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OPERADS OF DECORATED CLIQUES I: CONSTRUCTION AND QUOTIENTS

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Abstract. We introduce a functorial construction C which takes unitary magmas $\mathcal{M}$ as input and produces operads. The obtained operads involve configurations of chords labeled by elements of $\mathcal{M}$, called $\mathcal{M}$-decorated cliques and generalizing usual configurations of chords. By considering combinatorial subfamilies of $\mathcal{M}$-decorated cliques defined, for instance, by limiting the maximal number of crossing diagonals or the maximal degree of the vertices, we obtain suboperads and quotients of $\mathrm{C}\mathcal{M}$. This leads to a new hierarchy of operads containing, among others, operads on noncrossing configurations, Motzkin configurations, forests, dissections of polygons, and involutions. Moreover, the construction C leads to alternative definitions of the operads of simple and double multi-tildes, and of the gravity operad.

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Introduction

Configurations of chords on regular polygons are very classical combinatorial objects. Up to some restrictions or enrichments, sets of these objects can be put in bijection with several combinatorial families. For instance, it is well-known that triangulations [DLRS10], forming a particular subset of the set of all configurations of chords, are in one-to-one correspondence with binary trees, and a lot of structures and operations on binary trees translate nicely on triangulations. Indeed, among others, the rotation operation on binary trees [Knu98] is the covering relation of the Tamari order [HT72] and this operation translates as a diagonal flip in triangulations. Also, noncrossing configurations [FN99] form another interesting subfamily of such chord configurations. Natural generalizations of noncrossing configurations consist in allowing, with more or less restrictions, some crossing diagonals. One of these families is formed by the multi-triangulations [CP92] wherein the number of mutually crossing diagonals is bounded. In particular, the class of combinatorial objects in bijection with some configurations of chords is large enough in order to contain, among others, dissections of polygons, noncrossing partitions, permutations, and involutions.

On the other hand, coming historically from algebraic topology [May72, BV73], operads provide an abstraction of the notion of operators (of any arities) and their compositions. In more concrete terms, operads are algebraic structures abstracting the notion of planar rooted trees and their grafting operations (see [LV12] for a complete exposition of the theory and [Mén15] for an exposition focused on symmetric set-operads). The modern treatment of operads in algebraic combinatorics consists in regarding combinatorial objects like operators endowed with gluing operations mimicking the composition of operators. In the last years, a lot of combinatorial sets and combinatorial spaces have been endowed fruitfully with the structure of an operad (see for instance [Cha08] for an exposition of known interactions between operads and combinatorics, focused on trees, [LMN13, GLMN16], where operads abstracting operations in language theory are introduced, [CG14] for the study of an operad involving particular noncrossing configurations, [Gir15] for a general construction of operads on many combinatorial sets, [Gir16a], where operads are constructed from posets, and [CHN16], where operads on various species of trees are introduced). In most of the cases, this approach brings results about enumeration, helps to discover new statistics, and leads to establish new links (by morphisms) between different combinatorial sets or spaces. We can observe that most of the subfamilies of polygons endowed with configurations of chords discussed above are stable for several natural composition operations.

Even better, some of these can be described as the closure with respect to these composition operations of small sets of polygons. For this reason, operads are very promising candidates, among the modern algebraic structures, to study such objects under an algebraic and combinatorial flavor.

The purpose of this work is twofold. First, we are concerned with endowing the linear span of the configurations of chords with the structure of an operad. This leads to seeing these objects under a new light, stressing some of their combinatorial and algebraic properties. Second, we would provide a general construction of operads of configurations of chords rich enough so that it includes some already known operads. As a consequence, we obtain alternative definitions of existing operads and new interpretations of these. For this aim, we work here with $\mathcal{M}$-decorated cliques (or $\mathcal{M}$-cliques for short), that are complete graphs whose arcs are labeled by elements of $\mathcal{M}$, where $\mathcal{M}$ is a unitary magma. These objects are natural generalizations of configurations of chords since the arcs of any $\mathcal{M}$-clique labeled by the unit of $\mathcal{M}$ are considered as missing. The elements of $\mathcal{M}$ different from the unit allow moreover to handle chords of different colors. For instance, each usual noncrossing configuration $\mathfrak{c}$ can be encoded by an $\mathbb{N}_2$-clique $\mathfrak{p}$, where $\mathbb{N}_2$ is the cyclic additive unitary magma $\mathbb{Z}/2\mathbb{Z}$, wherein each arc labeled by $1 \in \mathbb{N}_2$ in $\mathfrak{p}$ denotes the presence of the same arc in $\mathfrak{c}$, and each arc labeled by $0 \in \mathbb{N}_2$ in $\mathfrak{p}$ denotes its absence in $\mathfrak{c}$. Our construction is materialized by a functor C from the category of unitary magmas to the category of operads. It builds, from any unitary magma $\mathcal{M}$, an operad $\mathsf{C}\mathcal{M}$ on $\mathcal{M}$-cliques. The partial composition $\mathfrak{p} \circ_i \mathfrak{q}$ of two $\mathcal{M}$-cliques $\mathfrak{p}$ and $\mathfrak{q}$ of $\mathsf{C}\mathcal{M}$ consists in gluing the ith edge of $\mathfrak{p}$ (with respect to a precise indexation) and a special arc of $\mathfrak{q}$, called the base, together to form a new $\mathcal{M}$-clique. The magmatic operation of $\mathcal{M}$ explains how to relabel the two overlapping arcs.

This operad $\mathsf{C}\mathcal{M}$ has a lot of properties, which can be apprehended both under a combinatorial and an algebraic point of view. First, many families of particular configurations of chords form quotients or suboperads of $\mathsf{C}\mathcal{M}$. We can for instance control the degrees of the vertices or the crossings between diagonals to obtain new operads. We can also forbid all diagonals, or some labels for the diagonals or the edges, or all nestings of diagonals, or even all cycles formed by arcs. All these combinatorial particularities and restrictions on $\mathcal{M}$-cliques behave well algebraically. Moreover, by using the fact that the direct sum of two ideals of an operad $\mathcal{O}$ is still an ideal of $\mathcal{O}$, these constructions can be mixed to get even more operads. For instance, it is well-known that Motzkin configurations, which are polygons with disjoint noncrossing diagonals, are enumerated by Motzkin numbers [Mot48]. Since a Motzkin configuration can be encoded by an $\mathcal{M}$-clique where all vertices are of degree at most 1 and no diagonal crosses another one, we obtain an operad $\mathrm{Mot}\,\mathcal{M}$ on colored Motzkin configurations which is both a quotient of $\mathrm{Deg}_1\,\mathcal{M}$, the quotient of $\mathrm{C}\mathcal{M}$ consisting in all $\mathcal{M}$-cliques such that all vertices are of degree at most 1, and of $\mathrm{Cro}_0\,\mathcal{M}$, the quotient (and suboperad) of $\mathrm{C}\mathcal{M}$ consisting in all noncrossing $\mathcal{M}$-cliques. We also get quotients of $\mathrm{C}\mathcal{M}$ involving, among others, Schröder trees, forests of paths, forests of trees, dissections of polygons, Lucas configurations, with colored versions for each of these. This leads to a new hierarchy of operads, wherein links between its components appear as surjective or injective operad morphisms. One of the most notable of these is built by considering the $\mathbb{D}_0$-cliques that have vertices of degree at most 1, where $\mathbb{D}_0$ is the multiplicative unitary magma on $\{0,1\}$. This is in fact the quotient $\mathrm{Deg}_1\,\mathbb{D}_0$ of $\mathrm{C}\mathbb{D}_0$ and involves involutions (or equivalently, standard Young tableaux by the Robinson–Schensted correspondence [Lot02]). To the best of our knowledge, $\mathrm{Deg}_1\,\mathbb{D}_0$ is the first nontrivial operad on these objects.

As an important remark at this stage, let us highlight that, if $\mathcal{M}$ is nontrivial, $\mathrm{C}\mathcal{M}$ is not a binary operad. Indeed, all its minimal generating sets are infinite and its generators have arbitrarily high arities. Furthermore, the construction C maintains some links with the operad RatFct of rational functions introduced by Loday [Lod10]. In fact, provided that $\mathcal{M}$ satisfies some conditions, each $\mathcal{M}$-clique encodes a rational function. This defines an operad morphism from $\mathrm{C}\mathcal{M}$ to RatFct. Moreover, the construction C allows to construct already known operads in original ways. For instance, for well-chosen unitary magmas $\mathcal{M}$, the operads $\mathrm{C}\mathcal{M}$ contain MT and DMT, two operads defined in [LMN13] respectively in [GLMN16] that involve multi-tildes and double multi-tildes, operators coming from formal language theory [CCM11]. The operads $\mathrm{C}\mathcal{M}$ also contains Grav, the gravity operad, a symmetric operad introduced by Getzler [Get94], seen here as a nonsymmetric one [AP15].

This text is organized as follows. Section 1 sets our notations, general definitions, and tools about nonsymmetric operads (since we deal only with nonsymmetric operads here, we call these simply operads) and configurations of chords. In Section 2, we introduce $\mathcal{M}$-cliques, the construction C, and study some of its properties. Then Section 3 is devoted to define several suboperads and quotients of $\mathrm{C}\mathcal{M}$. This leads to plenty of new operads on particular $\mathcal{M}$-cliques. Finally, in Section 4, we use the construction C to provide alternative definitions of some known operads.

This paper is an extended version of [Gir17], containing the proofs of the presented results.

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General notations and conventions. All the algebraic structures of this article have a field of characteristic zero $\mathbb{K}$ as ground field. For any set S, $\mathbb{K}\langle S\rangle$ denotes the linear span of the elements of S. For any integers a and c, $[a,c]$ denotes the set $\{b \in \mathbb{N} : a \leqslant b \leqslant c\}$ and $[n]$, the set $[1,n]$. The cardinality of a finite set S is denoted by $\#S$. If u is a word, its letters are indexed from left to right from 1 to its length $|u|$. If a is a letter, $|u|_a$ denotes the number of occurrences of a in u.

1. Elementary definitions and tools

We set here our notations and recall some definitions about operads and related structures. We also introduce some notations and definitions about configurations of chords in polygons.

1.1. **Nonsymmetric operads.** We adopt most of notations and conventions of [LV12] about operads. For the sake of completeness, we recall here the elementary notions about operads employed thereafter.

A *nonsymmetric operad in the category of vector spaces*, or a *nonsymmetric operad* for short, is a graded vector space

$$\mathcal{O} := \bigoplus_{n\geqslant 1} \mathcal{O}(n) \tag{1.1.1}$$

together with linear maps

$$\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(n+m-1), \qquad n, m \geqslant 1, i \in [n], \tag{1.1.2}$$

called *partial compositions*, and a distinguished element $\mathbb{1} \in \mathcal{O}(1)$, the *unit* of $\mathcal{O}$. This data has to satisfy the three relations

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z), \qquad x \in \mathcal{O}(n), y \in \mathcal{O}(m), z \in \mathcal{O}(k), i \in [n], j \in [m], \tag{1.1.3a}$$

$$(x \circ_i y) \circ_{j+m-1} z = (x \circ_j z) \circ_i y, \qquad x \in \mathcal{O}(n), y \in \mathcal{O}(m), z \in \mathcal{O}(k), i < j \in [n], \tag{1.1.3b}$$

$$\mathbb{1} \circ_1 x = x = x \circ_i \mathbb{1}, \qquad x \in \mathcal{O}(n), i \in [n]. \tag{1.1.3c}$$

Since we consider in this paper only nonsymmetric operads, we shall call these simply *operads*. Moreover, in this work, we shall only consider operads $\mathcal{O}$ for which $\mathcal{O}(1)$ has dimension 1.

If $\mathcal{O}$ is such that all $\mathcal{O}(n)$ have finite dimensions for all $n \geqslant 1$, the *Hilbert series* of $\mathcal{O}$ is the series $\mathcal{H}_{\mathcal{O}}(t)$ defined by

$$\mathcal{H}_{\mathcal{O}}(t) := \sum_{n\geqslant 1} \dim \mathcal{O}(n)\, t^n. \tag{1.1.4}$$

If x is an element of $\mathcal{O}$ such that $x \in \mathcal{O}(n)$ for an $n \geqslant 1$, we say that n is the *arity* of x and we denote it by $|x|$. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two operads, a linear map $\phi : \mathcal{O}_1 \to \mathcal{O}_2$ is an *operad morphism* if it respects arities, sends the unit of $\mathcal{O}_1$ to the unit of $\mathcal{O}_2$, and commutes with partial composition maps. We say that $\mathcal{O}_2$ is a *suboperad* of $\mathcal{O}_1$ if $\mathcal{O}_2$ is a graded subspace of $\mathcal{O}_1$, $\mathcal{O}_1$ and $\mathcal{O}_2$ have the same unit, and the partial compositions of $\mathcal{O}_2$ are the ones of $\mathcal{O}_1$ restricted on $\mathcal{O}_2$. For any subset G of $\mathcal{O}$, the *operad generated* by G is the smallest suboperad $\mathcal{O}^G$ of $\mathcal{O}$ containing G. If $\mathcal{O}^G = \mathcal{O}$ and G is minimal with respect to the inclusion among the subsets of G satisfying this property, G is a *minimal generating set* of $\mathcal{O}$ and its elements are *generators* of $\mathcal{O}$. An *operad ideal* of $\mathcal{O}$ is a graded subspace I of $\mathcal{O}$ such that, for any $x \in \mathcal{O}$ and $y \in I$, $x \circ_i y$ and $y \circ_j x$ are in I for all valid integers i and j. Given an operad ideal I of $\mathcal{O}$, one can define the *quotient operad* $\mathcal{O}/I$ of $\mathcal{O}$ by I in the usual way.

Let us recall and set some more definitions about operads. The *Hadamard product* between the two operads $\mathcal{O}_1$ and $\mathcal{O}_2$ is the operad $\mathcal{O}_1 * \mathcal{O}_2$ satisfying $(\mathcal{O}_1 * \mathcal{O}_2)(n) = \mathcal{O}_1(n) \otimes \mathcal{O}_2(n)$, and its partial composition is defined component-wise from the partial compositions of $\mathcal{O}_1$ and $\mathcal{O}_2$. An element x of $\mathcal{O}(2)$ is *associative* if $x \circ_1 x = x \circ_2 x$. An *antiautomorphism* of $\mathcal{O}$ is a graded vector space automorphism ϕ of $\mathcal{O}$ sending the unit of $\mathcal{O}$ to the unit of $\mathcal{O}$ and such that for any $x \in \mathcal{O}(n)$, $y \in \mathcal{O}$, and $i \in [n]$, $\phi(x \circ_i y) = \phi(x) \circ_{n-i+1} \phi(y)$. A *symmetry* of $\mathcal{O}$ is either an automorphism or an antiautomorphism of $\mathcal{O}$. The set of all symmetries of $\mathcal{O}$ forms a group for the map composition, called the *group of symmetries* of $\mathcal{O}$. A basis $B := \bigsqcup_{n \geqslant 1} B(n)$ of $\mathcal{O}$ is a *set-operad basis* if all partial compositions of elements of B belong to B. In this case, we say that $\mathcal{O}$ is a *set-operad* with respect to the basis B. Moreover, if all the maps

$$\circ_i^y : B(n) \to B(n+m-1), \qquad n, m \geqslant 1, i \in [n], y \in B(m), \tag{1.1.5}$$

defined by

$$\circ_i^y(x) = x \circ_i y, \qquad x \in B(n), \tag{1.1.6}$$

are injective, we say that B is a *basic set-operad basis* of $\mathcal{O}$. This notion is a slightly modified version of the original notion of basic set-operads introduced by Vallette [Val07]. Finally, $\mathcal{O}$ is *cyclic* (see [GK95]) if there is a map

$$\rho : \mathcal{O}(n) \to \mathcal{O}(n), \qquad n \geqslant 1, \tag{1.1.7}$$

satisfying, for all $x \in \mathcal{O}(n)$, $y \in \mathcal{O}(m)$, and $i \in [n]$,

$$\rho(\mathbb{1}) = \mathbb{1}, \tag{1.1.8a}$$

$$\rho^{n+1}(x) = x, \tag{1.1.8b}$$

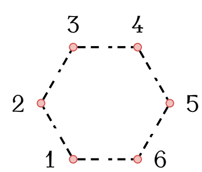
$$\rho(x \circ_i y) = \begin{cases} \rho(y) \circ_m \rho(x), & \text{if } i = 1, \\ \rho(x) \circ_{i-1} y, & \text{otherwise.} \end{cases} \tag{1.1.8c}$$

We call such a map ρ a *rotation map*.

1.2. **Configurations of chords.** Configurations of chords are very classical combinatorial objects defined as collections of diagonals and edges in regular polygons. The literature abounds of studies of various kinds of configurations. One can cite for instance [DLRS10] about triangulations, [FN99] about noncrossing configurations, and [CP92] about multi-triangulations. Combinatorial properties related with crossings and nestings in configurations of chords appear in [Jon05, CDD+07, RS10, SS12]. We provide here definitions about these objects and consider a generalization of configurations wherein the edges and diagonals are labeled by a set.

1.2.1. *Polygons.* A *polygon* of *size* $n \geqslant 1$ is a directed graph $\mathfrak{p}$ on the set of vertices $[n+1]$. An *arc* of $\mathfrak{p}$ is a pair of integers (x, y) with $1 \leqslant x < y \leqslant n+1$, a *diagonal* is an arc (x, y) different from $(x, x+1)$ and $(1, n+1)$, and an *edge* is an arc of the form $(x, x+1)$ and different from $(1, n+1)$. We denote by $\mathcal{A}_\mathfrak{p}$ (respectively $\mathcal{D}_\mathfrak{p}$, $\mathcal{E}_\mathfrak{p}$) the set of all arcs (respectively diagonals, edges) of $\mathfrak{p}$. For any $i \in [n]$, the *ith edge* of $\mathfrak{p}$ is the edge $(i, i+1)$, and the arc $(1, n+1)$ is the *base* of $\mathfrak{p}$.

In our graphical representations, each polygon is drawn so that its base is the bottommost segment, vertices are implicitly numbered from 1 to $n+1$ in clockwise direction, and the diagonals are not drawn. For example,

$\mathfrak{p} :=$ 

(1.2.1)

is a polygon of size 5. Its set of all diagonals is

$$\mathcal{D}_\mathfrak{p} = \{(1,3), (1,4), (1,5), (2,4), (2,5), (2,6), (3,5), (3,6), (4,6)\}, \tag{1.2.2}$$

its set of all edges is

$$\mathcal{E}_\mathfrak{p} = \{(1,2), (2,3), (3,4), (4,5), (5,6)\}, \tag{1.2.3}$$

and its set of all arcs is

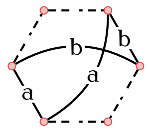
$$\mathcal{A}_\mathfrak{p} = \mathcal{D}_\mathfrak{p} \sqcup \mathcal{E}_\mathfrak{p} \sqcup \{(1,6)\}. \tag{1.2.4}$$

1.2.2. *Configurations.* For any set S, an *S-configuration* (or a *configuration* when S is known without ambiguity) is a polygon $\mathfrak{p}$ endowed with a partial function

$$\phi_\mathfrak{p} : \mathcal{A}_\mathfrak{p} \to S. \tag{1.2.5}$$

When $\phi_\mathfrak{p}((x,y))$ is defined, we say that the arc (x, y) is *labeled* and we write simply $\mathfrak{p}(x, y)$ instead of $\phi_\mathfrak{p}((x,y))$. If the base of $\mathfrak{p}$ is labeled, we write simply $\mathfrak{p}_0$ for $\mathfrak{p}(1, n+1)$, where n is the size of $\mathfrak{p}$. Finally, if the ith edge of $\mathfrak{p}$ is labeled, we write simply $\mathfrak{p}_i$ for $\mathfrak{p}(i, i+1)$.

In our graphical representations, we shall represent any S-configuration $\mathfrak{p}$ by drawing a polygon of the same size as the one of $\mathfrak{p}$ following the conventions explained before, and by labeling its arcs accordingly. For instance

$\mathfrak{p} :=$ 

(1.2.6)

is an $\{a, b\}$-configuration. The arcs $(1, 2)$ and $(1, 4)$ of $\mathfrak{p}$ are labeled by a, the arcs $(2, 5)$ and $(4, 5)$ are labeled by b, and the other arcs are unlabeled.

1.2.3. *Additional definitions.* Let us now provide some definitions and statistics on configurations. Let $\mathfrak{p}$ be a configuration of size n. The *skeleton* of $\mathfrak{p}$ is the undirected graph $\mathrm{skel}(\mathfrak{p})$ on the set of vertices $[n + 1]$ such that for any $x < y \in [n + 1]$, there is an arc $\{x, y\}$ in $\mathrm{skel}(\mathfrak{p})$ if (x, y) is labeled in $\mathfrak{p}$. The *degree* of a vertex x of $\mathfrak{p}$ is the number of vertices adjacent to x in $\mathrm{skel}(\mathfrak{p})$. The *degree* $\mathrm{degr}(\mathfrak{p})$ of $\mathfrak{p}$ is the maximal degree among its vertices. Two (non-necessarily labeled) diagonals (x, y) and (x', y') of $\mathfrak{p}$ are *crossing* if $x < x' < y < y'$ or $x' < x < y' < y$. The *crossing number* of a labeled diagonal (x, y) of $\mathfrak{p}$ is the number of labeled diagonals (x', y') such that (x, y) and (x', y') are crossing. The *crossing number* $\mathrm{cros}(\mathfrak{p})$ of $\mathfrak{p}$ is the maximal crossing among its labeled diagonals. If $\mathrm{cros}(\mathfrak{p}) = 0$, there are no crossing diagonals in $\mathfrak{p}$ and in this case, $\mathfrak{p}$ is *noncrossing*. A (non-necessarily labeled) arc (x', y') is *nested* in a (non-necessarily labeled) arc (x, y) of $\mathfrak{p}$ if $x \leqslant x' < y' \leqslant y$. We say that $\mathfrak{p}$ is *nesting-free* if for any labeled arcs (x, y) and (x', y') of $\mathfrak{p}$ such that (x', y') is nested in (x, y), $(x, y) = (x', y')$. Moreover, $\mathfrak{p}$ is *acyclic* if $\mathrm{skel}(\mathfrak{p})$ is acyclic, that is, there is no subset $\{x_1, \dots, x_k\}$ of $[n + 1]$ of cardinality $k \geqslant 3$ such that $\{x_i, x_{i+1}\}$ and $\{x_k, x_1\}$ are arcs in $\mathrm{skel}(\mathfrak{p})$ for all $i \in [k - 1]$. If $\mathfrak{p}$ has no labeled edges nor labeled base, $\mathfrak{p}$ is *white*. If $\mathfrak{p}$ has no labeled diagonals, $\mathfrak{p}$ is a *bubble*. A *triangle* is a configuration of size 2. Obviously, all triangles are bubbles, and all bubbles are noncrossing.

2. From unitary magmas to operads

We describe in this section our construction from unitary magmas to operads and study its main algebraic and combinatorial properties.

2.1. **Operads of decorated cliques.** We present here our main combinatorial objects, the decorated cliques. The construction C, which takes a unitary magma as input and produces an operad, is defined.

2.1.1. *Unitary magmas.* Recall first that a unitary magma is a set endowed with a binary operation $\star$ admitting a left and right unit $\mathbb{1}_{\mathcal{M}}$. For convenience, we denote by $\overline{\mathcal{M}}$ the set $\mathcal{M} \setminus \{\mathbb{1}_{\mathcal{M}}\}$. To explore some examples in this article, we shall mostly consider four sorts of unitary magmas: the additive unitary magma on all integers denoted by $\mathbb{Z}$, the cyclic additive unitary magma on $\mathbb{Z}/\ell\mathbb{Z}$ denoted by $\mathbb{N}_\ell$, the unitary magma

$$\mathbb{D}_\ell := \{\mathbb{1}, 0, \mathrm{d}_1, \dots, \mathrm{d}_\ell\}, \tag{2.1.1}$$

where $\mathbb{1}$ is the unit of $\mathbb{D}_\ell$, 0 is absorbing, and $\mathrm{d}_i \star \mathrm{d}_j = 0$ for all $i, j \in [\ell]$, and the unitary magma

$$\mathbb{E}_\ell := \{\mathbb{1}, \mathrm{e}_1, \dots, \mathrm{e}_\ell\}, \tag{2.1.2}$$

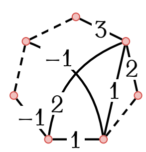
where $\mathbb{1}$ is the unit of $\mathbb{E}_\ell$ and $\mathrm{e}_i \star \mathrm{e}_j = \mathbb{1}$ for all $i, j \in [\ell]$. Observe that, since

$$\mathrm{e}_1 \star (\mathrm{e}_1 \star \mathrm{e}_2) = \mathrm{e}_1 \star \mathbb{1} = \mathrm{e}_1 \neq \mathrm{e}_2 = \mathbb{1} \star \mathrm{e}_2 = (\mathrm{e}_1 \star \mathrm{e}_1) \star \mathrm{e}_2, \tag{2.1.3}$$

all unitary magmas $\mathbb{E}_\ell$, $\ell \geqslant 2$, are not monoids.

2.1.2. *Decorated cliques.* An *$\mathcal{M}$-decorated clique* (or an *$\mathcal{M}$-clique* for short) is an $\mathcal{M}$-configuration $\mathfrak{p}$ such that all arcs of $\mathfrak{p}$ have labels. If the arc (x, y) of $\mathfrak{p}$ is labeled by an element different from $\mathbb{1}_{\mathcal{M}}$, we say that the arc (x, y) is *solid*. By convention, we require that the $\mathcal{M}$-clique of size 1 having its base labeled by $\mathbb{1}_{\mathcal{M}}$ is the only such object of size 1. The set of all $\mathcal{M}$-cliques is denoted by $\mathcal{C}_{\mathcal{M}}$.

In our graphical representations, we shall represent any $\mathcal{M}$-clique $\mathfrak{p}$ by following the drawing conventions of configurations explained in Section 1.2.2 with the difference that non-solid diagonals are not drawn. For instance,

$\mathfrak{p} :=$ 

(2.1.4)

is a $\mathbb{Z}$-clique such that, among others $\mathfrak{p}(1,2) = -1$, $\mathfrak{p}(1,5) = 2$, $\mathfrak{p}(3,7) = -1$, $\mathfrak{p}(5,7) = 1$, $\mathfrak{p}(2,3) = 0$ (because 0 is the unit of $\mathbb{Z}$), and $\mathfrak{p}(2,6) = 0$ (for the same reason).

Let us now provide some definitions and statistics on $\mathcal{M}$-cliques. The *underlying configuration* of $\mathfrak{p}$ is the $\overline{\mathcal{M}}$-configuration $\overline{\mathfrak{p}}$ of the same size as the one of $\mathfrak{p}$ and such that $\overline{\mathfrak{p}}(x, y) := \mathfrak{p}(x, y)$ for all solid arcs (x, y) of $\mathfrak{p}$, and all other arcs of $\overline{\mathfrak{p}}$ are unlabeled. The *skeleton*, (respectively *degree*, *crossing number*) of $\mathfrak{p}$ is the skeleton (respectively the degree, the crossing number) of $\overline{\mathfrak{p}}$. Moreover, $\mathfrak{p}$ is *nesting-free*, (respectively *acyclic*, *white*, an *$\mathcal{M}$-bubble*, an *$\mathcal{M}$-triangle*), if $\overline{\mathfrak{p}}$ is nesting-free (respectively acyclic, white, a bubble, a triangle). The set of all $\mathcal{M}$-bubbles (respectively $\mathcal{M}$-triangles) is denoted by $\mathcal{B}_{\mathcal{M}}$ (respectively $\mathcal{T}_{\mathcal{M}}$).

2.1.3. *Partial composition of $\mathcal{M}$-cliques.* From now, the *arity* of an $\mathcal{M}$-clique $\mathfrak{p}$ is its size and is denoted by $|\mathfrak{p}|$. For any unitary magma $\mathcal{M}$, we define the vector space

$$\mathrm{C}\mathcal{M} := \bigoplus_{n \geqslant 1} \mathrm{C}\mathcal{M}(n) = \mathbb{K}\left\langle \mathcal{C}_{\mathcal{M}} \right\rangle, \tag{2.1.5}$$

where $\mathrm{C}\mathcal{M}(n)$ is the linear span of all $\mathcal{M}$-cliques of arity n, $n \geqslant 1$. The set $\mathcal{C}_{\mathcal{M}}$ forms hence a basis of $\mathrm{C}\mathcal{M}$ called *fundamental basis*. Observe that the space

$\mathrm{C}\mathcal{M}(1)$ has dimension 1 since it is the linear span of the $\mathcal{M}$-clique. We endow $\mathrm{C}\mathcal{M}$ with partial composition maps

$$\circ_i : \mathrm{C}\mathcal{M}(n) \otimes \mathrm{C}\mathcal{M}(m) \to \mathrm{C}\mathcal{M}(n+m-1), \qquad n, m \geqslant 1, i \in [n], \tag{2.1.6}$$

defined linearly, in the fundamental basis, in the following way. Let $\mathfrak{p}$ and $\mathfrak{q}$ be two $\mathcal{M}$-cliques of respective arities n and m, and $i \in [n]$ be an integer. We set $\mathfrak{p} \circ_i \mathfrak{q}$ as the $\mathcal{M}$-clique of arity $n+m-1$ such that, for any arc (x, y) where $1 \leqslant x < y \leqslant n+m$,

$$(\mathfrak{p} \circ_i \mathfrak{q})(x, y) := \begin{cases} \mathfrak{p}(x, y), & \text{if } y \leqslant i, \\ \mathfrak{p}(x, y-m+1), & \text{if } x \leqslant i < i+m \leqslant y \\ & \qquad \text{and } (x, y) \neq (i, i+m), \\ \mathfrak{p}(x-m+1, y-m+1), & \text{if } i+m \leqslant x, \\ \mathfrak{q}(x-i+1, y-i+1), & \text{if } i \leqslant x < y \leqslant i+m \\ & \qquad \text{and } (x, y) \neq (i, i+m), \\ \mathfrak{p}_i \star \mathfrak{q}_0, & \text{if } (x, y) = (i, i+m), \\ \mathbb{1}_{\mathcal{M}}, & \text{otherwise.} \end{cases} \tag{2.1.7}$$

We recall that $\star$ denotes the operation of $\mathcal{M}$ and $\mathbb{1}_{\mathcal{M}}$ its unit. Graphically, $\mathfrak{p} \circ_i \mathfrak{q}$ is obtained by gluing the base of $\mathfrak{q}$ onto the ith edge of $\mathfrak{p}$ and by labeling this arc by $\mathfrak{p}_i \star \mathfrak{q}_0$, and by adding all required non solid diagonals on the graph thus obtained to become a clique (see Figure 1). For example, in $\mathrm{C}\mathbb{Z}$, we have the two partial

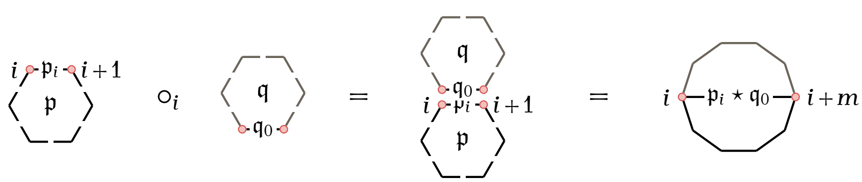
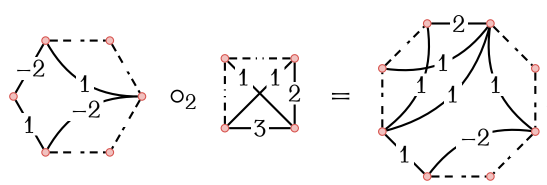
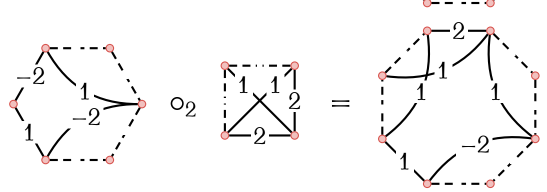


FIGURE 1. The partial composition of $\mathrm{C}\mathcal{M}$, described in graphical terms. Here, $\mathfrak{p}$ and $\mathfrak{q}$ are two $\mathcal{M}$-cliques. The arity of $\mathfrak{q}$ is m and i is an integer between 1 and $|\mathfrak{p}|$.

compositions



(2.1.8a)



(2.1.8b)

2.1.4. *Functorial construction from unitary magmas to operads.* If $\mathscr{M}_1$ and $\mathscr{M}_2$ are two unitary magmas and $\theta : \mathscr{M}_1 \to \mathscr{M}_2$ is a unitary magma morphism, we define

$$\mathrm{C}\theta : \mathrm{C}\mathscr{M}_1 \to \mathrm{C}\mathscr{M}_2 \tag{2.1.9}$$

as the linear map sending any $\mathscr{M}_1$-clique $\mathfrak{p}$ of arity n to the $\mathscr{M}_2$-clique $(\mathrm{C}\theta)(\mathfrak{p})$ of the same arity such that, for any arc (x, y) where $1 \leqslant x < y \leqslant n+1$,

$$((\mathrm{C}\theta)(\mathfrak{p}))(x, y) := \theta(\mathfrak{p}(x, y)). \tag{2.1.10}$$

Graphically, $(\mathrm{C}\theta)(\mathfrak{p})$ is the $\mathscr{M}_2$-clique obtained by relabeling each arc of $\mathfrak{p}$ by the image of its label by θ.

Theorem 2.1.1. *The construction* C *is a functor from the category of unitary magmas to the category of operads. Moreover,* C *respects injections and surjections.*

Proof. Let $\mathscr{M}$ be a unitary magma. The fact that $\mathrm{C}\mathscr{M}$ endowed with the partial composition (2.1.7) is an operad can be established by showing that the two associativity relations (1.1.3a) and (1.1.3b) of operads are satisfied. This is a technical but a simple verification. Since $\mathrm{C}\mathscr{M}(1)$ contains and this element is the unit for this partial composition, (1.1.3c) holds. Moreover, let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two unitary magmas and $\theta : \mathscr{M}_1 \to \mathscr{M}_2$ be a unitary magma morphism. The fact that the map $\mathrm{C}\theta$ defined in (2.1.10) is an operad morphism is straightforward to check. All this implies that C is a functor. Finally, the fact that C respects injections and surjections is also straightforward to verify. □

We call the construction C the *clique construction*, and $\mathrm{C}\mathscr{M}$ the *$\mathscr{M}$-clique operad*. Observe that the fundamental basis of $\mathrm{C}\mathscr{M}$ is a set-operad basis of $\mathrm{C}\mathscr{M}$. Besides, if $\mathscr{M}$ is the trivial unitary magma $\{\mathbb{1}_{\mathscr{M}}\}$, $\mathrm{C}\mathscr{M}$ is the linear span of all decorated cliques having only non-solid arcs. Thus, each space $\mathrm{C}\mathscr{M}(n)$, $n \geqslant 1$, is of dimension 1 and it follows from the definition of the partial composition of $\mathrm{C}\mathscr{M}$ that this operad is isomorphic to the associative operad As. The next result shows that the clique construction is compatible with the Cartesian product of unitary magmas.

Proposition 2.1.2. *Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two unitary magmas. Then* $\mathrm{C}(\mathscr{M}_1 \times \mathscr{M}_2)$ *is isomorphic to the Hadamard product of operads* $(\mathrm{C}\mathscr{M}_1) * (\mathrm{C}\mathscr{M}_2)$.

Proof. Let $\phi : (\mathrm{C}\mathscr{M}_1) * (\mathrm{C}\mathscr{M}_2) \to \mathrm{C}(\mathscr{M}_1 \times \mathscr{M}_2)$ be the linear map defined as follows. For any $\mathscr{M}_1$-clique $\mathfrak{p}$ of $\mathrm{C}\mathscr{M}_1$ and any $\mathscr{M}_2$-clique $\mathfrak{q}$ of $\mathrm{C}\mathscr{M}_2$ both of arity n, $\phi(\mathfrak{p} \otimes \mathfrak{q})$ is the $\mathscr{M}_1 \times \mathscr{M}_2$-clique defined, for any $1 \leqslant x < y \leqslant n+1$, by

$$(\phi(\mathfrak{p} \otimes \mathfrak{q}))(x, y) := (\mathfrak{p}(x, y), \mathfrak{q}(x, y)). \tag{2.1.11}$$

Let the linear map $\psi : \mathrm{C}(\mathscr{M}_1 \times \mathscr{M}_2) \to (\mathrm{C}\mathscr{M}_1) * (\mathrm{C}\mathscr{M}_2)$ defined, for any $\mathscr{M}_1 \times \mathscr{M}_2$-clique $\mathfrak{r}$ of $\mathrm{C}(\mathscr{M}_1 \times \mathscr{M}_2)$ of arity n, as follows. The $\mathscr{M}_1$-clique $\mathfrak{p}$ and the $\mathscr{M}_2$-clique

$\mathfrak{q}$ of arity n of the tensor $\mathfrak{p} \otimes \mathfrak{q} := \psi(\mathfrak{r})$ are defined, for any $1 \leqslant x < y \leqslant n+1$, by $\mathfrak{p}(x, y) := a$ and $\mathfrak{q}(x, y) := b$, where $(a, b) = \mathfrak{r}(x, y)$. Since we observe immediately that ψ is the inverse of ϕ, ϕ is a bijection. Moreover, it follows from the definition of the partial composition of clique operads that ϕ is an operad morphism. The statement of the proposition follows. □

2.2. **General properties.** We investigate here some properties of clique operads, as their dimensions, their minimal generating sets, the fact that they admit a cyclic operad structure, and describe their partial compositions over two alternative bases.

2.2.1. *Binary relations.* Let us start by remarking that, depending on the cardinality m of $\mathcal{M}$, the set of all $\mathcal{M}$-cliques can be interpreted as particular binary relations. If $m \geqslant 4$, let us set $\mathcal{M} = \{\mathbb{1}_{\mathcal{M}}, \mathtt{a}, \mathtt{b}, \mathtt{c}, \dots\}$ so that $\mathtt{a}$, $\mathtt{b}$, and $\mathtt{c}$ are distinguished pairwise distinct elements of $\mathcal{M}$ different from $\mathbb{1}_{\mathcal{M}}$. Given an $\mathcal{M}$-clique $\mathfrak{p}$ of arity $n \geqslant 2$, we build a binary relation $\mathfrak{R}$ on $[n+1]$ satisfying, for all $x < y \in [n+1]$,

$$\begin{aligned} x \,\mathfrak{R}\, y, &\quad \text{if } \mathfrak{p}(x, y) = \mathtt{a}, \\ y \,\mathfrak{R}\, x, &\quad \text{if } \mathfrak{p}(x, y) = \mathtt{b}, \\ x \,\mathfrak{R}\, y \text{ and } y \,\mathfrak{R}\, x, &\quad \text{if } \mathfrak{p}(x, y) = \mathtt{c}. \end{aligned} \tag{2.2.1}$$

In particular, if $m = 2$ (respectively $m = 3$, $m = 4$), $\mathcal{M} = \{\mathbb{1}, \mathtt{c}\}$ (respectively $\mathcal{M} = \{\mathbb{1}, \mathtt{a}, \mathtt{b}\}$, $\mathcal{M} = \{\mathbb{1}, \mathtt{a}, \mathtt{b}, \mathtt{c}\}$) and the set of all $\mathcal{M}$-cliques of arities $n \geqslant 2$ is in one-to-one correspondence with the set of all irreflexive and symmetric (respectively irreflexive and antisymmetric, irreflexive) binary relations on $[n+1]$. Therefore, the operads $\mathrm{C}\mathcal{M}$ can be interpreted as operads involving binary relations with more or less properties.

2.2.2. *Dimensions and minimal generating set.*

Proposition 2.2.1. *Let $\mathcal{M}$ be a finite unitary magma. For all $n \geqslant 2$,*

$$\dim \mathrm{C}\mathcal{M}(n) = m^{\binom{n+1}{2}}, \tag{2.2.2}$$

where $m := \#\mathcal{M}$.

Proof. By definition of the clique construction and of $\mathcal{M}$-cliques, the dimension of $\mathrm{C}\mathcal{M}(n)$ is the number of maps from the set $\left\{(x, y) \in [n+1]^2 : x < y\right\}$ to $\mathcal{M}$. Therefore, if $n \geqslant 2$, this implies (2.2.2). □

From Proposition 2.2.1, the first dimensions of $\mathrm{C}\mathcal{M}$ depending on $m := \#\mathcal{M}$ are

$$1, 1, 1, 1, 1, 1, 1, 1, \qquad m = 1, \tag{2.2.3a}$$

$$1, 8, 64, 1024, 32768, 2097152, 268435456, 68719476736, \qquad m = 2, \tag{2.2.3b}$$

$$1, 27, 729, 59049, 14348907, 10460353203, 22876792454961,$$
$$150094635296999121, \qquad m = 3, \tag{2.2.3c}$$

$$1, 64, 4096, 1048576, 1073741824, 4398046511104, 72057594037927936,$$
$$4722366482869645213696, \qquad m = 4. \tag{2.2.3d}$$

Except for the first terms, the second one forms Sequence A006125, the third one forms Sequence A047656, and the last one forms Sequence A053763 of [Slo].

Lemma 2.2.2. *Let $\mathcal{M}$ be a unitary magma, $\mathfrak{p}$ be an $\mathcal{M}$-clique of arity $n \geqslant 2$, and (x, y) be a diagonal of $\mathfrak{p}$. Then the following two assertions are equivalent:*
(i) the diagonal (x, y) is solid and its crossing number is 0, or (x, y) is not solid;
(ii) the $\mathcal{M}$-clique $\mathfrak{p}$ can be written as $\mathfrak{p} = \mathfrak{q} \circ_x \mathfrak{r}$, where $\mathfrak{q}$ is an $\mathcal{M}$-clique of arity $n + x - y + 1$ and $\mathfrak{r}$ is an $\mathcal{M}$-clique of arity $y - x$.

Proof. Assume first that *(i)* holds. Set $\mathfrak{q}$ as the $\mathcal{M}$-clique of arity $n + x - y + 1$ defined, for any arc (z, t) where $1 \leqslant z < t \leqslant n + x - y + 2$, by

$$\mathfrak{q}(z, t) := \begin{cases} \mathfrak{p}(z, t), & \text{if } t \leqslant x, \\ \mathfrak{p}(z, t + y - x - 1), & \text{if } x + 1 \leqslant t, \\ \mathfrak{p}(z + y - x - 1, t + y - x - 1), & \text{otherwise,} \end{cases} \tag{2.2.4}$$

and $\mathfrak{r}$ as the $\mathcal{M}$-clique of arity $y - x$ defined, for any arc (z, t) where $1 \leqslant z < t \leqslant y - x + 1$, by

$$\mathfrak{r}(z, t) := \begin{cases} \mathfrak{p}(z + x - 1, t + x - 1), & \text{if } (z, t) \neq (1, y - x + 1), \\ \mathbb{1}_{\mathcal{M}}, & \text{otherwise.} \end{cases} \tag{2.2.5}$$

By following the definition of the partial composition of $\mathrm{C}\mathcal{M}$, one obtains $\mathfrak{p} = \mathfrak{q} \circ_x \mathfrak{r}$, hence *(ii)* holds.

Assume conversely that *(ii)* holds. By definition of the partial composition of $\mathrm{C}\mathcal{M}$, the fact that $\mathfrak{p} = \mathfrak{q} \circ_x \mathfrak{r}$ implies that $\mathfrak{p}(x', y') = \mathbb{1}_{\mathcal{M}}$ for any arc (x', y') such that (x, y) and (x', y') are crossing. Therefore, *(i)* holds. □

Let $\mathcal{P}_{\mathcal{M}}$ be the set of all $\mathcal{M}$-cliques $\mathfrak{p}$ of arity $n \geqslant 2$ that do not satisfy the property of the statement of Lemma 2.2.2. In other words, $\mathcal{P}_{\mathcal{M}}$ is the set of all $\mathcal{M}$-cliques such that, for any (non-necessarily solid) diagonal (x, y) of $\mathfrak{p}$, there is at least one solid diagonal (x', y') of $\mathfrak{p}$ such that (x, y) and (x', y') are crossing. We call $\mathcal{P}_{\mathcal{M}}$ the set of all *prime $\mathcal{M}$-cliques*. Observe that, according to this description, all $\mathcal{M}$-triangles are prime.

Proposition 2.2.3. *Let $\mathcal{M}$ be a unitary magma. The set $\mathcal{P}_{\mathcal{M}}$ is a minimal generating set of $\mathrm{C}\mathcal{M}$.*

Proof. We show by induction on the arity that $\mathscr{P}_{\mathcal{M}}$ is a generating set of $\mathsf{C}\mathcal{M}$. Let $\mathfrak{p}$ be an $\mathcal{M}$-clique. If $\mathfrak{p}$ is of arity 1, $\mathfrak{p} = \circ\text{-}\text{-}\circ$ and hence $\mathfrak{p}$ trivially belongs to $(\mathsf{C}\mathcal{M})^{\mathscr{P}_{\mathcal{M}}}$ (recall that this notation stands for the suboperad of $\mathsf{C}\mathcal{M}$ generated by $\mathscr{P}_{\mathcal{M}}$). Let us assume that $\mathfrak{p}$ is of arity $n \geqslant 2$. First, if $\mathfrak{p} \in \mathscr{P}_{\mathcal{M}}$, then $\mathfrak{p} \in (\mathsf{C}\mathcal{M})^{\mathscr{P}_{\mathcal{M}}}$. Otherwise, $\mathfrak{p}$ is an $\mathcal{M}$-clique which satisfies the description of the statement of Lemma 2.2.2. Therefore, by this lemma, there are two $\mathcal{M}$-cliques $\mathfrak{q}$ and $\mathfrak{r}$ and an integer $x \in [|\mathfrak{p}|]$ such that $|\mathfrak{q}| < |\mathfrak{p}|$, $|\mathfrak{r}| < |\mathfrak{p}|$, and $\mathfrak{p} = \mathfrak{q} \circ_x \mathfrak{r}$. By induction hypothesis, $\mathfrak{q}$ and $\mathfrak{r}$ belong to $(\mathsf{C}\mathcal{M})^{\mathscr{P}_{\mathcal{M}}}$ and hence, $\mathfrak{p}$ also belongs to $(\mathsf{C}\mathcal{M})^{\mathscr{P}_{\mathcal{M}}}$.

Finally, by Lemma 2.2.2, if $\mathfrak{p}$ is a prime $\mathcal{M}$-clique, $\mathfrak{p}$ cannot be expressed as a partial composition of prime $\mathcal{M}$-cliques. Moreover, since the space $\mathsf{C}\mathcal{M}(1)$ is trivial, these arguments imply that $\mathscr{P}_{\mathcal{M}}$ is a minimal generating set of $\mathsf{C}\mathcal{M}$. □

Computer experiments tell us that, if $m := \#\mathcal{M} = 2$, the first numbers of prime $\mathcal{M}$-cliques are, size by size,

$$0, 8, 16, 352, 16448, 1380224. \tag{2.2.6}$$

Moreover, observe that the nth term of this sequence is divisible by m^{n+1} since the labels of the base and the edges of an $\mathcal{M}$-clique $\mathfrak{p}$ have no influence on the fact that $\mathfrak{p}$ is prime. This gives the sequence

$$0, 1, 1, 11, 257, 10783, \tag{2.2.7}$$

enumerating the first of them size by size. Besides, a prime $\mathcal{M}$-clique $\mathfrak{p}$ is *minimal* if any $\mathcal{M}$-clique obtained from $\mathfrak{p}$ by replacing a solid arc by a non-solid one is not prime. Of course, all minimal prime $\mathcal{M}$-cliques are white. Computer experiments show us that, if $m := \#\mathcal{M} = 2$, the numbers of minimal prime $\mathcal{M}$-cliques begin by

$$0, 1, 1, 5, 22, 119. \tag{2.2.8}$$

None of these sequences appear in [Slo] at this time.

2.2.3. *Associative elements.*

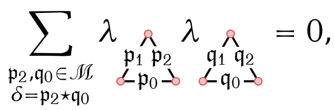
Proposition 2.2.4. *Let $\mathcal{M}$ be a unitary magma and f be an element of $\mathsf{C}\mathcal{M}(2)$ of the form*

$$f := \sum_{\mathfrak{p} \in \mathcal{T}_{\mathcal{M}}} \lambda_{\mathfrak{p}} \mathfrak{p}, \tag{2.2.9}$$

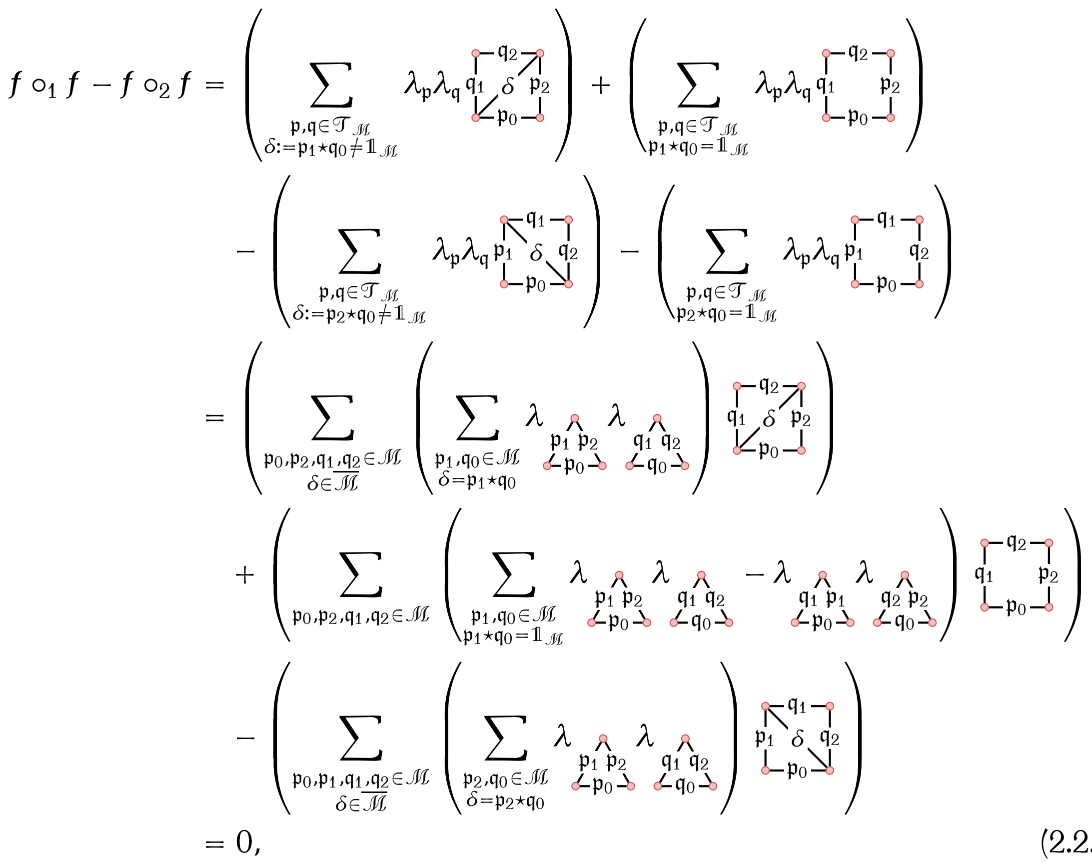
where the $\lambda_{\mathfrak{p}}$, $\mathfrak{p} \in \mathcal{T}_{\mathcal{M}}$, are coefficients of $\mathbb{K}$. Then f is associative if and only if

$$\sum_{\substack{\mathfrak{p}_1, \mathfrak{q}_0 \in \mathcal{M} \\ \delta = \mathfrak{p}_1 \star \mathfrak{q}_0}} \lambda_{\substack{\mathfrak{p}_1\ \mathfrak{p}_2 \\ \mathfrak{p}_0}} \lambda_{\substack{\mathfrak{q}_1\ \mathfrak{q}_2 \\ \mathfrak{q}_0}} = 0, \qquad \mathfrak{p}_0, \mathfrak{p}_2, \mathfrak{q}_1, \mathfrak{q}_2 \in \mathcal{M}, \delta \in \overline{\mathcal{M}}, \tag{2.2.10a}$$

$$\sum_{\substack{\mathfrak{p}_1, \mathfrak{q}_0 \in \mathcal{M} \\ \mathfrak{p}_1 \star \mathfrak{q}_0 = \mathbb{1}_{\mathcal{M}}}} \lambda_{\substack{\mathfrak{p}_1\ \mathfrak{p}_2 \\ \mathfrak{p}_0}} \lambda_{\substack{\mathfrak{q}_1\ \mathfrak{q}_2 \\ \mathfrak{q}_0}} - \lambda_{\substack{\mathfrak{q}_1\ \mathfrak{p}_1 \\ \mathfrak{p}_0}} \lambda_{\substack{\mathfrak{q}_2\ \mathfrak{p}_2 \\ \mathfrak{q}_0}} = 0, \qquad \mathfrak{p}_0, \mathfrak{p}_2, \mathfrak{q}_1, \mathfrak{q}_2 \in \mathcal{M}, \tag{2.2.10b}$$

 $\qquad \mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{q}_1, \mathfrak{q}_2 \in \mathcal{M}, \delta \in \overline{\mathcal{M}}.$ (2.2.10c)

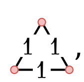
Proof. The element f defined in (2.2.9) is associative if and only if $f \circ_1 f - f \circ_2 f = 0$. Therefore, this property is equivalent to the fact that

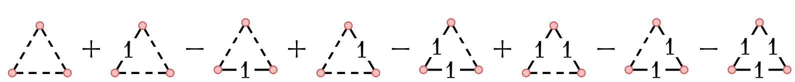


(2.2.11)

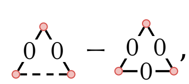
and hence, is equivalent to the fact that (2.2.10a), (2.2.10b), and (2.2.10c) hold. □

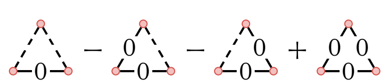
For instance, by Proposition 2.2.4, the binary elements

 (2.2.12a)

 (2.2.12b)

of $\mathrm{C}\mathbb{N}_2$ are associative, and the binary elements

 (2.2.13a)

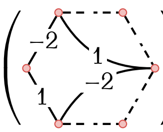
 (2.2.13b)

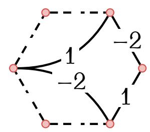
of $\mathrm{C}\mathbb{D}_0$ are associative.

2.2.4. *Symmetries.* Let $\mathrm{ref} : \mathrm{C}\mathcal{M} \to \mathrm{C}\mathcal{M}$ be the linear map sending any $\mathcal{M}$-clique $\mathfrak{p}$ of arity n to the $\mathcal{M}$-clique $\mathrm{ref}(\mathfrak{p})$ of the same arity such that, for any arc (x, y) where $1 \leqslant x < y \leqslant n + 1$,

$$(\mathrm{ref}(\mathfrak{p}))\,(x, y) := \mathfrak{p}(n - y + 2, n - x + 2). \tag{2.2.14}$$

Graphically, $\mathrm{ref}(\mathfrak{p})$ is the $\mathcal{M}$-clique obtained by applying on $\mathfrak{p}$ a reflection trough the vertical line passing by its base. For instance, in $\mathrm{C}\mathbb{Z}$ we have

ref (

) = 

. (2.2.15)

Proposition 2.2.5. *Let $\mathcal{M}$ be a unitary magma. Then the group of symmetries of $\mathrm{C}\mathcal{M}$ contains the map ref and all the maps $\mathrm{C}\theta$ where θ is a unitary magma automorphisms of $\mathcal{M}$.*

Proof. When θ is a unitary magma automorphism of $\mathcal{M}$, since by Theorem 2.1.1 C is a functor respecting bijections, $\mathrm{C}\theta$ is an operad automorphism of $\mathrm{C}\mathcal{M}$. Hence, $\mathrm{C}\theta$ belongs to the group of symmetries of $\mathrm{C}\mathcal{M}$. Moreover, the fact that ref belongs to the group of symmetries of $\mathrm{C}\mathcal{M}$ can be established by showing that this map is an antiautomorphism of $\mathrm{C}\mathcal{M}$, directly from the definition of the partial composition of $\mathrm{C}\mathcal{M}$ and that of ref. □

2.2.5. *Basic set-operad basis.* A unitary magma $\mathcal{M}$ is *right cancelable* if for any $x, y, z \in \mathcal{M}$, $y \star x = z \star x$ implies $y = z$.

Proposition 2.2.6. *Let $\mathcal{M}$ be a unitary magma. The fundamental basis of $\mathrm{C}\mathcal{M}$ is a basic set-operad basis if and only if $\mathcal{M}$ is right cancelable.*

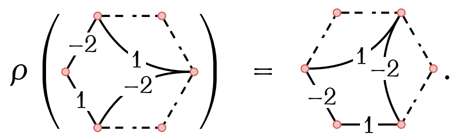
Proof. Assume first that $\mathcal{M}$ is right cancelable. Let $n \geqslant 1$, $i \in [n]$, and $\mathfrak{p}$, $\mathfrak{p}'$, and $\mathfrak{q}$ be three $\mathcal{M}$-cliques such that $\mathfrak{p}$ and $\mathfrak{p}'$ are of arity n. If $\circ_i^{\mathfrak{q}}(\mathfrak{p}) = \circ_i^{\mathfrak{q}}(\mathfrak{p}')$, we have $\mathfrak{p} \circ_i \mathfrak{q} = \mathfrak{p}' \circ_i \mathfrak{q}$. By definition of the partial composition map of $\mathrm{C}\mathcal{M}$, we have $\mathfrak{p}(x, y) = \mathfrak{p}'(x, y)$ for all arcs (x, y) where $1 \leqslant x < y \leqslant n + 1$ and $(x, y) \neq (i, i + 1)$. Moreover, we have $\mathfrak{p}_i \star \mathfrak{q}_0 = \mathfrak{p}'_i \star \mathfrak{q}_0$. Since $\mathcal{M}$ is right cancelable, this implies that $\mathfrak{p}_i = \mathfrak{p}'_i$, and hence, $\mathfrak{p} = \mathfrak{p}'$. This shows that the maps $\circ_i^{\mathfrak{q}}$ are injective and thus, that the fundamental basis of $\mathrm{C}\mathcal{M}$ is a basic set-operad basis.

Conversely, assume that the fundamental basis of $\mathrm{C}\mathcal{M}$ is a basic set-operad basis. Then, in particular, for all $n \geqslant 1$ and all $\mathcal{M}$-cliques $\mathfrak{p}$, $\mathfrak{p}'$, and $\mathfrak{q}$ such that $\mathfrak{p}$ and $\mathfrak{p}'$ are of arity n, $\circ_1^{\mathfrak{q}}(\mathfrak{p}) = \circ_1^{\mathfrak{q}}(\mathfrak{p}')$ implies $\mathfrak{p} = \mathfrak{p}'$. This is equivalent to the statement that $\mathfrak{p}_1 \star \mathfrak{q}_0 = \mathfrak{p}'_1 \star \mathfrak{q}_0$ implies $\mathfrak{p}_1 = \mathfrak{p}'_1$. This amount exactly to the statement that $\mathcal{M}$ is right cancelable. □

2.2.6. *Cyclic operad structure.* Let $\rho : \mathrm{C}\mathcal{M} \to \mathrm{C}\mathcal{M}$ be the linear map sending any $\mathcal{M}$-clique $\mathfrak{p}$ of arity n to the $\mathcal{M}$-clique $\rho(\mathfrak{p})$ of the same arity such that, for any arc (x, y) where $1 \leqslant x < y \leqslant n+1$,

$$(\rho(\mathfrak{p}))(x,y) := \begin{cases} \mathfrak{p}(x+1, y+1), & \text{if } y \leqslant n, \\ \mathfrak{p}(1, x+1), & \text{otherwise } (y = n+1). \end{cases} \tag{2.2.16}$$

Graphically, $\rho(\mathfrak{p})$ is the $\mathcal{M}$-clique obtained by applying a rotation of one step of $\mathfrak{p}$ in counterclockwise direction. For instance, in $\mathrm{C}\mathbb{Z}$ we have



(2.2.17)

Proposition 2.2.7. *Let $\mathcal{M}$ be a unitary magma. The map ρ is a rotation map of $\mathrm{C}\mathcal{M}$, endowing this operad with a cyclic operad structure.*

Proof. The fact that ρ is a rotation map for $\mathrm{C}\mathcal{M}$ follows from a technical but straightforward verification of the fact that Relations (1.1.8a), (1.1.8b), and (1.1.8c) hold. □

2.2.7. *Alternative bases.* If $\mathfrak{p}$ and $\mathfrak{q}$ are two $\mathcal{M}$-cliques of the same arity, the *Hamming distance* $\mathrm{h}(\mathfrak{p}, \mathfrak{q})$ between $\mathfrak{p}$ and $\mathfrak{q}$ is the number of arcs (x, y) such that $\mathfrak{p}(x,y) \neq \mathfrak{q}(x,y)$. Let $\preceq_{\mathrm{be}}$ be the partial order relation on the set of all $\mathcal{M}$-cliques, where, for any $\mathcal{M}$-cliques $\mathfrak{p}$ and $\mathfrak{q}$, one has $\mathfrak{p} \preceq_{\mathrm{be}} \mathfrak{q}$ if $\mathfrak{q}$ can be obtained from $\mathfrak{p}$ by replacing some labels $\mathbb{1}_{\mathcal{M}}$ of its edges or its base by other labels of $\mathcal{M}$. In the same way, let $\preceq_{\mathrm{d}}$ be the partial order on the same set, where $\mathfrak{p} \preceq_{\mathrm{d}} \mathfrak{q}$ if $\mathfrak{q}$ can be obtained from $\mathfrak{p}$ by replacing some labels $\mathbb{1}_{\mathcal{M}}$ of its diagonals by other labels of $\mathcal{M}$.

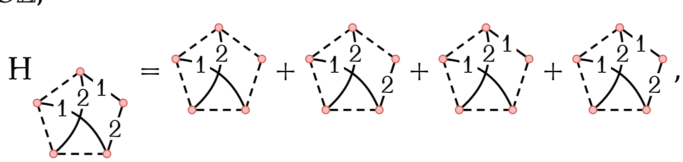
For all $\mathcal{M}$-cliques $\mathfrak{p}$, let us introduce the elements of $\mathrm{C}\mathcal{M}$ defined by

$$\mathrm{H}_{\mathfrak{p}} := \sum_{\substack{\mathfrak{p}' \in \mathcal{C}_{\mathcal{M}} \\ \mathfrak{p}' \preceq_{\mathrm{be}} \mathfrak{p}}} \mathfrak{p}', \tag{2.2.18a}$$

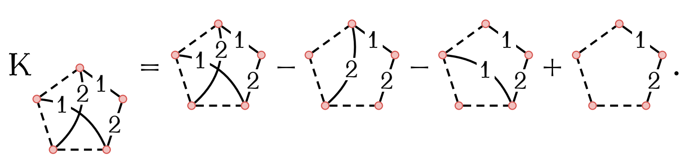
and

$$\mathrm{K}_{\mathfrak{p}} := \sum_{\substack{\mathfrak{p}' \in \mathcal{C}_{\mathcal{M}} \\ \mathfrak{p}' \preceq_{\mathrm{d}} \mathfrak{p}}} (-1)^{\mathrm{h}(\mathfrak{p}', \mathfrak{p})} \mathfrak{p}'. \tag{2.2.18b}$$

For instance, in $\mathrm{C}\mathbb{Z}$,



(2.2.19a)



(2.2.19b)

Since, by Möbius inversion, for any $\mathcal{M}$-clique $\mathfrak{p}$ we have

$$\sum_{\substack{\mathfrak{p}' \in \mathcal{C}_{\mathcal{M}} \\ \mathfrak{p}' \preceq_{\mathrm{be}} \mathfrak{p}}} (-1)^{\mathrm{h}(\mathfrak{p}', \mathfrak{p})} \mathrm{H}_{\mathfrak{p}'} = \mathfrak{p} = \sum_{\substack{\mathfrak{p}' \in \mathcal{C}_{\mathcal{M}} \\ \mathfrak{p}' \preceq_{\mathrm{d}} \mathfrak{p}}} \mathrm{K}_{\mathfrak{p}'} \tag{2.2.20}$$

by triangularity, the family of all the $\mathrm{H}_{\mathfrak{p}}$ (respectively $\mathrm{K}_{\mathfrak{p}}$) forms a basis of $\mathrm{C}\mathcal{M}$ called the *H-basis* (respectively the *K-basis*).

If $\mathfrak{p}$ is an $\mathcal{M}$-clique, $\mathrm{d}_0(\mathfrak{p})$ (respectively $\mathrm{d}_i(\mathfrak{p})$) is the $\mathcal{M}$-clique obtained by replacing the label of the base (respectively ith edge) of $\mathfrak{p}$ by $\mathbb{1}_{\mathcal{M}}$.

Proposition 2.2.8. *Let $\mathcal{M}$ be a unitary magma. The partial composition of $\mathrm{C}\mathcal{M}$ can be expressed in terms of the* H*-basis, for any $\mathcal{M}$-cliques $\mathfrak{p}$ and $\mathfrak{q}$ different from $\circ\!-\!-\!\circ$ and any valid integer i, by*

$$\mathrm{H}_{\mathfrak{p}} \circ_i \mathrm{H}_{\mathfrak{q}} = \begin{cases} \mathrm{H}_{\mathfrak{p} \circ_i \mathfrak{q}} + \mathrm{H}_{\mathrm{d}_i(\mathfrak{p}) \circ_i \mathfrak{q}} + \mathrm{H}_{\mathfrak{p} \circ_i \mathrm{d}_0(\mathfrak{q})} + \mathrm{H}_{\mathrm{d}_i(\mathfrak{p}) \circ_i \mathrm{d}_0(\mathfrak{q})}, & \text{if } \mathfrak{p}_i \neq \mathbb{1}_{\mathcal{M}} \text{ and } \mathfrak{q}_0 \neq \mathbb{1}_{\mathcal{M}}, \\ \mathrm{H}_{\mathfrak{p} \circ_i \mathfrak{q}} + \mathrm{H}_{\mathrm{d}_i(\mathfrak{p}) \circ_i \mathfrak{q}}, & \text{if } \mathfrak{p}_i \neq \mathbb{1}_{\mathcal{M}} \text{ and } \mathfrak{q}_0 = \mathbb{1}_{\mathcal{M}}, \\ \mathrm{H}_{\mathfrak{p} \circ_i \mathfrak{q}} + \mathrm{H}_{\mathfrak{p} \circ_i \mathrm{d}_0(\mathfrak{q})}, & \text{if } \mathfrak{p}_i = \mathbb{1}_{\mathcal{M}} \text{ and } \mathfrak{q}_0 \neq \mathbb{1}_{\mathcal{M}}, \\ \mathrm{H}_{\mathfrak{p} \circ_i \mathfrak{q}}, & \text{otherwise.} \end{cases} \tag{2.2.21}$$

Proof. From the definition of the H-basis, we have

$$\begin{aligned} \mathrm{H}_{\mathfrak{p}} \circ_i \mathrm{H}_{\mathfrak{q}} &= \sum_{\substack{\mathfrak{p}', \mathfrak{q}' \in \mathcal{C}_{\mathcal{M}} \\ \mathfrak{p}' \preceq_{\mathrm{be}} \mathfrak{p} \\ \mathfrak{q}' \preceq_{\mathrm{be}} \mathfrak{q}}} \mathfrak{p}' \circ_i \mathfrak{q}' \\ &= \sum_{\substack{\mathfrak{p}', \mathfrak{q}' \in \mathcal{C}_{\mathcal{M}} \\ \mathfrak{p}' \preceq_{\mathrm{be}} \mathfrak{p} \\ \mathfrak{q}' \preceq_{\mathrm{be}} \mathfrak{q} \\ \mathfrak{p}'_i \neq \mathbb{1}_{\mathcal{M}} \\ \mathfrak{q}'_0 \neq \mathbb{1}_{\mathcal{M}}}} \mathfrak{p}' \circ_i \mathfrak{q}' + \sum_{\substack{\mathfrak{p}', \mathfrak{q}' \in \mathcal{C}_{\mathcal{M}} \\ \mathfrak{p}' \preceq_{\mathrm{be}} \mathfrak{p} \\ \mathfrak{q}' \preceq_{\mathrm{be}} \mathfrak{q} \\ \mathfrak{p}'_i \neq \mathbb{1}_{\mathcal{M}} \\ \mathfrak{q}'_0 = \mathbb{1}_{\mathcal{M}}}} \mathfrak{p}' \circ_i \mathfrak{q}' + \sum_{\substack{\mathfrak{p}', \mathfrak{q}' \in \mathcal{C}_{\mathcal{M}} \\ \mathfrak{p}' \preceq_{\mathrm{be}} \mathfrak{p} \\ \mathfrak{q}' \preceq_{\mathrm{be}} \mathfrak{q} \\ \mathfrak{p}'_i = \mathbb{1}_{\mathcal{M}} \\ \mathfrak{q}'_0 \neq \mathbb{1}_{\mathcal{M}}}} \mathfrak{p}' \circ_i \mathfrak{q}' + \sum_{\substack{\mathfrak{p}', \mathfrak{q}' \in \mathcal{C}_{\mathcal{M}} \\ \mathfrak{p}' \preceq_{\mathrm{be}} \mathfrak{p} \\ \mathfrak{q}' \preceq_{\mathrm{be}} \mathfrak{q} \\ \mathfrak{p}'_i = \mathbb{1}_{\mathcal{M}} \\ \mathfrak{q}'_0 = \mathbb{1}_{\mathcal{M}}}} \mathfrak{p}' \circ_i \mathfrak{q}'. \end{aligned} \tag{2.2.22}$$

Let s_1 (respectively s_2, s_3, s_4) be the first (respectively second, third, fourth) summand of the right-hand side of (2.2.22). There are four cases to explore depending on whether the ith edge of $\mathfrak{p}$ and the base of $\mathfrak{q}$ are solid or not. From the definition of the H-basis and of the partial order relation $\preceq_{\mathrm{be}}$, we have

(a) if $\mathfrak{p}_i \neq \mathbb{1}_{\mathcal{M}}$ and $\mathfrak{q}_0 \neq \mathbb{1}_{\mathcal{M}}$, $s_1 = \mathrm{H}_{\mathfrak{p} \circ_i \mathfrak{q}}$, $s_2 = \mathrm{H}_{\mathfrak{p} \circ_i \mathrm{d}_0(\mathfrak{q})}$, $s_3 = \mathrm{H}_{\mathrm{d}_i(\mathfrak{p}) \circ_i \mathfrak{q}}$, and $s_4 = \mathrm{H}_{\mathrm{d}_i(\mathfrak{p}) \circ_i \mathrm{d}_0(\mathfrak{q})}$;
(b) if $\mathfrak{p}_i \neq \mathbb{1}_{\mathcal{M}}$ and $\mathfrak{q}_0 = \mathbb{1}_{\mathcal{M}}$, $s_1 = 0$, $s_2 = \mathrm{H}_{\mathfrak{p} \circ_i \mathfrak{q}}$, $s_3 = 0$, and $s_4 = \mathrm{H}_{\mathrm{d}_i(\mathfrak{p}) \circ_i \mathfrak{q}}$;
(c) if $\mathfrak{p}_i = \mathbb{1}_{\mathcal{M}}$ and $\mathfrak{q}_0 \neq \mathbb{1}_{\mathcal{M}}$, $s_1 = 0$, $s_2 = 0$, $s_3 = \mathrm{H}_{\mathfrak{p} \circ_i \mathfrak{q}}$, and $s_4 = \mathrm{H}_{\mathfrak{p} \circ_i \mathrm{d}_0(\mathfrak{q})}$;
(d) and if $\mathfrak{p}_i = \mathbb{1}_{\mathcal{M}}$ and $\mathfrak{q}_0 = \mathbb{1}_{\mathcal{M}}$, $s_1 = 0$, $s_2 = 0$, $s_3 = 0$, and $s_4 = \mathrm{H}_{\mathfrak{p} \circ_i \mathfrak{q}}$.

By assembling these cases together, we obtain the stated result. □

Proposition 2.2.9. *Let $\mathcal{M}$ be a unitary magma. The partial composition of $\mathrm{C}\mathcal{M}$ can be expressed in terms of the* K*-basis, for any $\mathcal{M}$-cliques $\mathfrak{p}$ and $\mathfrak{q}$ different from $\circ\!-\!-\!\circ$ and any valid integer i, by*

$$\mathrm{K}_{\mathfrak{p}} \circ_i \mathrm{K}_{\mathfrak{q}} = \begin{cases} \mathrm{K}_{\mathfrak{p} \circ_i \mathfrak{q}}, & \text{if } \mathfrak{p}_i \star \mathfrak{q}_0 = \mathbb{1}_{\mathcal{M}}, \\ \mathrm{K}_{\mathfrak{p} \circ_i \mathfrak{q}} + \mathrm{K}_{\mathrm{d}_i(\mathfrak{p}) \circ_i \mathrm{d}_0(\mathfrak{q})}, & \text{otherwise.} \end{cases} \tag{2.2.23}$$

Proof. Let m be the arity of $\mathfrak{q}$. From the definition of the K-basis and of the partial order relation $\preceq_{\mathrm{d}}$, we have

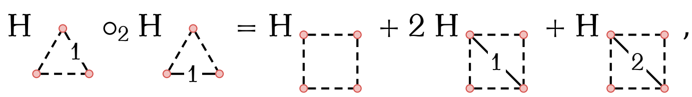
$$
\begin{aligned}
\mathsf{K}_{\mathfrak{p}} \circ_i \mathsf{K}_{\mathfrak{q}} &= \sum_{\substack{\mathfrak{p}',\mathfrak{q}' \in \mathcal{C}_{\mathcal{M}} \\ \mathfrak{p}' \preceq_{\mathrm{d}} \mathfrak{p} \\ \mathfrak{q}' \preceq_{\mathrm{d}} \mathfrak{q}}} (-1)^{\mathrm{h}(\mathfrak{p}',\mathfrak{p})+\mathrm{h}(\mathfrak{q}',\mathfrak{q})} \mathfrak{p}' \circ_i \mathfrak{q}' \\
&= \sum_{\substack{\mathfrak{p}',\mathfrak{q}' \in \mathcal{C}_{\mathcal{M}} \\ \mathfrak{p}' \circ_i \mathfrak{q}' \preceq_{\mathrm{d}} \mathfrak{p} \circ_i \mathfrak{q} \\ \mathfrak{p}'_i = \mathfrak{p}_i \\ \mathfrak{q}'_0 = \mathfrak{q}_0}} (-1)^{\mathrm{h}(\mathfrak{p}',\mathfrak{p})+\mathrm{h}(\mathfrak{q}',\mathfrak{q})} \mathfrak{p}' \circ_i \mathfrak{q}' \\
&= \sum_{\substack{\mathfrak{r} \in \mathcal{C}_{\mathcal{M}} \\ \mathfrak{r} \preceq_{\mathrm{d}} \mathfrak{p} \circ_i \mathfrak{q} \\ \mathfrak{r}(i, i+m-1) = \mathfrak{p}_i \star \mathfrak{q}_0}} (-1)^{\mathrm{h}(\mathfrak{r}, \mathfrak{p} \circ_i \mathfrak{q})} \mathfrak{r}.
\end{aligned}
\tag{2.2.24}
$$

If $\mathfrak{p}_i \star \mathfrak{q}_0 = \mathbb{1}_{\mathcal{M}}$, (2.2.24) is equal to $\mathsf{K}_{\mathfrak{p} \circ_i \mathfrak{q}}$. Otherwise, if $\mathfrak{p}_i \star \mathfrak{q}_0 \neq \mathbb{1}_{\mathcal{M}}$, we have

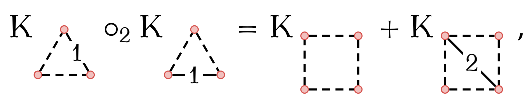
$$
\begin{aligned}
\sum_{\substack{\mathfrak{r} \in \mathcal{C}_{\mathcal{M}} \\ \mathfrak{r} \preceq_{\mathrm{d}} \mathfrak{p} \circ_i \mathfrak{q} \\ \mathfrak{r}(i, i+m-1) = \mathfrak{p}_i \star \mathfrak{q}_0}} (-1)^{\mathrm{h}(\mathfrak{r}, \mathfrak{p} \circ_i \mathfrak{q})} \mathfrak{r} &= \sum_{\substack{\mathfrak{r} \in \mathcal{C}_{\mathcal{M}} \\ \mathfrak{r} \preceq_{\mathrm{d}} \mathfrak{p} \circ_i \mathfrak{q}}} (-1)^{\mathrm{h}(\mathfrak{r}, \mathfrak{p} \circ_i \mathfrak{q})} \mathfrak{r} \; - \sum_{\substack{\mathfrak{r} \in \mathcal{C}_{\mathcal{M}} \\ \mathfrak{r} \preceq_{\mathrm{d}} \mathfrak{p} \circ_i \mathfrak{q} \\ \mathfrak{r}(i, i+m-1) \neq \mathfrak{p}_i \star \mathfrak{q}_0}} (-1)^{\mathrm{h}(\mathfrak{r}, \mathfrak{p} \circ_i \mathfrak{q})} \mathfrak{r} \\
&= \mathsf{K}_{\mathfrak{p} \circ_i \mathfrak{q}} - \sum_{\substack{\mathfrak{r} \in \mathcal{C}_{\mathcal{M}} \\ \mathfrak{r} \preceq_{\mathrm{d}} \mathrm{d}_i(\mathfrak{p}) \circ_i \mathrm{d}_0(\mathfrak{q})}} (-1)^{\mathrm{h}(\mathfrak{r}, \mathfrak{p} \circ_i \mathfrak{q})} \mathfrak{r} \\
&= \mathsf{K}_{\mathfrak{p} \circ_i \mathfrak{q}} - \sum_{\substack{\mathfrak{r} \in \mathcal{C}_{\mathcal{M}} \\ \mathfrak{r} \preceq_{\mathrm{d}} \mathrm{d}_i(\mathfrak{p}) \circ_i \mathrm{d}_0(\mathfrak{q})}} (-1)^{1+\mathrm{h}(\mathfrak{r}, \mathrm{d}_i(\mathfrak{p}) \circ_i \mathrm{d}_0(\mathfrak{q}))} \mathfrak{r} \\
&= \mathsf{K}_{\mathfrak{p} \circ_i \mathfrak{q}} + \mathsf{K}_{\mathrm{d}_i(\mathfrak{p}) \circ_i \mathrm{d}_0(\mathfrak{q})}.
\end{aligned}
$$

This proves the claimed formula for the partial composition of $\mathrm{C}\mathcal{M}$ over the K-basis. $\square$

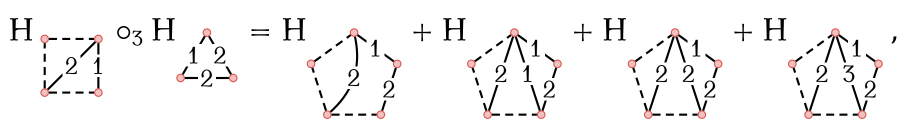
For instance, in $\mathrm{C}\mathbb{Z}$,



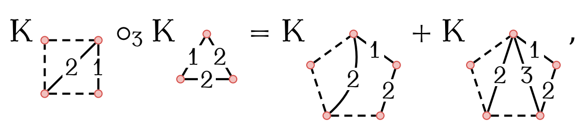
(2.2.25a)



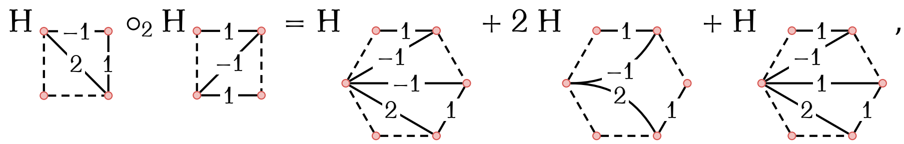
(2.2.25b)



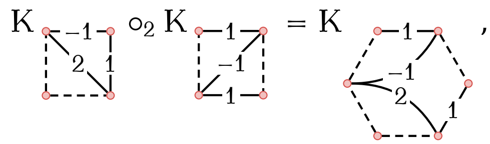
(2.2.25c)



(2.2.25d)

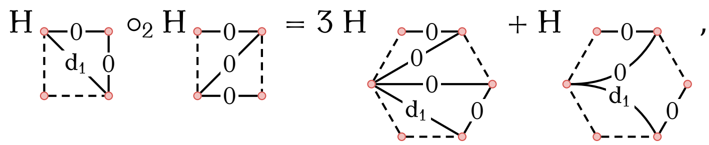


(2.2.25e)

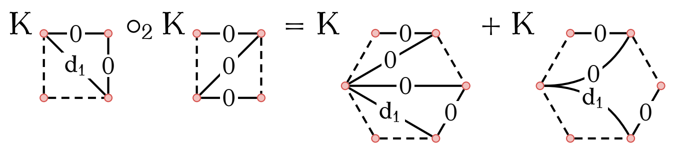


(2.2.25f)

and in $\mathbb{D}_1$,



(2.2.26a)



(2.2.26b)

2.2.8. *Rational functions.* The graded vector space of all commutative rational functions $\mathbb{K}(\mathbb{U})$, where $\mathbb{U}$ is the infinite commutative alphabet $\{u_1, u_2, \dots\}$, has the structure of an operad RatFct introduced by Loday [Lod10] and is defined as follows. Let $\mathrm{RatFct}(n)$ be the subspace $\mathbb{K}(u_1, \dots, u_n)$ of $\mathbb{K}(\mathbb{U})$ and

$$\mathrm{RatFct} := \bigoplus_{n \geqslant 1} \mathrm{RatFct}(n). \tag{2.2.27}$$

Observe that, since RatFct is a graded space, each rational function has an arity. Hence, by setting $f_1(u_1) := 1$ and $f_2(u_1, u_2) := 1$, f_1 is of arity 1 while f_2 is of arity 2, so that f_1 and f_2 are considered as different rational functions. The partial composition of two rational functions $f \in \mathrm{RatFct}(n)$ and $g \in \mathrm{RatFct}(m)$ is defined by

$$f \circ_i g := f\left(u_1, \dots, u_{i-1}, u_i + \dots + u_{i+m-1}, u_{i+m}, \dots, u_{n+m-1}\right) \, g\left(u_i, \dots, u_{i+m-1}\right). \tag{2.2.28}$$

The rational function f of $\mathrm{RatFct}(1)$ defined by $f(u_1) := 1$ is the unit of RatFct. As shown by Loday, this operad is (nontrivially) isomorphic to the operad Mould introduced by Chapoton [Cha07].

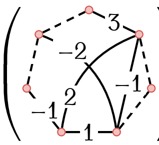
Let us assume that $\mathcal{M}$ is a *$\mathbb{Z}$-graded unitary magma*, that is, a unitary magma such that there exists a unitary magma morphism $\theta : \mathcal{M} \to \mathbb{Z}$. We say that θ is a *rank function* of $\mathcal{M}$. In this context, let

$$\mathrm{F}_\theta : \mathrm{C}\mathcal{M} \to \mathrm{RatFct} \tag{2.2.29}$$

be the linear map defined, for any $\mathcal{M}$-clique $\mathfrak{p}$, by

$$\mathrm{F}_\theta(\mathfrak{p}) := \prod_{(x,y) \in \mathcal{A}_{\mathfrak{p}}} \left(u_x + \dots + u_{y-1}\right)^{\theta(\mathfrak{p}(x,y))}. \tag{2.2.30}$$

For instance, by considering the unitary magma $\mathbb{Z}$ together with its identity map Id as rank function, we have

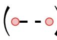
F_{Id} ()

$$= \frac{(u_1+u_2+u_3+u_4)^2\,(u_1+u_2+u_3+u_4+u_5+u_6)\,u_4^3}{u_1\,(u_3+u_4+u_5+u_6)^2\,(u_5+u_6)}. \tag{2.2.31}$$

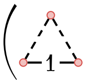
Theorem 2.2.10. *Let $\mathcal{M}$ be a $\mathbb{Z}$-graded unitary magma and θ be a rank function of $\mathcal{M}$. The map F_θ is an operad morphism from $\mathrm{C}\mathcal{M}$ to* RatFct.

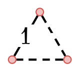
Proof. For the sake of brevity of notation, for all positive integers $x < y$, we denote by $\mathbb{U}_{x,y}$ the sums $u_x + \dots + u_{y-1}$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be two $\mathcal{M}$-cliques of respective arities n and m, and $i \in [n]$ be an integer. From the definition of the partial composition of $\mathrm{C}\mathcal{M}$, the one (see (2.2.28)) of RatFct, and the fact that θ is a unitary magma morphism, we have

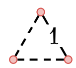
$$\begin{aligned}
\mathrm{F}_\theta(\mathfrak{p}) \circ_i \mathrm{F}_\theta(\mathfrak{q}) &= (\mathrm{F}_\theta(\mathfrak{p}))\,(u_1,\dots,u_{i-1},\mathbb{U}_{i,i+m},u_{i+m},\dots,u_{n+m-1})\;(\mathrm{F}_\theta(\mathfrak{q}))\,(u_i,\dots,u_{i+m-1}) \\
&= \left(\prod_{1\leqslant x<y\leqslant i-1} \mathbb{U}_{x,y}^{\theta(\mathfrak{p}(x,y))}\right)\left(\prod_{i+1\leqslant x<y\leqslant n+1} \mathbb{U}_{x+m-1,y+m-1}^{\theta(\mathfrak{p}(x,y))}\right)\mathbb{U}_{i,i+m}^{\theta(\mathfrak{p}_i)} \\
&\qquad\times\left(\prod_{1\leqslant x<y\leqslant m+1} \mathbb{U}_{x+i-1,y+i-1}^{\theta(\mathfrak{q}(x,y))}\right) \\
&= \left(\prod_{1\leqslant x<y\leqslant i-1} \mathbb{U}_{x,y}^{\theta(\mathfrak{p}(x,y))}\right)\left(\prod_{i+1\leqslant x<y\leqslant n+1} \mathbb{U}_{x+m-1,y+m-1}^{\theta(\mathfrak{p}(x,y))}\right)\mathbb{U}_{i,i+m}^{\theta(\mathfrak{p}_i)+\theta(\mathfrak{q}_0)} \\
&\qquad\times\left(\prod_{\substack{1\leqslant x<y\leqslant m+1\\(x,y)\neq(1,m+1)}} \mathbb{U}_{x+i-1,y+i-1}^{\theta(\mathfrak{q}(x,y))}\right) \\
&= \left(\prod_{1\leqslant x<y\leqslant i-1} \mathbb{U}_{x,y}^{\theta(\mathfrak{p}(x,y))}\right)\left(\prod_{i+1\leqslant x<y\leqslant n+1} \mathbb{U}_{x+m-1,y+m-1}^{\theta(\mathfrak{p}(x,y))}\right)\mathbb{U}_{i,i+m}^{\theta(\mathfrak{p}_i\star\mathfrak{q}_0)} \\
&\qquad\times\left(\prod_{\substack{1\leqslant x<y\leqslant m+1\\(x,y)\neq(1,m+1)}} \mathbb{U}_{x+i-1,y+i-1}^{\theta(\mathfrak{q}(x,y))}\right) \\
&= \prod_{(x,y)\in\mathcal{A}_{\mathfrak{p}\circ_i\mathfrak{q}}} \mathbb{U}_{x,y}^{\theta((\mathfrak{p}\circ_i\mathfrak{q})(x,y))} \\
&= \mathrm{F}_\theta(\mathfrak{p}\circ_i\mathfrak{q}).
\end{aligned}$$

Moreover, since $\theta(\mathbb{1}_{\mathcal{M}}) = 0$, we have F_θ () $= 1$, so that F_θ sends the unit of $\mathrm{C}\mathcal{M}$ to the unit of RatFct. Therefore, F_θ is an operad morphism. □

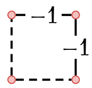
The operad morphism F_θ is not injective. Indeed, by considering the magma $\mathbb{Z}$ together with its identity map Id as rank function, we have for instance

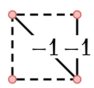
$\mathrm{F}_{\mathrm{Id}}\Big($ 

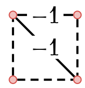
$-$ 

$-$ 

$\Big) = (u_1 + u_2) - u_1 - u_2 = 0,$ (2.2.32a)

$\mathrm{F}_{\mathrm{Id}}\Big($ 

$-$ 

$-$ 

$\Big) = \dfrac{1}{u_2u_3} - \dfrac{1}{(u_2+u_3)u_3} - \dfrac{1}{u_2(u_2+u_3)} = 0.$ (2.2.32b)

Proposition 2.2.11. *The subspace of* RatFct *of all Laurent polynomials on* $\mathbb{U}$ *is the image by* $\mathrm{F}_{\mathrm{Id}} : \mathrm{C}\mathbb{Z} \to$ RatFct *of the subspace of* $\mathrm{C}\mathbb{Z}$ *consisting in the linear span of all* $\mathbb{Z}$*-bubbles.*

Proof. First, by Theorem 2.2.10, F_{Id} is a well-defined operad morphism from $\mathrm{C}\mathbb{Z}$ to RatFct. Let $u_1^{\alpha_1} \dots u_n^{\alpha_n}$ be a Laurent monomial, where $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$ and $n \geqslant 1$. Consider also the $\mathbb{Z}$-clique $\mathfrak{p}_\alpha$ of arity $n+1$ satisfying

$$\mathfrak{p}_\alpha(x, y) := \begin{cases} \alpha_x, & \text{if } y = x + 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2.33}$$

Observe that $\mathfrak{p}_\alpha$ is a $\mathbb{Z}$-bubble. By definition of F_{Id}, we have $\mathrm{F}_{\mathrm{Id}}(\mathfrak{p}_\alpha) = u_1^{\alpha_1} \dots u_n^{\alpha_n}$. Now, since a Laurent polynomial is a linear combination of some Laurent monomials, by the linearity of F_{Id}, the statement of the proposition follows. □

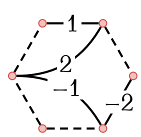
For any $n \geqslant 1$, let

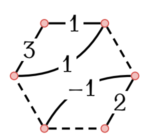
$$\star : \mathrm{C}\mathcal{M}(n) \otimes \mathrm{C}\mathcal{M}(n) \to \mathrm{C}\mathcal{M}(n) \tag{2.2.34}$$

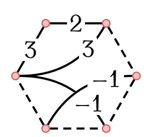
be the product defined for all $\mathcal{M}$-cliques $\mathfrak{p}$ and $\mathfrak{q}$ by

$$(\mathfrak{p} \star \mathfrak{q})(x, y) := \mathfrak{p}(x, y) \star \mathfrak{q}(x, y), \tag{2.2.35}$$

where (x, y) is any arc such that $1 \leqslant x < y \leqslant n + 1$, and then extended linearly. For instance, in $\mathrm{C}\mathbb{Z}$,



$\star$ 

$=$ 

. (2.2.36)

Proposition 2.2.12. *Let* $\mathcal{M}$ *be a* $\mathbb{Z}$*-graded unitary magma and* θ *be a rank function of* $\mathcal{M}$*. For any homogeneous elements* f *and* g *of* $\mathrm{C}\mathcal{M}$ *of the same arity,*

$$\mathrm{F}_\theta(f)\mathrm{F}_\theta(g) = \mathrm{F}_\theta(f \star g). \tag{2.2.37}$$

Proof. Let $\mathfrak{p}$ and $\mathfrak{q}$ be two $\mathcal{M}$-cliques of $\mathrm{C}\mathcal{M}$ of arity n. By definition of the operation $\star$ on $\mathrm{C}\mathcal{M}(n)$ and the fact that θ is a unitary magma morphism,

$$\mathrm{F}_\theta(\mathfrak{p})\mathrm{F}_\theta(\mathfrak{q}) = \left(\prod_{(x,y) \in \mathcal{A}_\mathfrak{p}} (u_x + \dots + u_{y-1})^{\theta(\mathfrak{p}(x,y))}\right)\left(\prod_{(x,y) \in \mathcal{A}_\mathfrak{q}} (u_x + \dots + u_{y-1})^{\theta(\mathfrak{q}(x,y))}\right)$$

$$= \prod_{1 \leqslant x < y \leqslant n+1} (u_x + \dots + u_{y-1})^{\theta(\mathfrak{p}(x,y)) + \theta(\mathfrak{q}(x,y))}$$
$$= \prod_{1 \leqslant x < y \leqslant n+1} (u_x + \dots + u_{y-1})^{\theta(\mathfrak{p}(x,y) * \mathfrak{q}(x,y))}$$
$$= \mathrm{F}_\theta(\mathfrak{p} \star \mathfrak{q}).$$

By the linearity of F_θ and of $\star$, (2.2.37) follows. □

Proposition 2.2.13. *Let $\mathfrak{p}$ be an $\mathcal{M}$-clique of $\mathrm{C}\mathbb{Z}$. Then*

$$\frac{1}{\mathrm{F}_{\mathrm{Id}}(\mathfrak{p})} = \mathrm{F}_{\mathrm{Id}}((\mathrm{C}\eta)(\mathfrak{p})), \tag{2.2.38}$$

where $\eta : \mathbb{Z} \to \mathbb{Z}$ is the unitary magma morphism defined by $\eta(x) := -x$ for all $x \in \mathbb{Z}$.

Proof. Observe that $(\mathrm{C}\eta)(\mathfrak{p})$ is the $\mathcal{M}$-clique obtained by relabeling each arc (x, y) of $\mathfrak{p}$ by $-\mathfrak{p}(x, y)$. Hence, since η is a unitary magma morphism, we have

$$\begin{aligned} \mathrm{F}_{\mathrm{Id}}((\mathrm{C}\eta)(\mathfrak{p})) &= \prod_{(x,y) \in \mathcal{A}_\mathfrak{p}} (u_x + \dots + u_{y-1})^{\theta(-\mathfrak{p}(x,y))} \\ &= \prod_{(x,y) \in \mathcal{A}_\mathfrak{p}} (u_x + \dots + u_{y-1})^{-\theta(\mathfrak{p}(x,y))} \\ &= \frac{1}{\mathrm{F}_{\mathrm{Id}}(\mathfrak{p})} \end{aligned}$$

as expected. □

3. Quotients and suboperads

We define here quotients and suboperads of $\mathrm{C}\mathcal{M}$, leading to the construction of some new operads involving various combinatorial objects which are, basically, $\mathcal{M}$-cliques with some restrictions.

3.1. **Main substructures.** Most of the natural subfamilies of $\mathcal{M}$-cliques that can be described by simple combinatorial properties as $\mathcal{M}$-cliques with restrained labels for the bases, edges, and diagonals, white $\mathcal{M}$-cliques, $\mathcal{M}$-cliques with a fixed maximal crossing number, $\mathcal{M}$-bubbles, $\mathcal{M}$-cliques with a fixed maximal value for their degrees, nesting-free $\mathcal{M}$-cliques, and acyclic $\mathcal{M}$-cliques inherit from the algebraic structure of operad of $\mathrm{C}\mathcal{M}$ and form quotients and suboperads of $\mathrm{C}\mathcal{M}$ (see Table 1). We construct and briefly study here these main substructures of $\mathrm{C}\mathcal{M}$.

Operad	Objects	Status with respect to $\mathsf{C}\mathcal{M}$
$\mathsf{Lab}_{B,E,D}\,\mathcal{M}$	$\mathcal{M}$-cliques with restricted labels	Suboperad
$\mathsf{Whi}\,\mathcal{M}$	White $\mathcal{M}$-cliques	Suboperad
$\mathsf{Cro}_k\,\mathcal{M}$	$\mathcal{M}$-cliques of crossings at most k	Suboperad and quotient
$\mathsf{Bub}\,\mathcal{M}$	$\mathcal{M}$-bubbles	Quotient
$\mathsf{Deg}_k\,\mathcal{M}$	$\mathcal{M}$-cliques of degree at most k	Quotient
$\mathsf{Nes}\,\mathcal{M}$	Nesting-free $\mathcal{M}$-cliques	Quotient
$\mathsf{Acy}\,\mathcal{M}$	Acyclic $\mathcal{M}$-cliques	Quotient

TABLE 1. Operads constructed as suboperads or quotients of $\mathsf{C}\mathcal{M}$. All these operads depend on a unitary magma $\mathcal{M}$ which has, in some cases, to satisfy some precise conditions. Some of these operads depend also on a nonnegative integer k or subsets B, E, and D of $\mathcal{M}$.

3.1.1. *Restricting the labels.* In what follows, if X and Y are two subsets of $\mathcal{M}$, $X \star Y$ denotes the set $\{x \star y : x \in X \text{ and } y \in Y\}$.

Let B, E, and D be three subsets of $\mathcal{M}$ and $\mathsf{Lab}_{B,E,D}\,\mathcal{M}$ be the subspace of $\mathsf{C}\mathcal{M}$ generated by all $\mathcal{M}$-cliques $\mathfrak{p}$ such that the bases of $\mathfrak{p}$ are labeled by B, all edges of $\mathfrak{p}$ are labeled by E, and all diagonals of $\mathfrak{p}$ are labeled by D.

Proposition 3.1.1. *Let $\mathcal{M}$ be a unitary magma and B, E, and D be three subsets of $\mathcal{M}$. If $\mathbb{1}_{\mathcal{M}} \in B$, $\mathbb{1}_{\mathcal{M}} \in D$, and $E \star B \subseteq D$, $\mathsf{Lab}_{B,E,D}\,\mathcal{M}$ is a suboperad of $\mathsf{C}\mathcal{M}$.*

Proof. First, since $\mathbb{1}_{\mathcal{M}} \in B$, the unit of $\mathsf{C}\mathcal{M}$ belongs to $\mathsf{Lab}_{B,E,D}\,\mathcal{M}$. Consider now two $\mathcal{M}$-cliques $\mathfrak{p}$ and $\mathfrak{q}$ of $\mathsf{Lab}_{B,E,D}\,\mathcal{M}$ and a partial composition $\mathfrak{r} := \mathfrak{p} \circ_i \mathfrak{q}$ for a valid integer i. By the definition of the partial composition of $\mathsf{C}\mathcal{M}$, the base of $\mathfrak{r}$ has the same label as the base of $\mathfrak{p}$, and all edges of $\mathfrak{r}$ have labels coming from the ones of $\mathfrak{p}$ and $\mathfrak{q}$. Moreover, all diagonals of $\mathfrak{r}$ are either non-solid, or come from diagonals of $\mathfrak{p}$ and $\mathfrak{q}$, or are the diagonal $\mathfrak{r}(i, i + |\mathfrak{q}|)$ which is labeled by $\mathfrak{p}_i \star \mathfrak{q}_0$. Since $\mathbb{1}_{\mathcal{M}} \in D$, $\mathfrak{p}_i \in E$, $\mathfrak{q}_0 \in B$, and $E \star B \subseteq D$, all the labels of these diagonals are in D. For these reasons, $\mathfrak{r}$ is in $\mathsf{Lab}_{B,E,D}\,\mathcal{M}$. This implies the statement of the proposition. □

Proposition 3.1.2. *Let $\mathcal{M}$ be a unitary magma and B, E, and D be three finite subsets of $\mathcal{M}$. For all $n \geqslant 2$,*

$$\dim \mathsf{Lab}_{B,E,D}\,\mathcal{M}(n) = b e^n d^{(n+1)(n-2)/2}, \tag{3.1.1}$$

where $b := \#B$, $e := \#E$, and $d := \#D$.

Proof. By Proposition 2.2.1, there are $m^{\binom{n+1}{2}}$ $\mathcal{M}$-cliques of arity n, where $m := \#\mathcal{M}$. Hence, there are $m^{\binom{n+1}{2}}/m^{n+1}$ $\mathcal{M}$-cliques of arity n with all edges and the base labeled by $\mathbb{1}_{\mathcal{M}}$. This also says that there are $d^{\binom{n+1}{2}}/d^{n+1}$ $\mathcal{M}$-cliques of arity n with all diagonals labeled by D and all edges and the base labeled by $\mathbb{1}_{\mathcal{M}}$. Since an $\mathcal{M}$-clique of $\mathrm{Lab}_{B,E,D}\,\mathcal{M}(n)$ has its n edges labeled by E and its base labeled by B, (3.1.1) follows. □

3.1.2. *White cliques.* Let $\mathrm{Whi}\,\mathcal{M}$ be the subspace of $\mathrm{C}\mathcal{M}$ generated by all white $\mathcal{M}$-cliques. Since, by definition of white $\mathcal{M}$-cliques,

$$\mathrm{Whi}\,\mathcal{M} = \mathrm{Lab}_{\{\mathbb{1}_{\mathcal{M}}\},\{\mathbb{1}_{\mathcal{M}}\},\mathcal{M}}\,\mathcal{M}, \tag{3.1.2}$$

by Proposition 3.1.1, $\mathrm{Whi}\,\mathcal{M}$ is a suboperad of $\mathrm{C}\mathcal{M}$. It follows from Proposition 3.1.2 that, if $\mathcal{M}$ is finite, the dimensions of $\mathrm{Whi}\,\mathcal{M}$ satisfy, for any $n \geqslant 2$,

$$\dim \mathrm{Whi}\,\mathcal{M}(n) = m^{(n+1)(n-2)/2}, \tag{3.1.3}$$

where $m := \#\mathcal{M}$.

3.1.3. *Restricting the crossings.* Let $k \geqslant 0$ be an integer and $\mathfrak{R}_{\mathrm{Cro}_k\,\mathcal{M}}$ be the subspace of $\mathrm{C}\mathcal{M}$ generated by all $\mathcal{M}$-cliques $\mathfrak{p}$ such that $\mathrm{cros}(\mathfrak{p}) \geqslant k+1$. As a quotient of graded vector spaces,

$$\mathrm{Cro}_k\,\mathcal{M} := \mathrm{C}\mathcal{M}/\mathfrak{R}_{\mathrm{Cro}_k\,\mathcal{M}} \tag{3.1.4}$$

is the linear span of all $\mathcal{M}$-cliques $\mathfrak{p}$ such that $\mathrm{cros}(\mathfrak{p}) \leqslant k$.

Proposition 3.1.3. *Let $\mathcal{M}$ be a unitary magma and $k \geqslant 0$ be an integer. Then the space $\mathrm{Cro}_k\,\mathcal{M}$ is a quotient operad of $\mathrm{C}\mathcal{M}$ and is isomorphic to the suboperad of $\mathrm{C}\mathcal{M}$ restricted to the subspace generated by all $\mathcal{M}$-cliques with crossing numbers no greater than k.*

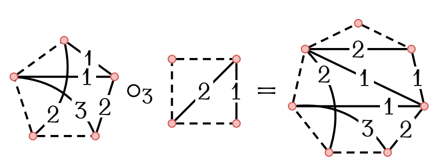
Proof. We first prove that $\mathrm{Cro}_k\,\mathcal{M}$ is a quotient of $\mathrm{C}\mathcal{M}$. For this, observe that, if $\mathfrak{p}$ and $\mathfrak{q}$ are two $\mathcal{M}$-cliques,

$$\mathrm{cros}(\mathfrak{p} \circ_i \mathfrak{q}) = \max\{\mathrm{cros}(\mathfrak{p}), \mathrm{cros}(\mathfrak{q})\} \tag{3.1.5}$$

for any valid integer i. For this reason, if $\mathfrak{p}$ is an $\mathcal{M}$-clique of $\mathfrak{R}_{\mathrm{Cro}_k\,\mathcal{M}}$, each clique obtained by a partial composition involving $\mathfrak{p}$ and other $\mathcal{M}$-cliques is still in $\mathfrak{R}_{\mathrm{Cro}_k\,\mathcal{M}}$. This proves that $\mathfrak{R}_{\mathrm{Cro}_k\,\mathcal{M}}$ is an operad ideal of $\mathrm{C}\mathcal{M}$ and hence, that $\mathrm{Cro}_k\,\mathcal{M}$ is a quotient of $\mathrm{C}\mathcal{M}$.

To prove the second part of the statement, consider two $\mathcal{M}$-cliques $\mathfrak{p}$ and $\mathfrak{q}$ of $\mathrm{Cro}_k\,\mathcal{M}$. By (3.1.5), all $\mathcal{M}$-cliques $\mathfrak{p} \circ_i \mathfrak{q}$ are still in $\mathrm{Cro}_k\,\mathcal{M}$, for all valid integers i. Moreover, the unit of $\mathrm{C}\mathcal{M}$ belongs to $\mathrm{Cro}_k\,\mathcal{M}$. This implies the desired property. □

For instance, in the operad $\mathrm{Cro}_2\,\mathbb{Z}$, we have



(3.1.6)

If $0 \leqslant k' \leqslant k$ are integers, by Proposition 3.1.3, $\mathrm{Cro}_k \mathcal{M}$ and $\mathrm{Cro}_{k'} \mathcal{M}$ are both quotients and suboperads of $\mathsf{C}\mathcal{M}$. First, since any $\mathcal{M}$-clique of $\mathrm{Cro}_{k'} \mathcal{M}$ is also an $\mathcal{M}$-clique of $\mathrm{Cro}_k \mathcal{M}$, $\mathrm{Cro}_{k'} \mathcal{M}$ is a suboperad of $\mathrm{Cro}_k \mathcal{M}$. Second, since $\mathfrak{R}_{\mathrm{Cro}_k \mathcal{M}}$ is a subspace of $\mathfrak{R}_{\mathrm{Cro}_{k'} \mathcal{M}}$, $\mathrm{Cro}_{k'} \mathcal{M}$ is a quotient of $\mathrm{Cro}_k \mathcal{M}$.

Observe that $\mathrm{Cro}_0 \mathcal{M}$ is the linear span of all noncrossing $\mathcal{M}$-cliques. We can see these objects as noncrossing configurations [FN99], where the edges and bases are colored by elements of $\mathcal{M}$ and the diagonals by elements of $\overline{\mathcal{M}}$. The operad $\mathrm{Cro}_0 \mathcal{M}$ has a lot of combinatorial and algebraic properties and will be studied in detail in [Gir18].

3.1.4. *Bubbles.* Let $\mathfrak{R}_{\mathrm{Bub}\, \mathcal{M}}$ be the subspace of $\mathsf{C}\mathcal{M}$ generated by all $\mathcal{M}$-cliques that are not bubbles. As a quotient of graded vector spaces,

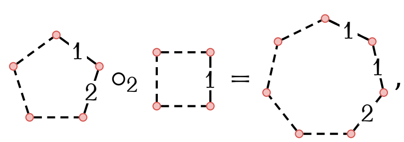
$$\mathrm{Bub}\, \mathcal{M} := \mathsf{C}\mathcal{M} / \mathfrak{R}_{\mathrm{Bub}\, \mathcal{M}} \tag{3.1.7}$$

is the linear span of all $\mathcal{M}$-bubbles.

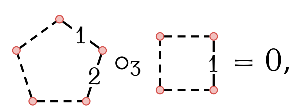
Proposition 3.1.4. *Let $\mathcal{M}$ be a unitary magma. Then the space $\mathrm{Bub}_{\mathcal{M}}$ is a quotient operad of $\mathsf{C}\mathcal{M}$.*

Proof. If $\mathfrak{p}$ and $\mathfrak{q}$ are two $\mathcal{M}$-cliques, all solid diagonals of $\mathfrak{p}$ and $\mathfrak{q}$ appear in $\mathfrak{p} \circ_i \mathfrak{q}$, for any valid integer i. For this reason, if $\mathfrak{p}$ is an $\mathcal{M}$-clique of $\mathfrak{R}_{\mathrm{Bub}\, \mathcal{M}}$, each $\mathcal{M}$-clique obtained by a partial composition involving $\mathfrak{p}$ and other $\mathcal{M}$-cliques is still in $\mathfrak{R}_{\mathrm{Bub}\, \mathcal{M}}$. This proves that $\mathfrak{R}_{\mathrm{Bub}\, \mathcal{M}}$ is an operad ideal of $\mathsf{C}\mathcal{M}$ and implies the statement of the proposition. □

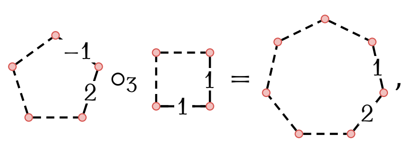
For instance, in the operad $\mathrm{Bub}\, \mathbb{Z}$, we have



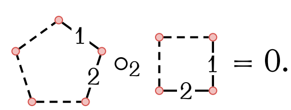
(3.1.8a)



(3.1.8c)



(3.1.8b)



(3.1.8d)

If $\mathcal{M}$ is finite, the dimensions of $\mathrm{Bub}\, \mathcal{M}$ satisfy, for any $n \geqslant 2$,

$$\dim \mathrm{Bub}\, \mathcal{M}(n) = m^{n+1}, \tag{3.1.9}$$

where $m := \# \mathcal{M}$.

3.1.5. *Restricting the degrees.* Let $k \geqslant 0$ be an integer and $\mathfrak{R}_{\mathrm{Deg}_k \mathcal{M}}$ be the subspace of $\mathsf{C}\mathcal{M}$ generated by all $\mathcal{M}$-cliques $\mathfrak{p}$ such that $\mathrm{degr}(\mathfrak{p}) \geqslant k+1$. As a quotient of graded vector spaces,

$$\mathrm{Deg}_k \mathcal{M} := \mathsf{C}\mathcal{M} / \mathfrak{R}_{\mathrm{Deg}_k \mathcal{M}} \tag{3.1.10}$$

is the linear span of all $\mathcal{M}$-cliques $\mathfrak{p}$ such that $\mathrm{degr}(\mathfrak{p}) \leqslant k$.

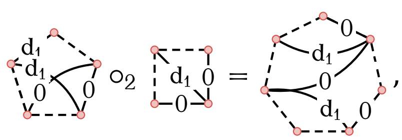
Proposition 3.1.5. *Let $\mathcal{M}$ be a unitary magma without nontrivial unit divisors and $k \geqslant 0$ be an integer. Then the space* $\mathrm{Deg}_k\,\mathcal{M}$ *is a quotient operad of* $\mathrm{C}\mathcal{M}$.

Proof. Since $\mathcal{M}$ has no nontrivial unit divisors, for any $\mathcal{M}$-cliques $\mathfrak{p}$ and $\mathfrak{q}$ of $\mathrm{C}\mathcal{M}$, each solid arc of $\mathfrak{p}$ (respectively $\mathfrak{q}$) gives rise to a solid arc in $\mathfrak{p} \circ_i \mathfrak{q}$, for any valid integer i. Hence,

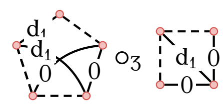
$$\mathrm{degr}(\mathfrak{p} \circ_i \mathfrak{q}) \geqslant \max\{\mathrm{degr}(\mathfrak{p}), \mathrm{degr}(\mathfrak{q})\}, \qquad (3.1.11)$$

and then, if $\mathfrak{p}$ is an $\mathcal{M}$-clique of $\mathfrak{R}_{\mathrm{Deg}_k\,\mathcal{M}}$, each $\mathcal{M}$-clique obtained by a partial composition involving $\mathfrak{p}$ and other $\mathcal{M}$-cliques is still in $\mathfrak{R}_{\mathrm{Deg}_k\,\mathcal{M}}$. This proves that $\mathfrak{R}_{\mathrm{Deg}_k\,\mathcal{M}}$ is an operad ideal of $\mathrm{C}\mathcal{M}$ and implies the statement of the proposition. □

For instance, in the operad $\mathrm{Deg}_3\,\mathbb{D}_2$ (observe that $\mathbb{D}_2$ is a unitary magma without nontrivial unit divisors), we have



(3.1.12a)

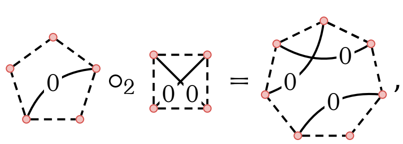


$= 0.$ (3.1.12b)

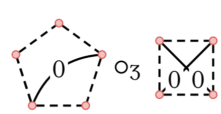
If $0 \leqslant k' \leqslant k$ are integers, by Proposition 3.1.5, $\mathrm{Deg}_k\,\mathcal{M}$ and $\mathrm{Deg}_{k'}\,\mathcal{M}$ are both quotient operads of $\mathrm{C}\mathcal{M}$. Moreover, since $\mathfrak{R}_{\mathrm{Deg}_k\,\mathcal{M}}$ is a subspace of $\mathfrak{R}_{\mathrm{Deg}_{k'}\,\mathcal{M}}$, $\mathrm{Deg}_{k'}\,\mathcal{M}$ is a quotient operad of $\mathrm{Deg}_k\,\mathcal{M}$.

Observe that $\mathrm{Deg}_0\,\mathcal{M}$ is the linear span of all $\mathcal{M}$-cliques without solid arcs. If $\mathfrak{p}$ and $\mathfrak{q}$ are such $\mathcal{M}$-cliques, all partial compositions $\mathfrak{p} \circ_i \mathfrak{q}$ are equal to the unique $\mathcal{M}$-clique without solid arcs of arity $|\mathfrak{p}| + |\mathfrak{q}| - 1$. For this reason, $\mathrm{Deg}_0\,\mathcal{M}$ is the associative operad As.

Any skeleton of an $\mathcal{M}$-clique of arity n of $\mathrm{Deg}_1\,\mathcal{M}$ can be seen as a partition of the set $[n+1]$ into singletons or pairs. Therefore, $\mathrm{Deg}_1\,\mathcal{M}$ can be seen as an operad on such colored partitions, where each pair of the partitions has one color from the set $\overline{\mathcal{M}}$. In the operad $\mathrm{Deg}_1\,\mathbb{D}_0$ (observe that $\mathbb{D}_0$ is the only unitary magma without nontrivial unit divisors on two elements), we have for instance



(3.1.13a)



$= 0.$ (3.1.13b)

By seeing each solid arc (x, y) of an $\mathcal{M}$-clique $\mathfrak{p}$ of $\mathrm{Deg}_1\,\mathbb{D}_0$ of arity n as the transposition exchanging the letter x and the letter y, we can interpret $\mathfrak{p}$ as an involution of $\mathfrak{S}_{n+1}$ made of the product of these transpositions. Hence, $\mathrm{Deg}_1\,\mathbb{D}_0$ can be seen as an operad on involutions. Under this point of view, the partial compositions (3.1.13a) and (3.1.13b) translate on permutations as

$$42315 \circ_2 3412 = 6452317, \qquad (3.1.14a) \qquad\qquad 42315 \circ_3 3412 = 0. \qquad (3.1.14b)$$

Equivalently, by the Robinson–Schensted correspondence (see for instance [Lot02]), $\mathrm{Deg}_1 \mathbb{D}_0$ is an operad on standard Young tableaux. The dimensions of the operad $\mathrm{Deg}_1 \mathbb{D}_0$ begin by

$$1, 4, 10, 26, 76, 232, 764, 2620, \tag{3.1.15}$$

and form, except for the first terms, Sequence **A000085** of [Slo]. Moreover, if $\#\mathcal{M} = 3$, the dimensions of $\mathrm{Deg}_1 \mathcal{M}$ begin by

$$1, 7, 25, 81, 331, 1303, 5937, 26785, \tag{3.1.16}$$

and form, except for the first terms, Sequence **A047974** of [Slo].

Besides, any skeleton of an $\mathcal{M}$-clique of $\mathrm{Deg}_2 \mathcal{M}$ can be seen as a *thunderstorm graph*, i.e., a graph where connected components are cycles or paths. Therefore, $\mathrm{Deg}_2 \mathcal{M}$ can be seen as an operad on such colored graphs, where the arcs of the graphs have one color from the set $\overline{\mathcal{M}}$. If $\#\mathcal{M} = 2$, the dimensions of this operad begin by

$$1, 8, 41, 253, 1858, 15796, 152219, 1638323, \tag{3.1.17}$$

and form, except for the first terms, Sequence **A136281** of [Slo].

3.1.6. *Nesting-free cliques.* Let $\mathfrak{R}_{\mathrm{Nes}\,\mathcal{M}}$ be the subspace of $\mathrm{C}\mathcal{M}$ generated by all $\mathcal{M}$-cliques that are not nesting-free. As a quotient of graded vector spaces,

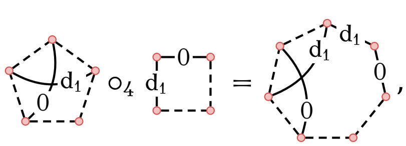
$$\mathrm{Nes}\,\mathcal{M} := \mathrm{C}\mathcal{M}/\mathfrak{R}_{\mathrm{Nes}\,\mathcal{M}} \tag{3.1.18}$$

is the linear span of all nesting-free $\mathcal{M}$-cliques.

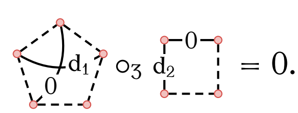
Proposition 3.1.6. *Let $\mathcal{M}$ be a unitary magma without nontrivial unit divisors. Then the space $\mathrm{Nes}\,\mathcal{M}$ is a quotient operad of $\mathrm{C}\mathcal{M}$.*

Proof. Since $\mathcal{M}$ has no nontrivial unit divisors, for any $\mathcal{M}$-cliques $\mathfrak{p}$ and $\mathfrak{q}$ of $\mathrm{C}\mathcal{M}$, each solid arc of $\mathfrak{p}$ (respectively $\mathfrak{q}$) gives rise to a solid arc in $\mathfrak{p} \circ_i \mathfrak{q}$, for any valid integer i. For this reason, if $\mathfrak{p}$ is an $\mathcal{M}$-clique of $\mathfrak{R}_{\mathrm{Nes}\,\mathcal{M}}$, $\mathfrak{p}$ is not nesting-free and each $\mathcal{M}$-clique obtained by a partial composition involving $\mathfrak{p}$ and other $\mathcal{M}$-cliques is still not nesting-free and thus, belongs to $\mathfrak{R}_{\mathrm{Nes}\,\mathcal{M}}$. This proves that $\mathfrak{R}_{\mathrm{Nes}\,\mathcal{M}}$ is an operad ideal of $\mathrm{C}\mathcal{M}$ and implies the statement of the proposition. □

For instance, in the operad $\mathrm{Nes}\,\mathbb{D}_2$,



(3.1.19a)



(3.1.19b)

Observe that in the same way as considering $\mathcal{M}$-cliques of crossing numbers k or less leads to quotients $\mathrm{Cro}_k \mathcal{M}$ of $\mathrm{C}\mathcal{M}$ (see Section 3.1.3), it is possible to define analogous quotients $\mathrm{Nes}_k \mathcal{M}$ spanned by $\mathcal{M}$-cliques having solid arcs that nest at most k other ones.

Recall that a *Dyck path* of *size* n is a word u on $\{\mathsf{a}, \mathsf{b}\}$ of length $2n$ such that $|u|_{\mathsf{a}} = |u|_{\mathsf{b}}$ and, for each prefix v of u, $|v|_{\mathsf{a}} \geqslant |v|_{\mathsf{b}}$.

Lemma 3.1.7. *Let $\mathcal{M}$ be a finite unitary magma without nontrivial unit divisors. For all $n \geqslant 2$, the set of all $\mathcal{M}$-cliques of $\mathrm{Nes}\,\mathcal{M}(n)$ is in one-to-one correspondence with the set of all Dyck paths of size $n+1$ wherein letters a at even positions are colored by $\overline{\mathcal{M}}$. Moreover, there is a correspondence between these two sets that sends any $\mathcal{M}$-clique of $\mathrm{Nes}\,\mathcal{M}(n)$ with k solid edges to a Dyck path with exactly k letters a at even positions, for any $0 \leqslant k \leqslant n$.*

Proof. In this proof, we denote by a_c the letter a of a Dyck path colored by $c \in \overline{\mathcal{M}}$. Given an $\mathcal{M}$-clique $\mathfrak{p}$ of $\mathrm{Nes}\,\mathcal{M}(n)$, we decorate each vertex x of $\mathfrak{p}$ by

(1) $\mathsf{a}\mathsf{a}_c$ if x has one outcoming arc and no incoming arc, where c is the label of the outcoming arc from x;
(2) $\mathsf{b}\mathsf{b}$ if x has no outcoming arc and one incoming arc;
(3) $\mathsf{b}\mathsf{a}_c$ if x has both one outcoming arc and one incoming arc, where c is the label of the outcoming arc from x;
(4) $\mathsf{a}\mathsf{b}$ otherwise.

Let ϕ be the map sending $\mathfrak{p}$ to the word obtained by concatenating the decorations of the vertices of $\mathfrak{p}$ thus described, read from 1 to $n+1$.

We show that ϕ is a bijection between the two sets of the statement of the lemma. First, observe that, since $\mathfrak{p}$ is nesting-free, for each vertex y of $\mathfrak{p}$, there is at most one incoming arc to y and one outcoming arc from y. For this reason, for any vertex y of $\mathfrak{p}$, the total number of incoming arcs to vertices $x \leqslant y$ of $\mathfrak{p}$ is smaller than or equal to the total number of outcoming arcs to vertices $x \leqslant y$ of $\mathfrak{p}$, and the total number of vertices having an incoming arc is equal to the total number of vertices having an outcoming arc in $\mathfrak{p}$. Thus, by forgetting the colorings of its letters, the word $\phi(\mathfrak{p})$ is a Dyck path.

Besides, given a Dyck path u of size $n+1$ wherein letters a at even positions are colored by $\overline{\mathcal{M}}$, one can build a unique $\mathcal{M}$-clique $\mathfrak{p}$ of $\mathrm{Nes}\,\mathcal{M}(n)$ such that $\phi(\mathfrak{p}) = u$. Indeed, by reading the letters of u two by two, we know the number of outcoming and incoming arcs for each vertex of $\mathfrak{p}$. Since $\mathfrak{p}$ is nesting-free, there is one unique way to connect these vertices by solid diagonals without creating nestings of arcs. Moreover, by *(1)*, *(2)*, *(3)*, and *(4)*, the colors of the letters a at even positions allow to label the solid arcs of $\mathfrak{p}$. Hence ϕ is a bijection as claimed.

Finally, by definition of ϕ, we observe that, if $\mathfrak{p}$ has exactly k solid arcs, the Dyck path $\phi(\mathfrak{p})$ has exactly k occurrences of the letter a at even positions. This implies the complete statement of the lemma. □

Let $\mathrm{nar}(n, k)$ be the *Narayana number* [Nar55] defined for all $0 \leqslant k \leqslant n-2$ by

$$\mathrm{nar}(n, k) := \frac{1}{k+1}\binom{n-2}{k}\binom{n-1}{k}. \tag{3.1.20}$$

The number of Dyck paths of size $n-1$ and exactly k occurrences of the factor ab is $\mathrm{nar}(n, k)$. Equivalently, this is also the number of binary trees with n leaves and exactly k internal nodes having an internal node as a left child.

Proposition 3.1.8. *Let $\mathcal{M}$ be a finite unitary magma without nontrivial unit divisors. For all $n \geqslant 2$,*

$$\dim \mathrm{Nes}\, \mathcal{M}(n) = \sum_{0 \leqslant k \leqslant n} (m-1)^k \, \mathrm{nar}(n+2, k), \tag{3.1.21}$$

where $m := \# \mathcal{M}$.

Proof. It is known from [Sul98] that the number of Dyck paths of size $n+1$ with k occurrences of the letter a at even positions is the Narayana number $\mathrm{nar}(n+2, k)$. Hence, by using this property together with Lemma 3.1.7, we obtain that the number of nesting-free $\mathcal{M}$-cliques of size n with k solid arcs is $(m-1)^k \, \mathrm{nar}(n+2, k)$. Therefore, since a nesting-free $\mathcal{M}$-clique of arity n can have at most n solid arcs, (3.1.21) holds. □

The skeletons of the $\mathcal{M}$-cliques of $\mathrm{Nes}\, \mathcal{M}$ of arities greater than 1 are the graphs such that, if $\{x, y\}$ and $\{x', y'\}$ are two arcs such that $x \leqslant x' < y' \leqslant y$, then $x = x'$ and $y = y'$. Therefore, $\mathrm{Nes}\, \mathcal{M}$ can be seen as an operad on such colored graphs, where the arcs of the graphs have one color from the set $\overline{\mathcal{M}}$. Equivalently, as Lemma 3.1.7 shows, $\mathrm{Nes}\, \mathcal{M}$ can be seen as an operad of Dyck paths where letters a at even positions are colored by $\overline{\mathcal{M}}$.

By Proposition 3.1.8, if $\# \mathcal{M} = 2$, the dimensions of $\mathrm{Nes}\, \mathcal{M}$ begin by

$$1, 5, 14, 42, 132, 429, 1430, 4862, \tag{3.1.22}$$

and form, except for the first terms, Sequence A000108 of [Slo]. If $\# \mathcal{M} = 3$, the dimensions of $\mathrm{Nes}\, \mathcal{M}$ begin by

$$1, 11, 45, 197, 903, 4279, 20793, 103049, \tag{3.1.23}$$

and form, except for the first terms, Sequence A001003 of [Slo]. If $\# \mathcal{M} = 4$, the dimensions of $\mathrm{Nes}\, \mathcal{M}$ begin by

$$1, 19, 100, 562, 3304, 20071, 124996, 793774, \tag{3.1.24}$$

and form, except for the first terms, Sequence A007564 of [Slo].

3.1.7. *Acyclic decorated cliques.* Let $\mathfrak{R}_{\mathrm{Acy}\, \mathcal{M}}$ be the subspace of $\mathrm{C}\mathcal{M}$ generated by all $\mathcal{M}$-cliques that are not acyclic. As a quotient of graded vector spaces,

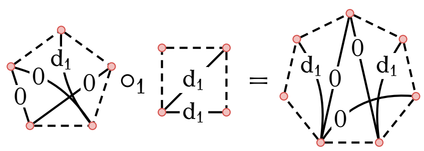
$$\mathrm{Acy}\, \mathcal{M} := \mathrm{C}\mathcal{M} / \mathfrak{R}_{\mathrm{Acy}\, \mathcal{M}} \tag{3.1.25}$$

is the linear span of all acyclic $\mathcal{M}$-cliques.

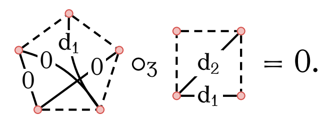
Proposition 3.1.9. *Let $\mathcal{M}$ be a unitary magma without nontrivial unit divisors. Then the space $\mathrm{Acy}\, \mathcal{M}$ is a quotient operad of $\mathrm{C}\mathcal{M}$.*

Proof. Since $\mathcal{M}$ has no nontrivial unit divisors, for any $\mathcal{M}$-cliques $\mathfrak{p}$ and $\mathfrak{q}$ of $\mathrm{C}\mathcal{M}$, each solid arc of $\mathfrak{p}$ (respectively $\mathfrak{q}$) gives rise to a solid arc in $\mathfrak{p} \circ_i \mathfrak{q}$, for any valid integer i. For this reason, if $\mathfrak{p}$ is an $\mathcal{M}$-clique of $\mathfrak{R}_{\mathrm{Acy}\,\mathcal{M}}$, $\mathfrak{p}$ is not acyclic and each $\mathcal{M}$-clique obtained by a partial composition involving $\mathfrak{p}$ and other $\mathcal{M}$-cliques is still not acyclic and thus, belongs to $\mathfrak{R}_{\mathrm{Acy}\,\mathcal{M}}$. This proves that $\mathfrak{R}_{\mathrm{Acy}\,\mathcal{M}}$ is an operad ideal of $\mathrm{C}\mathcal{M}$ and implies the statement of the proposition. □

For instance, in the operad $\mathrm{Acy}\,\mathbb{D}_2$,



(3.1.26a)



(3.1.26b)

The skeletons of the $\mathcal{M}$-cliques of $\mathrm{Acy}\,\mathcal{M}$ of arities greater than 1 are acyclic graphs or equivalently, forests of non-rooted trees. Therefore, $\mathrm{Acy}\,\mathcal{M}$ can be seen as an operad on colored forests of trees, where the edges of the trees of the forests have one color from the set $\overline{\mathcal{M}}$. If $\#\mathcal{M} = 2$, the dimensions of $\mathrm{Acy}\,\mathcal{M}$ begin by

$$1, 7, 38, 291, 2932, 36961, 561948, 10026505, \tag{3.1.27}$$

and form, except for the first terms, Sequence A001858 of [Slo].

3.2. **Secondary substructures.** Some more substructures of $\mathrm{C}\mathcal{M}$ are constructed and briefly studied here. They are constructed by mixing some of the constructions of the seven main substructures of $\mathrm{C}\mathcal{M}$ defined in Section 3.1 in the following sense.

For any operad $\mathcal{O}$ and operad ideals $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of $\mathcal{O}$, the space $\mathfrak{R}_1 + \mathfrak{R}_2$ is still an operad ideal of $\mathcal{O}$, and $\mathcal{O}/(\mathfrak{R}_1 + \mathfrak{R}_2)$ is a quotient of both $\mathcal{O}/\mathfrak{R}_1$ and $\mathcal{O}/\mathfrak{R}_2$. Moreover, if $\mathcal{O}'$ is a suboperad of $\mathcal{O}$ and $\mathfrak{R}$ is an operad ideal of $\mathcal{O}$, the space $\mathfrak{R} \cap \mathcal{O}'$ is an operad ideal of $\mathcal{O}'$, and $\mathcal{O}'/(\mathfrak{R} \cap \mathcal{O}')$ is a quotient of $\mathcal{O}'$ and a suboperad of $\mathcal{O}/\mathfrak{R}$. For these reasons (straightforwardly provable), we can combine the constructions of the previous section to build plenty new suboperads and quotients of $\mathrm{C}\mathcal{M}$ (see Table 2).

3.2.1. *Colored white noncrossing configurations.* If $\mathcal{M}$ is a unitary magma, let

$$\mathrm{WNC}\,\mathcal{M} := \mathrm{Whi}\,\mathcal{M}/\mathfrak{R}_{\mathrm{Cro}_0\,\mathcal{M}} \cap \mathrm{Whi}\,\mathcal{M}. \tag{3.2.1}$$

The $\mathcal{M}$-cliques of $\mathrm{WNC}\,\mathcal{M}$ are white noncrossing $\mathcal{M}$-cliques.

If $\#\mathcal{M} = 2$, the dimensions of $\mathrm{WNC}\,\mathcal{M}$ begin by

$$1, 1, 3, 11, 45, 197, 903, 4279, \tag{3.2.2}$$

and form Sequence A001003 of [Slo]. If $\#\mathcal{M} = 3$, the dimensions of $\mathrm{WNC}\,\mathcal{M}$ begin by

$$1, 1, 5, 31, 215, 1597, 12425, 99955, \tag{3.2.3}$$

Operad	Objects	Ideal of $\mathrm{C}\mathcal{M}$
$\mathrm{WNC}\,\mathcal{M}$	White noncrossing cliques	$\mathfrak{R}_{\mathrm{Cro}_0\,\mathcal{M}} \cap \mathrm{Whi}\,\mathcal{M}$
$\mathrm{Pat}\,\mathcal{M}$	Forests of paths	$\mathfrak{R}_{\mathrm{Deg}_2\,\mathcal{M}} + \mathfrak{R}_{\mathrm{Acy}\,\mathcal{M}}$
$\mathrm{For}\,\mathcal{M}$	Forests	$\mathfrak{R}_{\mathrm{Cro}_0\,\mathcal{M}} + \mathfrak{R}_{\mathrm{Acy}\,\mathcal{M}}$
$\mathrm{Mot}\,\mathcal{M}$	Motzkin configurations	$\mathfrak{R}_{\mathrm{Cro}_0\,\mathcal{M}} + \mathfrak{R}_{\mathrm{Deg}_1\,\mathcal{M}}$
$\mathrm{Dis}\,\mathcal{M}$	Dissections of polygons	$\left(\mathfrak{R}_{\mathrm{Cro}_0\,\mathcal{M}} + \mathfrak{R}_{\mathrm{Deg}_1\,\mathcal{M}}\right) \cap \mathrm{Whi}\,\mathcal{M}$
$\mathrm{Luc}\,\mathcal{M}$	Lucas configurations	$\mathfrak{R}_{\mathrm{Bub}\,\mathcal{M}} + \mathfrak{R}_{\mathrm{Deg}_1\,\mathcal{M}}$

TABLE 2. Operads obtained as quotients of $\mathrm{C}\mathcal{M}$ by mixing certain ideals of $\mathrm{C}\mathcal{M}$. All these operads depend on a unitary magma $\mathcal{M}$ which has, in some cases, to satisfy some precise conditions.

and form Sequence A269730 of [Slo]. Observe that these dimensions are shifted versions the ones of the γ-polytridendriform operads TDendr_γ [Gir16b] with $\gamma := \#\mathcal{M} - 1$.

3.2.2. *Colored forests of paths.* If $\mathcal{M}$ is a unitary magma without nontrivial unit divisors, let

$$\mathrm{Pat}\,\mathcal{M} := \mathrm{C}\mathcal{M}/\left(\mathfrak{R}_{\mathrm{Deg}_2\,\mathcal{M}} + \mathfrak{R}_{\mathrm{Acy}\,\mathcal{M}}\right). \tag{3.2.4}$$

The skeletons of the $\mathcal{M}$-cliques of $\mathrm{Pat}\,\mathcal{M}$ are forests of non-rooted trees that are paths. Therefore, $\mathrm{Pat}\,\mathcal{M}$ can be seen as an operad on such colored graphs, where the arcs of the graphs have one color from the set $\overline{\mathcal{M}}$.

If $\#\mathcal{M} = 2$, the dimensions of $\mathrm{Pat}\,\mathcal{M}$ begin by

$$1, 7, 34, 206, 1486, 12412, 117692, 1248004, \tag{3.2.5}$$

an form, except for the first terms, Sequence A011800 of [Slo].

3.2.3. *Colored forests.* If $\mathcal{M}$ is a unitary magma without nontrivial unit divisors, let

$$\mathrm{For}\,\mathcal{M} := \mathrm{C}\mathcal{M}/\left(\mathfrak{R}_{\mathrm{Cro}_0\,\mathcal{M}} + \mathfrak{R}_{\mathrm{Acy}\,\mathcal{M}}\right). \tag{3.2.6}$$

The skeletons of the $\mathcal{M}$-cliques of $\mathrm{For}\,\mathcal{M}$ are forests of rooted trees having no arcs $\{x, y\}$ and $\{x', y'\}$ satisfying $x < x' < y < y'$. Therefore, $\mathrm{For}\,\mathcal{M}$ can be seen as an operad on such colored forests, where the edges of the forests have one color from the set $\overline{\mathcal{M}}$. If $\#\mathcal{M} = 2$, the dimensions of $\mathrm{For}\,\mathcal{M}$ begin by

$$1, 7, 33, 81, 1083, 6854, 45111, 305629, \tag{3.2.7}$$

and form, except for the first terms, Sequence A054727, of [Slo].

3.2.4. *Colored Motzkin configurations.* If $\mathcal{M}$ is a unitary magma without nontrivial unit divisors, let

$$\mathrm{Mot}\,\mathcal{M} := \mathrm{C}\mathcal{M}/\left(\mathfrak{R}_{\mathrm{Cro}_0\,\mathcal{M}} + \mathfrak{R}_{\mathrm{Deg}_1\,\mathcal{M}}\right). \tag{3.2.8}$$

The skeletons of the $\mathcal{M}$-cliques of $\mathrm{Mot}\,\mathcal{M}$ are configurations of non-intersecting chords on a circle. Equivalently, these objects are graphs of involutions (see Section 3.1.5) having no arcs $\{x, y\}$ and $\{x', y'\}$ satisfying $x < x' < y < y'$. These objects are enumerated by Motzkin numbers [Mot48]. Therefore, $\mathrm{Mot}\,\mathcal{M}$ can be seen as an operad on such colored graphs, where the arcs of the graphs have one color from the set $\overline{\mathcal{M}}$. If $\#\mathcal{M} = 2$, the dimensions of $\mathrm{Mot}\,\mathcal{M}$ begin by

$$1, 4, 9, 21, 51, 127, 323, 835, \tag{3.2.9}$$

and form, except for the first terms, Sequence A001006, of [Slo].

3.2.5. *Colored dissections of polygons.* If $\mathcal{M}$ is a unitary magma without nontrivial unit divisors, let

$$\mathrm{Dis}\,\mathcal{M} := \mathrm{Whi}\,\mathcal{M}/\left(\left(\mathfrak{R}_{\mathrm{Cro}_0\,\mathcal{M}} + \mathfrak{R}_{\mathrm{Deg}_1\,\mathcal{M}}\right) \cap \mathrm{Whi}\,\mathcal{M}\right). \tag{3.2.10}$$

The skeletons of the $\mathcal{M}$-cliques of $\mathrm{Dis}\,\mathcal{M}$ are *strict dissections of polygons*, that are graphs of Motzkin configurations with no arcs of the form $\{x, x + 1\}$ or $\{1, n + 1\}$, where $n + 1$ is the number of vertices of the graphs. Therefore, $\mathrm{Dis}\,\mathcal{M}$ can be seen as an operad on such colored graphs, where the arcs of the graphs have one color from the set $\overline{\mathcal{M}}$. If $\#\mathcal{M} = 2$, the dimensions of $\mathrm{Dis}\,\mathcal{M}$ begin by

$$1, 1, 3, 6, 13, 29, 65, 148, \tag{3.2.11}$$

and form, except for the first terms, Sequence A093128 of [Slo].

3.2.6. *Colored Lucas configurations.* If $\mathcal{M}$ is a unitary magma without nontrivial unit divisors, let

$$\mathrm{Luc}\,\mathcal{M} := \mathrm{C}\mathcal{M}/\left(\mathfrak{R}_{\mathrm{Bub}\,\mathcal{M}} + \mathfrak{R}_{\mathrm{Deg}_1\,\mathcal{M}}\right). \tag{3.2.12}$$

The skeletons of the $\mathcal{M}$-cliques of $\mathrm{Luc}\,\mathcal{M}$ are graphs such that all vertices are of degree at most 1 and all arcs are of the form $\{x, x + 1\}$ or $\{1, n + 1\}$, where $n + 1$ is the number of vertices of the graphs. Therefore, $\mathrm{Luc}\,\mathcal{M}$ can be seen as an operad on such colored graphs, where the arcs of the graphs have one color from the set $\overline{\mathcal{M}}$. If $\#\mathcal{M} = 2$, the dimensions of $\mathrm{Luc}\,\mathcal{M}$ begin by

$$1, 4, 7, 11, 18, 29, 47, 76, \tag{3.2.13}$$

and form, except for the first terms, Sequence A000032 of [Slo].

3.3. **Relations between substructures.** The suboperads and quotients of $\mathsf{C}\mathcal{M}$ constructed in Sections 3.1 and 3.2 are linked by injective or surjective operad morphisms. To establish these, we begin with the following lemma.

Lemma 3.3.1. *Let $\mathcal{M}$ be a unitary magma. Then:*

(i) the space $\mathfrak{R}_{\mathrm{Acy}\,\mathcal{M}}$ is a subspace of $\mathfrak{R}_{\mathrm{Deg}_1\,\mathcal{M}}$;
(ii) the spaces $\mathfrak{R}_{\mathrm{Nes}\,\mathcal{M}}$ and $\mathfrak{R}_{\mathrm{Bub}\,\mathcal{M}}$ are subspaces of $\mathfrak{R}_{\mathrm{Deg}_0\,\mathcal{M}}$;
(iii) the spaces $\mathfrak{R}_{\mathrm{Cro}_0\,\mathcal{M}}$ and $\mathfrak{R}_{\mathrm{Deg}_2\,\mathcal{M}}$ are subspaces of $\mathfrak{R}_{\mathrm{Bub}\,\mathcal{M}}$;
(iv) the spaces $\mathfrak{R}_{\mathrm{Deg}_2\,\mathcal{M}}$ and $\mathfrak{R}_{\mathrm{Acy}\,\mathcal{M}}$ are subspaces of $\mathfrak{R}_{\mathrm{Nes}\,\mathcal{M}}$.

Proof. All the spaces appearing in the statement of the lemma are subspaces of $\mathsf{C}\mathcal{M}$ generated by some subfamilies of $\mathcal{M}$-cliques. Therefore, to prove the assertions of the lemma, we shall prove inclusions of adequate subfamilies of such objects.

If $\mathfrak{p}$ is an $\mathcal{M}$-clique of $\mathfrak{R}_{\mathrm{Acy}\,\mathcal{M}}$, by definition, $\mathfrak{p}$ has a cycle formed by solid arcs. Hence, $\mathfrak{p}$ has in particular a solid arc and a vertex of degree 2 or more. For this reason, since $\mathfrak{R}_{\mathrm{Deg}_1\,\mathcal{M}}$ is the linear span of all $\mathcal{M}$-cliques of degree 2 or more, $\mathfrak{p}$ is in $\mathfrak{R}_{\mathrm{Deg}_1\,\mathcal{M}}$. This implies *(i)*.

If $\mathfrak{p}$ is an $\mathcal{M}$-clique of $\mathfrak{R}_{\mathrm{Nes}\,\mathcal{M}}$ or $\mathfrak{R}_{\mathrm{Bub}\,\mathcal{M}}$, by definition, $\mathfrak{p}$ has in particular a solid arc. Hence, since $\mathfrak{R}_{\mathrm{Deg}_0\,\mathcal{M}}$ is the linear span of all $\mathcal{M}$-cliques with at least one vertex with a positive degree, $\mathfrak{p}$ is in $\mathfrak{R}_{\mathrm{Deg}_0\,\mathcal{M}}$. This implies *(ii)*.

If $\mathfrak{p}$ is an $\mathcal{M}$-clique of $\mathfrak{R}_{\mathrm{Cro}_0\,\mathcal{M}}$ or $\mathfrak{R}_{\mathrm{Deg}_2\,\mathcal{M}}$, $\mathfrak{p}$ has in particular a solid diagonal. Indeed, if $\mathfrak{p}$ is in $\mathfrak{R}_{\mathrm{Cro}_0\,\mathcal{M}}$ this property is immediate. If $\mathfrak{p}$ is in $\mathfrak{R}_{\mathrm{Deg}_2\,\mathcal{M}}$, since $\mathfrak{p}$ has a vertex x of degree 3 or more, the skeleton of $\mathfrak{p}$ has three arcs $\{x, y_1\}$, $\{x, y_2\}$, and $\{x, y_3\}$ with $y_i \neq x - 1$, $y_i \neq x + 1$, and $y_i \neq |\mathfrak{p}| + 1$ for at least one $i \in [3]$, so that the arc $(\min\{x, y_i\}, \max\{x, y_i\})$ is a solid diagonal of $\mathfrak{p}$. For this reason, since $\mathfrak{R}_{\mathrm{Bub}\,\mathcal{M}}$ is the linear span of all $\mathcal{M}$-cliques with at least one solid diagonal, $\mathfrak{p}$ is in $\mathfrak{R}_{\mathrm{Bub}\,\mathcal{M}}$. This implies *(iii)*.

If $\mathfrak{p}$ is an $\mathcal{M}$-clique of $\mathfrak{R}_{\mathrm{Deg}_2\,\mathcal{M}}$ or $\mathfrak{R}_{\mathrm{Acy}\,\mathcal{M}}$, $\mathfrak{p}$ has in particular a solid arc nested in another one. Indeed, if $\mathfrak{p}$ is in $\mathfrak{R}_{\mathrm{Deg}_2\,\mathcal{M}}$, since $\mathfrak{p}$ has a vertex x of a degree 3 or more, the skeleton of $\mathfrak{p}$ has three arcs $\{x, y_1\}$, $\{x, y_2\}$, and $\{x, y_3\}$. One can check that, for all relative orders between the vertices x, y_1, y_2, and y_3, one of these arcs is nested in another one, so that $\mathfrak{p}$ is not nesting-free. If $\mathfrak{p}$ is in $\mathfrak{R}_{\mathrm{Acy}\,\mathcal{M}}$, $\mathfrak{p}$ contains a cycle formed by solid arcs. Let x_1, x_2, $\dots$, x_k, $k \geqslant 3$, be the vertices of $\mathfrak{p}$ that form this cycle. We can assume without loss of generality that $x_1 \leqslant x_i$ for all $i \in [k]$ and thus, that (x_1, x_2) and (x_1, x_k) are solid arcs of $\mathfrak{p}$ being part of the cycle. Then, if $x_2 < x_k$, since $x_1 \leqslant x_1 < x_2 \leqslant x_k$, the arc (x_1, x_2) is nested in (x_1, x_k). Otherwise, $x_k < x_2$, and since $x_1 \leqslant x_1 < x_k \leqslant x_2$, the arc (x_1, x_k) is nested in (x_1, x_2). For these reasons, since $\mathfrak{R}_{\mathrm{Nes}\,\mathcal{M}}$ is the linear span of all $\mathcal{M}$-cliques that are non nesting-free, $\mathfrak{p}$ is in $\mathfrak{R}_{\mathrm{Nes}\,\mathcal{M}}$. This implies *(iv)*. □

3.3.1. *Relations between the main substructures.* Here we list and explain the morphisms between the main substructures of $\mathrm{C}\mathcal{M}$. First, Lemma 3.3.1 implies that there are surjective operad morphisms from $\mathrm{Acy}\,\mathcal{M}$ to $\mathrm{Deg}_1\,\mathcal{M}$, from $\mathrm{Nes}\,\mathcal{M}$ to $\mathrm{Deg}_0\,\mathcal{M}$, from $\mathrm{Bub}\,\mathcal{M}$ to $\mathrm{Deg}_0\,\mathcal{M}$, from $\mathrm{Cro}_0\,\mathcal{M}$ to $\mathrm{Bub}\,\mathcal{M}$, from $\mathrm{Deg}_2\,\mathcal{M}$ to $\mathrm{Bub}\,\mathcal{M}$, from $\mathrm{Deg}_2\,\mathcal{M}$ to $\mathrm{Nes}\,\mathcal{M}$, and from $\mathrm{Acy}\,\mathcal{M}$ to $\mathrm{Nes}\,\mathcal{M}$. Second, if B, E, and D are subsets of $\mathcal{M}$ such that $\mathbb{1}_{\mathcal{M}} \in B$, $\mathbb{1}_{\mathcal{M}} \in E$, and $E \star B \subseteq D$, $\mathrm{Whi}\,\mathcal{M}$ is a suboperad of $\mathrm{Lab}_{B,E,D}\,\mathcal{M}$. Finally, there is a surjective operad morphism from $\mathrm{Whi}\,\mathcal{M}$ to the associative operad As sending any $\mathcal{M}$-clique $\mathfrak{p}$ of $\mathrm{Whi}\,\mathcal{M}$ to the unique basis element of As of the same arity as the one of $\mathfrak{p}$. The relations between the main suboperads and quotients of $\mathrm{C}\mathcal{M}$ built here are summarized in the diagram of operad morphisms of Figure 2.

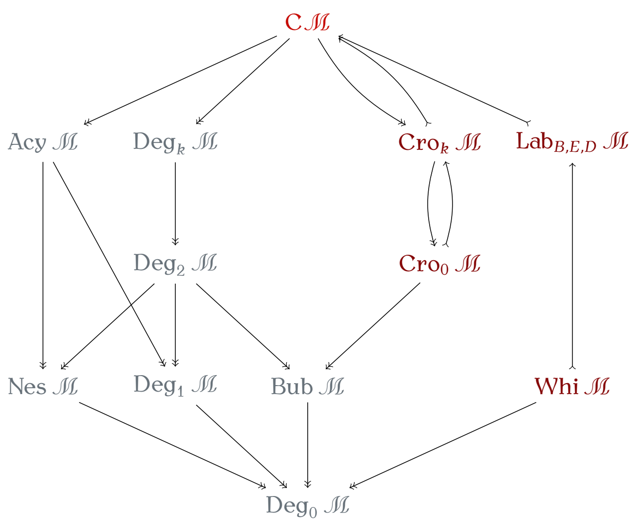


FIGURE 2. The diagram of the main suboperads and quotients of $\mathrm{C}\mathcal{M}$. Arrows $\rightarrowtail$ (respectively $\twoheadrightarrow$) are injective (respectively surjective) operad morphisms. Here, $\mathcal{M}$ is a unitary magma without nontrivial unit divisors, k is a positive integer, and B, E, and D are subsets of $\mathcal{M}$ such that $\mathbb{1}_{\mathcal{M}} \in B$, $\mathbb{1}_{\mathcal{M}} \in E$, and $E \star B \subseteq D$.

3.3.2. *Relations between the secondary and main substructures.* Here we list and explain the morphisms between the secondary and main substructures of $\mathrm{C}\mathcal{M}$. First, immediately from their definitions, $\mathrm{WNC}\,\mathcal{M}$ is a suboperad of $\mathrm{Cro}_0\,\mathcal{M}$ and a quotient of $\mathrm{Whi}\,\mathcal{M}$, $\mathrm{Pat}\,\mathcal{M}$ is both a quotient of $\mathrm{Deg}_2\,\mathcal{M}$ and $\mathrm{Acy}\,\mathcal{M}$, $\mathrm{For}\,\mathcal{M}$ is both a quotient of $\mathrm{Cro}_0\,\mathcal{M}$ and $\mathrm{Acy}\,\mathcal{M}$, $\mathrm{Mot}\,\mathcal{M}$ is both a quotient of $\mathrm{Cro}_0\,\mathcal{M}$ and $\mathrm{Deg}_1\,\mathcal{M}$, $\mathrm{Dis}\,\mathcal{M}$ is a suboperad of $\mathrm{Mot}\,\mathcal{M}$ and a quotient of $\mathrm{WNC}\,\mathcal{M}$, and $\mathrm{Luc}\,\mathcal{M}$ is both a quotient of $\mathrm{Bub}\,\mathcal{M}$ and $\mathrm{Deg}_1\,\mathcal{M}$. Moreover, since by Lemma 3.3.1, $\mathfrak{R}_{\mathrm{Acy}\,\mathcal{M}}$ is a subspace of $\mathfrak{R}_{\mathrm{Deg}_1\,\mathcal{M}}$, $\mathfrak{R}_{\mathrm{Deg}_2\,\mathcal{M}}$ and $\mathfrak{R}_{\mathrm{Acy}\,\mathcal{M}}$ are subspaces of $\mathfrak{R}_{\mathrm{Nes}\,\mathcal{M}}$, and $\mathfrak{R}_{\mathrm{Cro}_0\,\mathcal{M}}$ is a subspace of $\mathfrak{R}_{\mathrm{Bub}\,\mathcal{M}}$, we have that $\mathfrak{R}_{\mathrm{Deg}_2\,\mathcal{M}} + \mathfrak{R}_{\mathrm{Acy}\,\mathcal{M}}$ is a subspace of both $\mathfrak{R}_{\mathrm{Deg}_1\,\mathcal{M}}$ and $\mathfrak{R}_{\mathrm{Nes}\,\mathcal{M}}$, $\mathfrak{R}_{\mathrm{Cro}_0\,\mathcal{M}} + \mathfrak{R}_{\mathrm{Acy}\,\mathcal{M}}$ is a subspace of $\mathfrak{R}_{\mathrm{Cro}_0\,\mathcal{M}} + \mathfrak{R}_{\mathrm{Deg}_1\,\mathcal{M}}$, and $\mathfrak{R}_{\mathrm{Cro}_0\,\mathcal{M}} + \mathfrak{R}_{\mathrm{Deg}_1\,\mathcal{M}}$ is a subspace of $\mathfrak{R}_{\mathrm{Bub}\,\mathcal{M}} + \mathfrak{R}_{\mathrm{Deg}_1\,\mathcal{M}}$. For these reasons, there are surjective operad morphisms from $\mathrm{Pat}\,\mathcal{M}$ to $\mathrm{Deg}_1\,\mathcal{M}$, from $\mathrm{Pat}\,\mathcal{M}$ to $\mathrm{Nes}\,\mathcal{M}$, from $\mathrm{For}\,\mathcal{M}$ to $\mathrm{Mot}\,\mathcal{M}$, and from $\mathrm{Mot}\,\mathcal{M}$ to $\mathrm{Luc}\,\mathcal{M}$. The relations between the secondary suboperads and quotients of $\mathrm{C}\mathcal{M}$ built here are summarized in the diagram of operad morphisms of Figure 3.

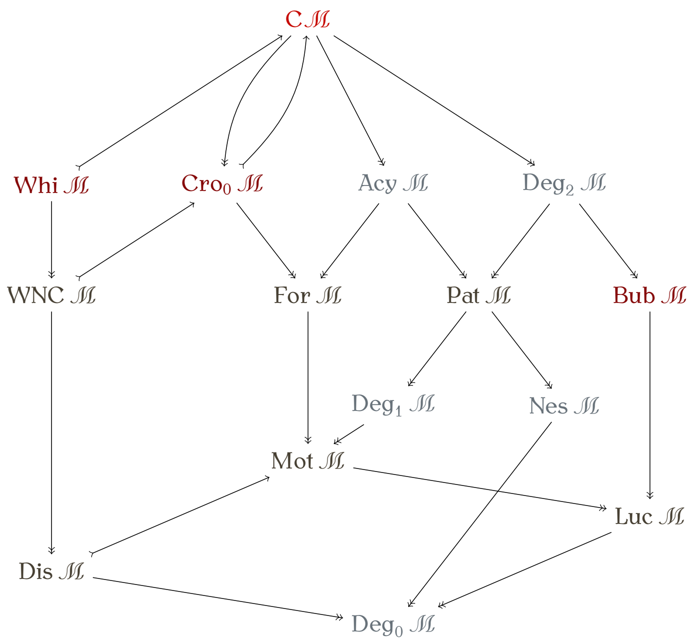


FIGURE 3. The diagram of the secondary suboperads and quotients of $\mathrm{C}\mathcal{M}$ together with some of their related main suboperads and quotients of $\mathrm{C}\mathcal{M}$. Arrows $\rightarrowtail$ (respectively $\twoheadrightarrow$) are injective (respectively surjective) operad morphisms. Here, $\mathcal{M}$ is a unitary magma without nontrivial unit divisors.

4. Concrete constructions

The clique construction provides alternative definitions of known operads. We explore here the cases of the operads MT and DMT of multi-tildes and double multi-tildes, and the gravity operad Grav.

4.1. Operads from language theory. We provide constructions of two operads coming from formal language theory by using the clique construction.

4.1.1. *Multi-tildes.* Multi-tildes are operators introduced in [CCM11] in the context of formal language theory as a convenient way to express regular languages. Let, for any $n \geqslant 1$, P_n be the set

$$P_n := \left\{(x, y) \in [n]^2 : x \leqslant y\right\}. \tag{4.1.1}$$

A *multi-tilde* is a pair $(n, \mathfrak{s})$, where n is a positive integer and $\mathfrak{s}$ is a subset of P_n. The *arity* of the multi-tilde $(n, \mathfrak{s})$ is n.

As shown in [LMN13], the graded (by the arity) collection of all multi-tildes admits a very natural structure of an operad. This operad, denoted by MT, is defined as follows. The partial composition $(n, \mathfrak{s}) \circ_i (m, \mathfrak{t})$, $i \in [n]$, of two multi-tildes $(n, \mathfrak{s})$ and $(m, \mathfrak{t})$ is defined by

$$(n, \mathfrak{s}) \circ_i (m, \mathfrak{t}) := \left(n + m - 1, \{\mathrm{sh}_i^m(x, y) : (x, y) \in \mathfrak{s}\} \cup \left\{\mathrm{sh}_0^i(x, y) : (x, y) \in \mathfrak{t}\right\}\right), \tag{4.1.2}$$

where

$$\mathrm{sh}_j^p(x, y) := \begin{cases} (x, y), & \text{if } y \leqslant i - 1, \\ (x, y + p - 1), & \text{if } x \leqslant i \leqslant y, \\ (x + p - 1, y + p - 1), & \text{otherwise.} \end{cases} \tag{4.1.3}$$

For instance, we have

$$(5, \{(1, 5), (2, 4), (4, 5)\}) \circ_4 (6, \{(2, 2), (4, 6)\}) = (10, \{(1, 10), (2, 9), (4, 10), (5, 5), (7, 9)\}), \tag{4.1.4a}$$

$$(5, \{(1, 5), (2, 4), (4, 5)\}) \circ_5 (6, \{(2, 2), (4, 6)\}) = (10, \{(1, 10), (2, 4), (4, 10), (6, 6), (8, 10)\}). \tag{4.1.4b}$$

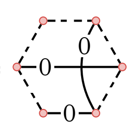
Observe that the multi-tilde $(1, \emptyset)$ is the unit of MT.

Let $\phi_{\mathrm{MT}} : \mathrm{MT} \to \mathrm{C}\mathbb{D}_0$ be the linear map defined as follows. For any multi-tilde $(n, \mathfrak{s})$ different from $(1, \{(1, 1)\})$, $\phi_{\mathrm{MT}}((n, \mathfrak{s}))$ is the $\mathbb{D}_0$-clique of arity n defined, for any $1 \leqslant x < y \leqslant n + 1$, by

$$\phi_{\mathrm{MT}}((n, \mathfrak{s}))(x, y) := \begin{cases} 0, & \text{if } (x, y - 1) \in \mathfrak{s}, \\ \mathbb{1}, & \text{otherwise.} \end{cases} \tag{4.1.5}$$

For instance,

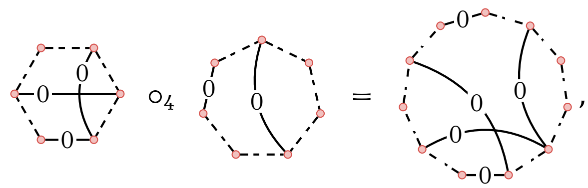
$$\phi_{\mathrm{MT}}((5, \{(1, 5), (2, 4), (4, 5)\})) = \tag{4.1.6}$$



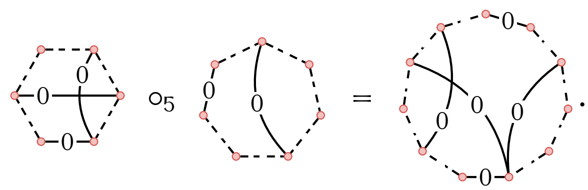
Proposition 4.1.1. *The operad* $\mathrm{C}\mathbb{D}_0$ *is isomorphic to the suboperad of* MT *consisting in the linear span of all multi-tildes except the nontrivial multi-tilde* $(1, \{(1, 1)\})$ *of arity* 1. *Moreover,* ϕ_{MT} *is an isomorphism between these two operads.*

Proof. A direct consequence of the definition (4.1.5) of ϕ_{MT} is that this map is an isomorphism of vector spaces. Moreover, it follows from the definitions of the partial compositions of MT and $\mathrm{C}\mathbb{D}_0$ that ϕ_{MT} is an operad morphism. □

By Proposition 4.1.1, we can interpret the partial compositions (4.1.4a) and (4.1.4b) of multi-tildes as partial compositions of $\mathbb{D}_0$-cliques. This yields



(4.1.7a)



(4.1.7b)

4.1.2. *Double multi-tildes.* Double multi-tildes are natural generalizations of multi-tildes, introduced in [GLMN16]. A *double multi-tilde* is a triple $(n, \mathfrak{s}, \mathfrak{t})$, where $(n, \mathfrak{t})$ and $(n, \mathfrak{s})$ are both multi-tildes of the same arity n. The *arity* of the double multi-tilde $(n, \mathfrak{s}, \mathfrak{t})$ is n. As shown in [GLMN16], the linear span of all double multi-tildes admits a structure of an operad. This operad, denoted by DMT, is defined as follows. For any $n \geqslant 1$, $\mathrm{DMT}(n)$ is the linear span of all double multi-tildes of arity n and the partial composition $(n, \mathfrak{s}, \mathfrak{t}) \circ_i (m, \mathfrak{u}, \mathfrak{v})$, $i \in [n]$, of two double multi-tildes $(n, \mathfrak{s}, \mathfrak{t})$ and $(m, \mathfrak{u}, \mathfrak{v})$ is defined linearly by

$$(n, \mathfrak{s}, \mathfrak{t}) \circ_i (m, \mathfrak{u}, \mathfrak{v}) := (n, \mathfrak{s} \circ_i \mathfrak{u}, \mathfrak{t} \circ_i \mathfrak{v}), \tag{4.1.8}$$

where the two partial compositions $\circ_i$ of the right member of (4.1.8) are the ones of MT. We can observe that DMT is isomorphic to the Hadamard product $\mathrm{MT} * \mathrm{MT}$. For instance, we have

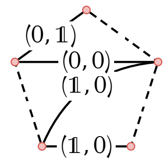
$$\begin{aligned}(3, \{(2,2)\}, \{(1,2), (1,3)\}) \circ_2 (2, \{(1,1)\}, \{(1,2)\}) \\ = (4, \{(2,2), (2,3)\}, \{(1,3), (1,4), (2,3)\}).\end{aligned} \tag{4.1.9}$$

The unit of DMT is $(1, \emptyset, \emptyset)$.

Consider now the operad $\mathrm{C}\mathbb{D}_0^2$ and let $\phi_{\mathrm{DMT}} : \mathrm{DMT} \to \mathrm{C}\mathbb{D}_0^2$ be the linear map defined as follows. The image by ϕ_{DMT} of $(1, \emptyset, \emptyset)$ is the unit of $\mathrm{C}\mathbb{D}_0^2$ and, for any double multi-tilde $(n, \mathfrak{s}, \mathfrak{t})$ of arity $n \geqslant 2$, $\phi_{\mathrm{DMT}}((n, \mathfrak{s}, \mathfrak{t}))$ is the $\mathbb{D}_0^2$-clique of arity n defined, for any $1 \leqslant x < y \leqslant n+1$, by

$$\phi_{\mathrm{DMT}}((n, \mathfrak{s}, \mathfrak{t}))(x, y) := \begin{cases} (0, \mathbb{1}), & \text{if } (x, y-1) \in \mathfrak{s} \text{ and } (x, y-1) \notin \mathfrak{t}, \\ (\mathbb{1}, 0), & \text{if } (x, y-1) \notin \mathfrak{s} \text{ and } (x, y-1) \in \mathfrak{t}, \\ (0, 0), & \text{if } (x, y-1) \in \mathfrak{s} \text{ and } (x, y-1) \in \mathfrak{t}, \\ (\mathbb{1}, \mathbb{1}), & \text{otherwise.} \end{cases} \tag{4.1.10}$$

For instance,

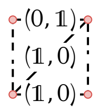
$$\phi_{\mathrm{DMT}}((4, \{(2,2),(2,3)\}, \{(1,3),(1,4),(2,3)\})) =$$ 

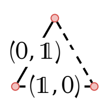
. (4.1.11)

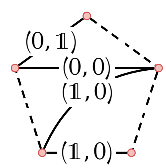
Proposition 4.1.2. *The operad* $\mathcal{C}\mathbb{D}_0^2$ *is isomorphic to the suboperad of* DMT *consisting in the linear span of all double multi-tildes except the three nontrivial double multi-tildes of arity* 1*. Moreover,* ϕ_{DMT} *is an isomorphism between these two operads.*

Proof. There are two ways to prove the first assertion of the statement of the proposition. On the one hand, this property follows from Proposition 2.1.2 and Proposition 4.1.1. On the other hand, the whole statement of the proposition is a direct consequence of the definition (4.1.10) of ϕ_{DMT}, showing that ϕ_{DMT} is an isomorphism of vector spaces, and, from the definitions of the partial compositions of DMT and $\mathcal{C}\mathbb{D}_0^2$ showing that ϕ_{DMT} is an operad morphism. □

By Proposition 4.1.2, we can interpret the partial composition (4.1.9) of double multi-tildes as a partial composition of $\mathbb{D}_0^2$-cliques. This gives



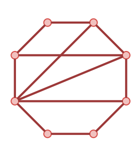
$\circ_2$ 

$=$ 

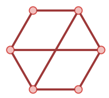
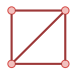
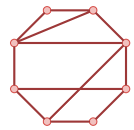
. (4.1.12)

4.2. **Gravity operad.** The *operad of gravity chord diagrams* Grav is an operad defined in [AP15]. This operad is the nonsymmetric version (obtained by forgetting the actions of the symmetric groups) of the gravity operad, a symmetric operad introduced by Getzler [Get94]. Let us describe this operad.

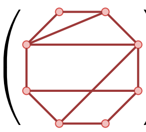
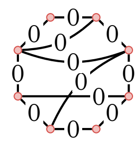
A *gravity chord diagram* is a $\{\star\}$-configuration $\mathfrak{c}$, where $\star$ is any symbol, satisfying the following conditions. By denoting by n the size of $\mathfrak{c}$, all the edges and the base of $\mathfrak{c}$ are labeled (by $\star$), and if (x, y) and (x', y') are two labeled crossing diagonals of $\mathfrak{c}$ such that $x < x'$, the arc (x', y) is not labeled. In other words, the quadrilateral formed by the vertices x, x', y, and y' of $\mathfrak{c}$ is such that its side (x', y) is unlabeled. For instance,


(4.2.1)

is a gravity chord diagram of arity 7 having four labeled diagonals (observe in particular that, as required, the arc $(3, 5)$ is not labeled). For any $n \geqslant 2$, $\mathrm{Grav}(n)$ is the linear span of all gravity chord diagrams of size n. Moreover, $\mathrm{Grav}(1)$ is the linear span of the singleton containing the only polygon of size 1 where its only arc is not labeled. The partial composition of Grav is defined graphically as follows. For any gravity chord diagrams $\mathfrak{c}$ and $\mathfrak{d}$ of respective arities n and m, and $i \in [n]$, the gravity chord diagram $\mathfrak{c} \circ_i \mathfrak{d}$ is obtained by gluing the base of $\mathfrak{d}$ onto the ith edge of $\mathfrak{c}$, so that the arc $(i, i+m)$ of $\mathfrak{c} \circ_i \mathfrak{d}$ is labeled. For example,

 $\circ_3$  $=$  . (4.2.2)

Let $\phi_{\mathrm{Grav}} : \mathrm{Grav} \to \mathcal{C}\mathbb{D}_0$ be the linear map defined in the following way. For any gravity chord diagram $\mathfrak{c}$, $\phi_{\mathrm{Grav}}(\mathfrak{c})$ is the $\mathbb{D}_0$-clique of $\mathcal{C}\mathbb{D}_0$ obtained by replacing all labeled arcs of $\mathfrak{c}$ by arcs labeled by 0 and all unlabeled arcs by arcs labeled by $\mathbb{1}$. For instance,

ϕ_{Grav} () $=$  . (4.2.3)

Let us say that an $\mathcal{M}$-clique $\mathfrak{p}$ satisfies the *gravity condition* if $\mathfrak{p} =$ ∘--∘, or $\mathfrak{p}$ has only solid edges and bases, and for all crossing diagonals (x, y) and (x', y') of $\mathfrak{p}$ such that $x < x'$, $\mathfrak{p}(x, y) \neq \mathbb{1}_{\mathcal{M}} \neq \mathfrak{p}(x', y')$ implies $\mathfrak{p}(x', y) = \mathbb{1}_{\mathcal{M}}$.

Proposition 4.2.1. *The linear span of all $\mathbb{D}_0$-cliques satisfying the gravity condition forms a suboperad of $\mathcal{C}\mathbb{D}_0$ isomorphic to* Grav*. Moreover, ϕ_{Grav} is an isomorphism between these two operads.*

Proof. Let us denote by $\mathcal{O}_{\mathrm{Grav}}$ the subspace of $\mathcal{C}\mathbb{D}_0$ described in the statement of the proposition. First of all, it follows from the definition of the partial composition of $\mathcal{C}\mathbb{D}_0$ that $\mathcal{O}_{\mathrm{Grav}}$ is closed under the partial composition operation (this property can be also seen as a consequence of the fact that the partial composition of two gravity chord diagrams is still a gravity chord diagram [AP15]). Hence, and since $\mathcal{O}_{\mathrm{Grav}}$ contains the unit of $\mathcal{C}\mathbb{D}_0$, $\mathcal{O}_{\mathrm{Grav}}$ is an operad. Second, observe that the image of ϕ_{Grav} is the underlying space of $\mathcal{O}_{\mathrm{Grav}}$ and, from the definition of the partial composition of Grav, one can check that ϕ_{Grav} is an operad morphism. Finally, since ϕ_{Grav} is a bijection from Grav to $\mathcal{O}_{\mathrm{Grav}}$, the statement of the proposition follows. □

Proposition 4.2.1 shows hence that the operad Grav can be built through the clique construction. Moreover, as explained in [AP15], Grav contains the nonsymmetric version of the Lie operad, the symmetric operad describing the category of Lie algebras. This nonsymmetric version of the Lie operad as been introduced in [ST09]. Since Lie is contained in Grav as the subspace of all gravity chord diagrams having the maximal number of labeled diagonals for each arity, Lie can be built through the clique construction as the suboperad of $\mathcal{C}\mathbb{D}_0$ containing all the $\mathbb{D}_0$-cliques that are images by ϕ_{Grav} of such maximal gravity chord diagrams.

Besides, this alternative construction of Grav leads to the following generalization for any unitary magma $\mathcal{M}$ of the gravity operad. Let $\mathrm{Grav}_{\mathcal{M}}$ be the linear span of all $\mathcal{M}$-cliques satisfying the gravity condition. It follows from the definition of the partial composition of $\mathrm{C}\mathcal{M}$ that $\mathrm{Grav}_{\mathcal{M}}$ is an operad. Moreover, observe that, if $\mathcal{M}$ has nontrivial unit divisors, $\mathrm{Grav}_{\mathcal{M}}$ is not a free operad.

Conclusion and perspectives

This work presents and studies the clique construction C, producing operads from unitary magmas. We have seen that C has many both algebraic and combinatorial properties. Among its most notable ones, $\mathrm{C}\mathcal{M}$ admits several quotients involving combinatorial families of decorated cliques, and contains some already studied operads. Let us address here some open questions.

First, we have for the time being no formula to enumerate prime (respectively white prime, minimal prime) $\mathcal{M}$-cliques (see (2.2.6) (respectively (2.2.7), (2.2.8)) for $\#\mathcal{M} = 2$). Obtaining these forms a first combinatorial question.

If $\mathcal{M}$ is a $\mathbb{Z}$-graded unitary magma, a link between $\mathrm{C}\mathcal{M}$ and the operad of rational functions RatFct has been developed in Section 2.2.8 by means of a morphism F_θ between these two operads. We have observed that F_θ is not injective (see (2.2.32a) and (2.2.32b)). A description of the kernel of F_θ, even if $\mathcal{M}$ is the unitary magma $\mathbb{Z}$, seems not easy to obtain. Trying to obtain this description is a second perspective of this work.

Here is a third perspective. In Section 3, we have defined and briefly studied some quotients and suboperads of $\mathrm{C}\mathcal{M}$. In particular, we have considered the quotient $\mathrm{Deg}_1 \mathcal{M}$ of $\mathrm{C}\mathcal{M}$, involving $\mathcal{M}$-cliques of degree at most 1. As mentioned, $\mathrm{Deg}_1 \mathbb{D}_0$ is an operad defined on the linear span of involutions (except the nontrivial involution of $\mathfrak{S}_2$). A complete study of this operad seems worth considering, including a description of a minimal generating set, a presentation by generators and relations, a description of its partial composition on the H-basis and on the K-basis, and a realization of this operad in terms of standard Young tableaux.

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