d} & \mathsf{u\,d} & \mathsf{d\,d} \\ 1\,1 & 1\,1 & 1\,1 & 1\,2 & 1\,1 & 1\,1 & 1\,2 & 1\,1 & 1\,1 \end{pmatrix}.$$

Theorem 12. *Let $\alpha \in \mathbb{N}_0$. For nonnegative integers $n_1, \ldots, n_k$, the linearization coefficient*

$$\mathcal{L}_q \left(\prod_{k=1}^{m} L_{n_k}^{(\alpha)}(x; y \,|\, q) \right) \tag{4.7}$$

is a polynomial in $\mathbb{N}[y, q]$.

Proof. In view of the orthogonality (4.2), it suffices to prove the $m = 3$ case. Indeed, we can derive the following explicit formula from [17, Theorem 1]:

$$\begin{aligned}
&\mathcal{L}_q(L_{n_1}^{(\alpha)}(x; y \mid q) L_{n_2}^{(\alpha)}(x; y \mid q) L_{n_3}^{(\alpha)}(x; y \mid q)) \\
&\quad = n_1!_q\, n_2!_q\, n_3!_q \sum_{s \geq \max(n_1, n_2, n_3)} y^s \begin{bmatrix} s \\ n_1 + n_2 + n_3 - 2s, s - n_3, s - n_2, s - n_1 \end{bmatrix}_q \\
&\qquad \times \begin{bmatrix} \alpha + s \\ s \end{bmatrix}_q \sum_{k \geq 0} \begin{bmatrix} n_1 + n_2 + n_3 - 2s \\ k \end{bmatrix}_q y^k q^{\binom{k+1}{2} + \binom{n_1+n_2+n_3-2s-k}{2} + k\alpha}, \qquad (4.8)
\end{aligned}$$

where the q-multinomial coefficients

$$\begin{bmatrix} a + b + c + d \\ a, b, c, d \end{bmatrix}_q = \frac{(a + b + c + d)!_q}{a!_q\, b!_q\, c!_q\, d!_q}$$

are known to be polynomials in $\mathbb{N}[q]$ for integral $a, b, c, d \geq 0$, see [13]. Hence, the right-hand side of (4.8) is a polynomial in $\mathbb{N}[y, q]$, and we are done. □

For arbitrary α, a combinatorial interpretation of (4.7) was given by Foata and Zeilberger [8] with $y = q = 1$, and generalized by the second author [23] to $q = 1$ (see also [24]), while for $\alpha = 0$ a combinatorial interpretation of (4.7) was given by Kasraoui et al. [17]. Thus, the following problem suggests itself.

Problem. *What is the combinatorial interpretation of* (4.7) *for* $\alpha \in \mathbb{N}_0$ *unifying the two special cases with* $\alpha = 0$ *or* $q = 1$*?*

5. Connection with rook polynomials and matching polynomials

In this section we show how the model of α-Laguerre configurations is connected with the models of non-attacking rook placements and matchings of complete bipartite graphs.

5.1. **Interpretation in rook polynomials.** An m by n board B is a subset of an $m \times n$ grid of cells (or squares). A rook is a chessboard piece which takes on rows and columns. If r_k is the number of ways of putting k non-attacking rooks on this board, then the ordinary rook polynomial is defined by

$$R_{m,n}(x) = \sum_k r_k x^k.$$

Thus, the Laguerre polynomials (1.2) can be written as

$$L_n^{(\alpha)}(x) = (-1)^n n!\, R_{n,n+\alpha}(-x^{-1}). \qquad (5.1)$$

A *k-rook placement* on a board B is a subset $C \subset B$ of k cells such that no two cells are in the same row or column of B. We refer the reader to Riordan's classical book [20, Chapters 7 and 8] for many problems formulated in terms of configurations of non-attacking rooks on "chessboards" of various shapes.

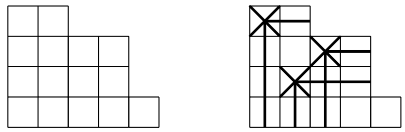

FIGURE 3. The Ferrers board of shape $\mu = (4, 4, 3, 3, 1)$ and a placement C of three non-attacking rooks with $\mathsf{inv}(C) = 3$.

We label the rows of the grid from top to bottom and the columns from left to right in the same way as referring to the entries of an $m \times n$ matrix. Recall that an *integer partition* is a sequence of positive integers $\mu := (\mu_1, \mu_2, \ldots, \mu_l)$ such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l > 0$. We also use the notation $\mu = (n_1^{m_1}, \ldots, n_k^{m_k})$ to denote the partition with m_i parts equal to n_i for $i = 1, \ldots, k$. For convenience, we shall identify μ with its Ferrers board B_μ, which is defined as the subset $\{(i, j) : 1 \leq i \leq \mu_j,\ 1 \leq j \leq l\}$ of $\mathbb{N} \times \mathbb{N}$. For a placement C of rooks on B_μ, the inversion number $\mathsf{inv}(C)$ is defined as follows: for each rook (cell) in C cross out all the cells which are below or to the right of the rook; then $\mathsf{inv}(C)$ is the number of squares of F_μ that are not crossed out. An example is shown in Figure 3.

Definition 13. *For integers $n, k \geq 0$ and $\alpha \geq -1$, let $\mathbf{m} = (m_0, \ldots, m_\alpha)$ and $\mathbf{n} = (n_1, \ldots, n_k)$ be nonnegative integer sequences such that $m_0+m_1+\cdots+m_\alpha+n_1+\cdots+n_k = n$ with $m_i \geq 0$ and $n_j \geq 1$. We define $\mathcal{B}_{n,k}^{(\alpha)}(\mathbf{m}; \mathbf{n})$ as the set of $n \times n$ squares of color shape $B := (B^{(1)}; B^{(2)})$ with*

$$B^{(1)} :=(n^{m_0}, \ldots, n^{m_\alpha}), \tag{5.2a}$$

$$B^{(2)} :=(n^{n_1}, \ldots, n^{n_k}). \tag{5.2b}$$

By convention, if $\alpha = -1$ (respectively $k = 0$), then $B^{(1)} = \emptyset$ (respectively $B^{(2)} = \emptyset$). Let

$$\mathsf{cw}(B) = \sum_{i=0}^{\alpha} m_i \quad \textit{and} \quad \mathsf{cd}(B) = \sum_{i=0}^{\alpha} i \cdot m_i.$$

Let $\mathcal{BC}_{n,k}^{(\alpha)}(\mathbf{m}; \mathbf{n})$ denote the set of all ordered pairs $\mathcal{R} = (B, C)$, where $B \in \mathcal{B}_{n,k}^{(\alpha)}(\mathbf{m}; \mathbf{n})$ and C is an n-rook placement on B such that

$$\min(C \cap B_1^{(2)}) < \min(C \cap B_2^{(2)}) < \cdots < \min(C \cap B_k^{(2)}), \tag{5.3}$$

where $\min(C \cap B_1^{(2)})$ is the minimum row index of cells in $C \cap B_1^{(2)}$. For each block $B_i^{(2)} = (n^{n_i})$, we define $\mathsf{ind}(C \cap B_i^{(2)})$ as the number of rooks in $C \cap B_i^{(2)}$ whose column indices are greater than the column index of the rook which has the maximum row index in $B_i^{(2)}$, and let $\mathsf{ind}(\mathcal{R}) = \sum_{i=1}^{k} \mathsf{ind}(C \cap B_i^{(2)})$. Let

$$\mathcal{BC}_{n,k}^{(\alpha)} = \bigcup_{\mathbf{m},\mathbf{n}} \mathcal{BC}_{n,k}^{(\alpha)}(\mathbf{m}; \mathbf{n}) \qquad \textit{with} \qquad \sum_{i=0}^{\alpha} m_i + \sum_{j=1}^{k} n_j = n.$$

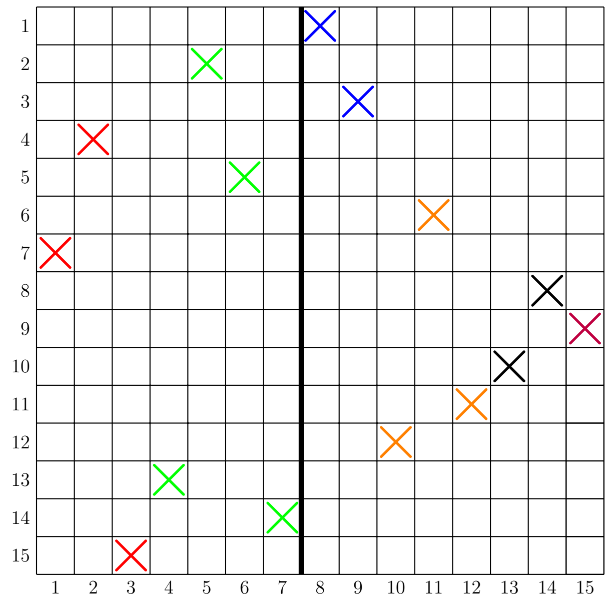

FIGURE 4. The colored rook configuration $\mathcal{R}$ corresponding to the 1-Laguerre configuration in Figure 2 with $(\mathbf{m};\mathbf{n}) = ((3,4);(2,3,2,1))$.

An element $\mathcal{R} = (B, C) \in \mathcal{BC}_{n,h}^{(\alpha)}$ is called a colored rook configuration.

Remark 14. One can imagine that each column of a board in $\mathcal{BC}_{n,k}^{(\alpha)}(\mathbf{m};\mathbf{n})$ is colored with colors in $\{0, 1, \ldots, \alpha + k\}$ from left to right as follows: the first m_0 columns get color 0, the next m_1 columns get color 1, ..., the last n_k columns get color $\alpha + k$.

Theorem 15. *The coefficient $\ell_{n,k}^{(\alpha)}(y;q)$ in (1.8) is the following generating polynomial of colored rook configurations in $\mathcal{BC}_{n,k}^{(\alpha)}$:*

$$\ell_{n,k}^{(\alpha)}(y;q) = \sum_{\mathcal{R}=(B,C)\in\mathcal{BC}_{n,k}^{(\alpha)}} y^{\mathsf{cw}(B)+\mathsf{ind}(\mathcal{R})} q^{\mathsf{inv}(C)+\mathsf{cd}(B)-\mathsf{ind}(\mathcal{R})}.$$

Proof. Let $\mathcal{LC}_{n,k}^{(\alpha)}(\mathbf{m};\mathbf{n})$ be the set of $\rho := (\sigma_0, \ldots, \sigma_\alpha;\, \lambda_1, \ldots, \lambda_k) \in \mathcal{LC}_{n,k}^{(\alpha)}$ such that $|\rho| = (|\sigma_0|, \ldots, |\sigma_\alpha|;\, |\lambda_1|, \ldots, |\lambda_k|) = (\mathbf{m};\mathbf{n})$. We define the map $\phi : \mathcal{LC}_{n,k}^{(\alpha)}(\mathbf{m};\mathbf{n}) \longrightarrow \mathcal{BC}_{n,k}^{(\alpha)}(\mathbf{m};\mathbf{n})$ by $\phi(\rho) = (B, C)$ for $\rho = (\sigma_0, \ldots, \sigma_\alpha;\, \lambda_1, \ldots, \lambda_k) \in \mathcal{LC}_{n,k}^{(\alpha)}(\mathbf{m};\mathbf{n})$ as follows:

(i) The colored board $B = (B^{(1)}, B^{(2)})$ is given by

$$B^{(1)} = (n^{|\sigma_0|}, \ldots, n^{|\sigma_\alpha|}) \quad \text{and} \quad B^{(2)} = (n^{|\lambda_1|}, n^{|\lambda_2|}, \ldots, n^{|\lambda_k|}).$$

(ii) If $w := \hat{\sigma}_0\hat{\sigma}_1 \cdots \hat{\sigma}_\alpha \lambda_1 \lambda_2 \cdots \lambda_k = w_1 \ldots w_n$, which is a permutation of $[n]$, let $C = \{(j, w_j) : j \in [n]\}$.

It is clear that $\phi(\rho) \in \mathcal{BC}_{n,k}^{(\alpha)}$, and the procedure is reversible. Hence ϕ is a bijection. It is easy to verify that $\mathsf{inv}(C) = \mathsf{inv}(\rho)$, $\mathsf{ind}(B_i^{(2)}) = \mathsf{rl}(\lambda_i)$, and $\mathsf{cd}(B) = \sum_{i=0}^{\alpha} i|\sigma_i|$, which implies that

$$|\underline{\sigma}| + \mathsf{rl}(\lambda) = \mathsf{cw}(B) + \mathsf{ind}(\mathcal{R});$$
$$\mathsf{inv}(\underline{\sigma}.\underline{\lambda}) - \mathsf{rl}(\lambda) + \mathsf{inv}(\underline{\underline{\sigma}}) = \mathsf{inv}(C) + \mathsf{cd}(B) - \mathsf{ind}(\mathcal{R}).$$

The result then follows from Theorem 7. □

Example 16. Let $\rho = ((7\,4)(15), (13\,2\,5)(14); 1\,3, 12\,6\,11, 10\,8, 9) \in \mathcal{LC}_{15,4}^{(1)}$. Then ϕ maps ρ to the placement of 15 rooks on the board $B = (15^7; 15^2, 10^3, 8^2, 7)$ shown in Figure 4. We find $\mathsf{cw}(B) = 1$, $\mathsf{cd}(B) = 4$; $\mathsf{inv}(C) = 52$ and $\mathsf{ind}(\mathcal{R}) = 3$.

5.2. **Interpretation in matching polynomials.** Recall that a *matching* of a graph G is a set of edges without common vertices. For any graph G with n vertices, the *matching polynomial* of G is defined by

$$m(G, x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k m_k x^{n-2k},$$

where m_k is the number of k-edge matchings of G. Let $K_{n,m}$ denote the set of complete bipartite graphs on the two disjoint sets $A = [n]$ and $B = \{1', \dots, m'\}$, that is, there is an edge (a, b) if and only if $a \in A$ and $b \in B$. From the explicit formula (1.2) it is quite easy to derive the connection formula

$$m(K_{n,n+\alpha}, x) = x^\alpha L_n^{(\alpha)}(x^2), \qquad \alpha \ge -1. \tag{5.4}$$

Godsil and Gutman [12] proved (5.4) by showing that the matching polynomials satisfy the same three-term recurrence relation (1.3). Here we give a simple bijection between our α-Laguerre configuration model and the above matching model of complete bipartite graphs. Let $\mathcal{M}_{n,m}^{n-k}$ be the set of matchings of $K_{n,m}$ with $n-k$ edges.

Proposition 17. *For integers $n, k \ge 1$ and $\alpha \ge -1$, there exists an explicit bijection $\phi :\ \mathcal{LC}_{n,k}^{(\alpha)} \longrightarrow \mathcal{M}_{n,n+\alpha}^{n-k}$.*

Proof. We construct such a bijection ϕ. Let $\rho = (\sigma_0, \sigma_1, \dots, \sigma_\alpha; \lambda_1, \lambda_2, \dots, \lambda_k) \in \mathcal{LC}_{n,k}^{(\alpha)}$ be an α-Laguerre configuration. We define a matching γ of $K_{n,n+\alpha}$ such that $(a, b') \in A \times B$ is an edge in γ if and only if (a, b) satisfies one of the following three conditions:

(1) $\sigma_0(a) = b$, i.e., the image of a is b through the action of permutation σ_0;
(2) a and b are consecutive letters in the word $\hat{\sigma}_1(n+1)\hat{\sigma}_2(n+2)\dots\hat{\sigma}_\alpha(n+\alpha)$;
(3) a and b are consecutive letters in the word λ_j for some $j \in [k]$.

By convention, if $\alpha = -1$ (respectively $\alpha = 0$) there are no words of types (1) and (2) (respectively type (2)). Since $\sigma_0\sigma_1 \dots \sigma_\alpha\lambda_1 \dots \lambda_k$ is a permutation of $[n]$, it is clear that there are $n-k$ such edges (a, b'). The above procedure is obviously reversible. □

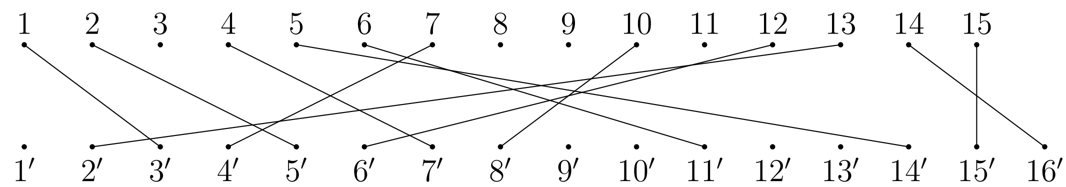

Figure 5. The matching corresponding to the 1-Laguerre configuration in Figure 2

Example 18. For the 1-Laguerre configuration

$$\rho = ((7\,4)(15), (13\,2\,5)(14); 1\,3, 12\,6\,11, 10\,8, 9) \in \mathcal{LC}^{(1)}_{15,4}$$

in Example 8, the corresponding matching γ of $K^{11}_{15,16}$ is shown in Figure 5.

Remark 19. We leave it to the interested reader to find the (q, y)-version of the above matching polynomials for (q, y)-Laguerre polynomials $L^{(\alpha)}_n(x; y\,|\,q)$.

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Univ Lyon, Université Lyon1, UMR 5208 du CNRS, Institut Camille Jordan, F-69622, Villeurbanne Cedex, France
Email address: qpan@math.univ-lyon1.fr

Univ Lyon, Université Lyon1, UMR 5208 du CNRS, Institut Camille Jordan, F-69622, Villeurbanne Cedex, France
Email address: zeng@math.univ-lyon1.fr