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A NOTE ON: RECTANGULAR SCHRÖDER PARKING FUNCTIONS COMBINATORICS

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Abstract. We study Schröder paths drawn in an (m, n) rectangle, for any positive integers m and n. We get explicit enumeration formulas, closely linked to those for the corresponding (m, n)-Dyck paths. Moreover, we study a Schröder version of (m, n)-parking functions, and associated (q, t)-analogs.

Contents

1. Introduction

Many objects are enumerated by Schröder numbers:

$$1, 2, 6, 22, 90, 394, 1806, \ldots \qquad \text{(sequence A006318 in [13])},$$

but the most classical are probably paths from $(0, 0)$ to (n, n) with steps either $(1, 0)$, $(0, 1)$, or $(1, 1)$, never going below the diagonal (although they may touch it). These are generalizations of Dyck paths (for which only the first two types of steps are allowed), well-known to be enumerated by the Catalan numbers. The aim of this work is to investigate properties of an analogous notion for the $m \times n$-rectangle, rather than for the $n \times n$-square, the latter corresponding to the classical case. Our (m, n)-Schröder paths (defined in Section 2) have the same kind of steps, but we change the endpoint to (m, n), and use the diagonal of the rectangle to crop the set of allowed paths. Partial results are already known, in the case where m and n are coprime, in particular when $m = rn+1$ (see [14]). We first obtain (see Corollary 2)

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a symmetric function enumeration formula for the general case (no coprimality condition). This result, as well as the methods used to obtain it, closely parallels the case of Dyck paths studied in [2]. Next, we consider the notion of Schröder parking functions, i.e., labeled rectangular Schröder paths, and investigate associated q-analogs and (q,t)-analogs of our enumeration formulas; with the parameter q accounting for the area between the path and the diagonal.

2. Schröder path enumeration

Schröder paths. Our **(m,n)-Schröder paths** are sequences

$$(0,0)=(x_0,y_0),(x_1,y_1),\ldots,(x_N,y_N)=(m,n)$$

with (x_k,y_k) in $\mathbb{N}\times\mathbb{N}$, and such that

- $(x_{k+1},y_{k+1})=(x_k,y_k)+(s,t)$, with $(s,t)\in\{(0,1),(1,1),(1,0)\}$,
- such that $my_k-nx_k\geq 0$, for all k.

Depending on (s,t) being equal to $(0,1)$, $(1,1)$, or $(1,0)$ in the first condition, we say that we have an **up**, **diagonal**, or **right** step. The inequality of the second condition ensures that the paths does not go below the diagonal.

Schröder polynomials. Recall that the enumerating polynomials for (n,n)-Schröder paths are given by the formula

$$S_n(y):=\sum_{k=0}^{n}\frac{1}{n-k+1}\binom{n}{k}\binom{2n-k}{n}y^k, \tag{1}$$

with $S_0=1$. These “Schröder polynomials” have a long and interesting history. The coefficient of y^k, denoted by $S_n^{(k)}$, enumerates Schröder paths having k diagonal steps. Hence, the case $k=0$ corresponds to the usual notion of Dyck paths, well-known to be counted by the Catalan numbers

$$S_n^{(0)}=\frac{1}{n+1}\binom{2n}{n}.$$

Small values of these polynomials are

$$\begin{aligned}
S_1(y)&=1+y,\\
S_2(y)&=2+3\,y+y^2,\\
S_3(y)&=5+10\,y+6\,y^2+y^3,\\
S_4(y)&=14+35\,y+30\,y^2+10\,y^3+y^4,\\
S_5(y)&=42+126\,y+140\,y^2+70\,y^3+15\,y^4+y^5.
\end{aligned}$$

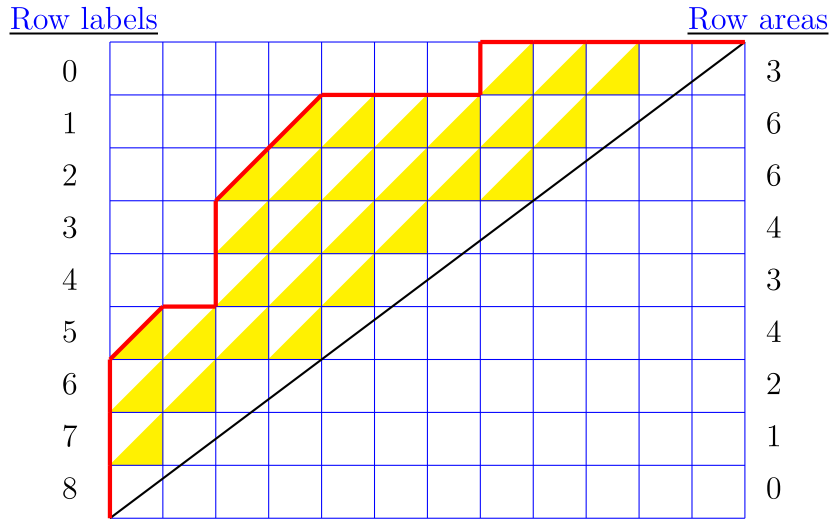

FIGURE 1. The $(12, 9)$-Schröder path $000\overline{0}22\overline{2}\overline{3}7$.

We consider here an (m, n)-rectangular analog of these polynomials, for any pair (m, n) of positive integers. Just as for Dyck paths, the case when m and n are coprime is somewhat simpler. To make this more apparent notation-wise, we often write our pairs of integers in the form $(m, n) = (ac, bc)$, with (a, b) coprime. Hence c is the greatest common divisor of m and n. In particular, the (a, b)-case corresponds to the coprime situation. We recall that the number of (a, b)-Dyck paths is simply given by the formula

$$C_{a,b} = \frac{1}{a+b}\binom{a+b}{a}.$$

The general case of (m, n)-Dyck path enumeration, $C_{m,n}$ giving the number of such paths, is probably best presented in generating series format as

$$\sum_{d\geq 0} C_{ad,bd} z^d = \exp\left(\sum_{j>1} \frac{1}{a}\binom{ja+jb}{ja}\frac{z^j}{j}\right). \tag{2}$$

The proof of this formula, going back to 1954, is due to Bizley [7], who attributes it to Grossman [10].

Sequence encoding. It will be practical to encode our paths as sequences $\alpha = a_0 a_1 \cdots a_{n-1}$ of barred or unbarred integers, with one a_i for each up or diagonal step of the path, reading them from top to bottom. For up steps we set $a_i = k$, whereas diagonal steps are encoded $a_i = \overline{k}$ (barred steps). In both cases, k is equal to to the number of entire "cells" that lie to the left of the unique up (or diagonal) step at height

$n-i$. The a_i are the **parts** of α. In this encoding, $\alpha = a_0 a_1 \cdots a_{n-1}$ corresponds to an (m,n)-Schröder path if and only if

(1) $a_0 \leq a_1 \leq \ldots \leq a_{n-1}$, (with the order $0 < \overline{0} < \cdots < k < \overline{k}$),
(2) if $a_i = \overline{k}$, then necessarily $a_i < a_{i+1}$, and
(3) for each i, we have $a_i \leq \overline{\lfloor i\, m/n \rfloor}$.

Each unbarred k, between 0 and m, occurs with some **multiplicity**[1] n_k in a path α. Removing 0-multiplicities, we obtain the **(multiplicity) composition** $\gamma(\alpha)$ of the sequence α, reading these multiplicities in increasing values of k. For example,

$$\gamma(0\overline{0}11\overline{1}244\overline{4}) = (1, 2, 1, 2).$$

Clearly $\gamma(\alpha)$ is a composition of $n-k$, where k stands for the number of diagonal steps in α. The parts of $\gamma(\alpha)$ may be understood as the lengths of **risers** (also called vertical runs) in the path. These are maximal sequences of consecutive up-steps.

Any (m,n)-Schröder path may be obtained by either barring or not the rightmost part of a given size in the analogous word encoding of an (m,n)-Dyck path. As we will see this makes the enumeration of Schröder paths easy, once we set up the right tools.

Symmetric function weight. As we will come to see later, it is interesting to consider a weighted enumeration of Schröder paths, with the weight lying in the graded ring

$$\Lambda = \bigoplus_{d \geq 0} \Lambda_d$$

of symmetric "functions" (polynomials in a countable set of variables $\boldsymbol{x} = x_1, x_2, x_3, \ldots$). Recall that the degree d homogeneous component Λ_d affords as a linear basis the set

$$\{e_\mu(\boldsymbol{x}) \mid \mu \vdash d\}$$

of elementary symmetric functions, with

$$e_\mu(\boldsymbol{x}) := e_{\mu_1}(\boldsymbol{x}) e_{\mu_2}(\boldsymbol{x}) \cdots e_{\mu_\ell}(\boldsymbol{x}),$$

for $\mu = \mu_1 \mu_2 \ldots \mu_\ell$ running over the set of partitions of d. In turn, each factor $e_k(\boldsymbol{x})$ is characterized by the generating function identity

$$\sum_{k \geq 0} e_k(\boldsymbol{x}) z^k = \prod_{i \geq 1} (1 + x_i\, z).$$

[1]possibly equal to 0.

with $e_0(\boldsymbol{x}) := 1$. It easily follows that

$$e_k(\boldsymbol{x}+y) = e_k(\boldsymbol{x}) + e_{k-1}(\boldsymbol{x})\, y, \tag{3}$$

where $\boldsymbol{x}+y$ means that we add a new variable y to those occurring in $\boldsymbol{x}$.

With these notions at hand, we now simply set

$$S_{m,n}(\boldsymbol{x};y) := \sum_{\alpha} \alpha(\boldsymbol{x})\, y^{\mathrm{diag}(\alpha)}, \quad \text{with} \quad \alpha(\boldsymbol{x}) := \prod_{k\in\gamma(\alpha)} e_k(\boldsymbol{x}), \tag{4}$$

where the sum is over the set of (m,n)-Schröder paths α, with $\mathrm{diag}(\alpha)$ denoting the number of diagonal steps in α. Likewise, we denote by

$$S^{(k)}_{m,n}(\boldsymbol{x}) := \sum_{\mathrm{diag}(\alpha)=k} \alpha(\boldsymbol{x}),$$

the symmetric function enumerator of (m,n)-Schröder paths with exactly k diagonal steps, so that $S_{m,n}(\boldsymbol{x};y) = \sum_k S^{(k)}_{m,n}(\boldsymbol{x})\, y^k$. For example, we have

$$\begin{aligned}
S_{1,1}(\boldsymbol{x};y) &= e_1(\boldsymbol{x}) + y,\\
S_{2,2}(\boldsymbol{x};y) &= (e_{11}(\boldsymbol{x}) + e_2(\boldsymbol{x})) + 3\, e_1(\boldsymbol{x})\, y + y^2,\\
S_{3,3}(\boldsymbol{x};y) &= (e_{111}(\boldsymbol{x}) + 3\, e_{21}(\boldsymbol{x}) + e_3(\boldsymbol{x}))\\
&\qquad +(6\, e_{11}(\boldsymbol{x}) + 4\, e_2(\boldsymbol{x}))y + 6\, e_1(\boldsymbol{x})y^2 + y^3.
\end{aligned}$$

Observe that, for all r and n, we have

$$S_{rn+1,n}(\boldsymbol{x};y) = S_{rn,n}(\boldsymbol{x};y), \tag{5}$$

since the last step of $(rn+1,n)$ must necessarily be a right step, and the coprimality of $rn+1$ and n implies that staying above the $(rn+1,n)$-diagonal insures as well that we stay above the (rn,n)-diagonal. Hence we get the same set of paths.

To make some expressions more compact, we shall use "plethystic notation", recalling that we have

$$e_n[m\,\boldsymbol{x}] := \sum_{\nu\vdash n} (-1)^{n-\ell(\nu)}\, m^{\ell(\nu)} \frac{p_\nu(\boldsymbol{x})}{z_\nu}, \tag{6}$$

with m considered as a "constant" in the calculation of plethysm. This means that $p[m\boldsymbol{x}] = m\, p[\boldsymbol{x}]$. Recall that it is usual, for any partition ν of n, to denote its length by $\ell(\nu)$, and to use the notation

$$z_\nu := 1^{d_1} d_1! 2^{d_2} d_2! \dots n^{d_n} d_n!,$$

where d_i is the number of copies of the part i in ν.

Main results. Recall from [2] that we have a Bizley-like formula for the symmetric function enumeration

$$\mathcal{C}_{m,n}(\boldsymbol{x}) := \sum_{\mathrm{diag}(\alpha)=0} \alpha(\boldsymbol{x})$$

of (m,n)-Dyck paths, namely

$$\sum_{d\geq 0} \mathcal{C}_{ad,bd}(\boldsymbol{x})\, z^d = \exp\left(\sum_{j\geq 1} e_{jb}[j\, a\, \boldsymbol{x}]\, \frac{z^j}{aj}\right), \tag{7}$$

with a and j considered as constants in the plethysm. We can exploit this formula to get one for our weighted enumeration of Schröder paths. Indeed,

Proposition 1. *For all m and n, we have*

$$S_{m,n}(\boldsymbol{x}; y) = \mathcal{C}_{m,n}(\boldsymbol{x} + y). \tag{8}$$

Proof. Let us fix the dimensions m, n of the rectangular box. Recall that, for any given $k \geq 0$, $S^{(k)}_{m,n}$ stands for the set of (m,n)-Schröder paths with exactly k diagonal steps. We denote by $\mathcal{C}_{m,n}$ the set $S^{(0)}_{m,n}$ of (m,n)-Dyck paths.

For $\beta \in \mathcal{C}_{m,n}$, let us denote by $\gamma(\beta) = (\gamma_1, \ldots, \gamma_j)$ the composition giving the lengths of the vertical runs of β. In view of (4) and (3), we may write

$$\begin{aligned}
\mathcal{C}_{m,n}[\boldsymbol{x} + y] &= \sum_{\beta\in\mathcal{C}_{m,n}} \prod_{\gamma_i\in\gamma(\beta)} e_{\gamma_i}[\boldsymbol{x} + y] \\
&= \sum_{\beta\in\mathcal{C}_{m,n}} \prod_{\gamma_i\in\gamma(\beta)} (e_{\gamma_i}[\boldsymbol{x}] + y\, e_{\gamma_i - 1}[\boldsymbol{x}]) \\
&= \sum_{\beta\in\mathcal{C}_{m,n}} \sum_{k\geq 0} y^k \sum_{\delta} e_{\delta}[y],
\end{aligned} \tag{9}$$

where the last sum is over all possible compositions δ obtained by reducing exactly k of the parts of $\gamma(\beta)$ by 1.

Let us now consider the projection

$$\pi \ :\ S^{(k)}_{m,n} \longrightarrow \mathcal{C}_{m,n}$$

which replaces any diagonal step by a North step followed by an East step, i.e., an (outer) corner. For a Dyck path β, let us denote by $S^{(k)}_{m,n}(\beta) := \pi^{-1}(\beta)$ the inverse image of β under π. It is clear that the sets $S^{(k)}_{m,n}(\beta)$, for β running over $\mathcal{C}_{m,n}$, are the blocks of a partition of

$S^{(k)}_{m,n}$. We may thus rewrite the inner sum in (9) as

$$\sum_{\delta} e_\delta[y] = \sum_{\alpha\in S^{(k)}_{m,n}(\beta)} e_{\gamma(\alpha)}[\boldsymbol{x}],$$

and get

$$\begin{aligned}\mathcal{C}_{m,n}[\boldsymbol{x}+y] &= \sum_{\beta\in\mathcal{C}_{m,n}} \sum_{k\geq 0} y^k \sum_{\alpha\in S^{(k)}_{m,n}(\beta)} e_{\gamma(\alpha)}[\boldsymbol{x}]\\ &= \sum_{k\geq 0} y^k \sum_{\alpha\in S^{(k)}_{m,n}} e_{\gamma(\alpha)}[\boldsymbol{x}] = S_{m,n}(\boldsymbol{x};y)\end{aligned}$$

as announced. □

From Equation (7), we may derive the following reformulation of Proposition 1.

Corollary 2. *The generating function of rectangular Schröder polynomials is*

$$\sum_{d\geq 0} S_{ad,bd}(\boldsymbol{x})\, z^d = \exp\left(\sum_{j\geq 1} e_{jb}[ja\,(\boldsymbol{x}+y)]\, \frac{z^j}{aj}\right). \tag{10}$$

For example, for any a and b coprime, we get

$$\begin{aligned}S_{a,b}(\boldsymbol{x}+y) &= \frac{1}{a} e_b[a\,(\boldsymbol{x}+y)],\\ S_{2a,2b}(\boldsymbol{x}+y) &= \frac{1}{2a} e_{2b}[2a\,(\boldsymbol{x}+y)] + \frac{1}{2a^2}\, e_b[a\,(\boldsymbol{x}+y)]^2,\\ S_{3a,3b}(\boldsymbol{x}+y) &= \frac{1}{3a} e_{3b}[3a\,(\boldsymbol{x}+y)] + \frac{1}{2a^2}\, e_b[a\,(\boldsymbol{x}+y)]\, e_{2b}[2a\,(\boldsymbol{x}+y)]\\ &\quad + \frac{1}{6a^3}\, e_b[a\,(\boldsymbol{x}+y)]^3.\end{aligned}$$

Recall the following well-known expansion[2] of (6) in terms of elementary symmetric functions:

$$e_n[m\,\boldsymbol{x}] = \sum_{\nu\vdash n} \binom{m}{\ell(\nu)} \binom{\ell(\nu)}{d_\nu} e_\nu(\boldsymbol{x}),$$

where $\binom{\ell(\nu)}{d_\nu}$ stands for the multinomial coefficient

$$\binom{\ell(\nu)}{d_\nu} := \binom{\ell(\nu)}{d_1, d_2, \dots, d_n},$$

[2]This is essentially an instance of the dual Cauchy formula.

with each d_i equal to the multiplicity of the part i in ν. Using this formula, we get

$$\begin{aligned} S_{a,b}(\boldsymbol{x};y) &= \frac{1}{a} e_b[a\,(\boldsymbol{x}+y)] \\ &= \frac{1}{a}\sum_{k=0}^{b} e_{b-k}[a\,\boldsymbol{x}]\, e_k[a\,y] \\ &= \frac{1}{a}\sum_{k=0}^{b} \binom{a}{k}\, e_{b-k}[a\,\boldsymbol{x}]\, y^k \\ &= \frac{1}{a}\sum_{k} y^k \sum_{\nu\vdash b-k} \binom{a}{k}\binom{a}{\ell(\nu)}\binom{\ell(\nu)}{d_\nu} e_\nu(\boldsymbol{x}). \end{aligned} \tag{11}$$

Moreover, we may write the coefficient of y^k in (11) as the following integer coefficient linear combination of the $e_\nu(\boldsymbol{x})$:

$$S_{a,b}^{(k)}(\boldsymbol{x}) = \sum_{\nu\vdash b-k} \frac{1}{a}\binom{a}{k}\binom{a}{\ell(\nu)}\binom{\ell(\nu)}{d_\nu} e_\nu(\boldsymbol{x}).$$

Since $\langle e_\mu(\boldsymbol{x}), \sum_{j\geq 0} e_j(\boldsymbol{x})\rangle = 1$ for all partition μ, we immediately get

$$S_{a,b}^{(k)} = \Big\langle S_{a,b}^{(k)}(\boldsymbol{x}), \sum_{j\geq 0} e_j(\boldsymbol{x})\Big\rangle,$$

where $\langle -,-\rangle$ stands for the usual scalar product on symmetric function[3]. Otherwise stated, for a and b coprime,

$$S_{a,b}^{(k)} = \sum_{\nu\vdash b-k} \frac{1}{a}\binom{a}{k}\binom{a}{\ell(\nu)}\binom{\ell(\nu)}{d_\nu}.$$

In particular, in view of (5), this covers the classical case ($m=n$) as well as the generalized version ($m=rn$) of [14]. One also deduces from Proposition 1 the following generalization of a result of Haglund [9].

Proposition 3. *For all m and n, we have*

$$S_{m,n}^{(k)} = \Big\langle \mathcal{C}_{m,n}(\boldsymbol{x}), e_{n-k}(\boldsymbol{x}) h_k(\boldsymbol{x})\Big\rangle.$$

Proof. We start by recalling the symmetric function identity

$$f(\boldsymbol{x}+y) = \sum_{k\geq 0} y^k\, h_k^\perp\, f(\boldsymbol{x}), \tag{12}$$

where $h_k^\perp$ stands for the dual of the operator of multiplication by $h_k(\boldsymbol{x})$ with respect to the symmetric function scalar product. Equation (12)

[3] For which $\langle p_\mu, p_\nu\rangle = z_\mu\, \delta_{\mu,\nu}$

may be checked by computation for the basis element $f = e_\lambda$ for λ a partition $\lambda = (\lambda_1, \dots, \lambda_t)$, by using the classical fact that

$$h_k^\perp(e_\lambda) = \sum_{|\rho|=k} \prod_{i=1}^{t} e_{\lambda_i - \rho_i},$$

where the sum is over sequences ρ such that $0 \le \rho_i \le 1$.

It follows directly from (8) that

$$\begin{aligned}
\sum_k S_{m,n}^{(k)}\, y^k &= \Big\langle \mathcal{C}_{m,n}(\boldsymbol{x} + y), \sum_{k\ge 0} e_k(\boldsymbol{x}) \Big\rangle, \\
&= \Big\langle \sum_{k\ge 0} y^k\, h_k^\perp\, \mathcal{C}_{m,n}(\boldsymbol{x}), \sum_{j\ge 0} e_j(\boldsymbol{x}) \Big\rangle, \\
&= \Big\langle \mathcal{C}_{m,n}(\boldsymbol{x}), \sum_{k\ge 0} y^k h_k(\boldsymbol{x}) \sum_{j\ge 0} e_j(\boldsymbol{x}) \Big\rangle, \\
&= \sum_{k\ge 0} \Big\langle \mathcal{C}_{m,n}(\boldsymbol{x}), h_k(\boldsymbol{x}) e_{n-k}(\boldsymbol{x}) \Big\rangle\, y^k.
\end{aligned}$$

The last equality comes from the fact that $\mathcal{C}_{m,n}(\boldsymbol{x})$ is homogeneous of degree n, hence all terms of the wrong degree vanish in the scalar product. Evidently we get the announced result by comparing powers of y of the same degree in both sides of the identity obtained. □

Area enumerator. The i^{th} **row area** of a path α in $\mathcal{S}_n^{(r)}$ is the integer

$$\mathrm{area}_i(\alpha) := \lfloor i\, m/n \rfloor - |a_i|,$$

where we set $|\overline{k}| := k$. Summing over all indices i between 1 and n, we get the **area** of α:

$$\mathrm{area}(\alpha) := \sum_{i=0}^{n-1} \mathrm{area}_i(\alpha).$$

This generalizes a notion of area on Schröder paths introduced for the case $m = n$ in [8] (and further studied in [3]) to (m, n)-Schröder paths. Following the presentation of [9], this may also be understood as the number of "upper" triangles lying below the path and above the diagonal line (as illustrated in Figure 1). These triangles are also called **area triangles**. We have the area q-enumerator symmetric function

$$S_{m,n}(\boldsymbol{x}; y, q) := \sum_\alpha \alpha(\boldsymbol{x})\, q^{\mathrm{area}(\alpha)} y^{\mathrm{diag}(\alpha)}.$$

Keeping up with our previous notational conventions, we also set

$$S_{m,n}^{(k)}(q) := \sum_{\mathrm{diag}(\alpha)=k} q^{\mathrm{area}(\alpha)},$$

and

$$\mathcal{C}_{m,n}(\boldsymbol{x};q) := \sum_{\mathrm{diag}(\alpha)=0} \alpha(\boldsymbol{x})\, q^{\mathrm{area}(\alpha)}.$$

Proposition 4. *For all m and n, we have*

$$S_{m,n}(\boldsymbol{x};y,q) = \mathcal{C}_{m,n}(\boldsymbol{x}+y;q), \tag{13}$$

and

$$S^{(k)}_{m,n}(q) = \Big\langle \mathcal{C}_{m,n}(\boldsymbol{x};q), e_{n-k}(\boldsymbol{x})h_k(\boldsymbol{x})\Big\rangle. \tag{14}$$

Proof. This comes easily from the fact that the projection π in the proof of Proposition 1 preserves the area. □

From a result of [11], it follows that $\mathcal{C}_{rn+1,n}(\boldsymbol{x};q) = \mathcal{C}_{rn,n}(\boldsymbol{x};q) = \nabla^r(e_n)\big|_{t=1}$, where ∇ is a Macdonald "eigenoperator" introduced in [5]. By this, we mean that its eigenfunctions are the (combinatorial) q,t-Macdonald polynomials. Thus, a special instance of (14) may be formulated as

$$S^{(k)}_{rn,n}(q) = \Big\langle \nabla^r(e_n)\big|_{t=1}, e_{n-k}(\boldsymbol{x})h_k(\boldsymbol{x})\Big\rangle. \tag{15}$$

In this way, we get back the case $t=1$ of Proposition 1 in [9].

3. Constant term formula

The following constant term formula adds an extra parameter to our story. We conjecture that it corresponds to a (q,t)-enumeration of (m,n)-Schröder parking functions, with t accounting for a "dinv"-statistic, still to be defined.

Conjecture 5. *We have*

$$S_{m,n}(\boldsymbol{x};y,q,t) =$$

$$\mathrm{CT}_{z_m,\dots,z_0}\left(\frac{1}{\boldsymbol{z}_{m,n}}\prod_{i=1}^{m}\frac{z_i(1+y\,z_i)}{z_i-q\,z_{i+1}}\Omega'(\boldsymbol{x};z_i)\prod_{j=i+1}^{m}\frac{(z_i-z_j)(z_i-qt\,z_j)}{(z_i-qz_j)(z_i-tz_j)}\right),$$

where $\Omega'(\boldsymbol{x};z) := \sum_{k\geq 0} e_k(\boldsymbol{x})\,z^k$, and $\boldsymbol{z}_{m,n} := \prod_{i=0}^{n-1} z_{\lfloor im/n\rfloor}$.

We recall that some care must be used in evaluating multivariate constant term expressions. Indeed, the order in which successive constant terms are taken does have an impact on the overall result. This is why, in the above formula, the indices appearing after "CT" specify that this should be done starting with z_m, and then going down to z_0. For example, we have

$$\begin{aligned}
S_{2,2}(\boldsymbol{x};y,q,t) &= (s_2+(q+t)\,s_{11}) + (q+t+1)\,s_1\,y + y^2,\\
S_{2,3}(\boldsymbol{x};y,q,t) &= (s_{21}+(q+t)\,s_{111}) + (s_2+(q+t+1)s_{11})\,y + s_1\,y^2,\\
S_{2,4}(\boldsymbol{x};y,q,t) &= (s_{22}+(q+t)\,s_{211}+(q^2+qt+t^2)\,s_{1111})\\
&\quad +((q+t+1)\,s_{21}+(q^2+qt+t^2+q+t)\,s_{111})\,y\\
&\quad +(s_2+(q+t)\,s_{11})\,y^2.
\end{aligned}$$

We underline that Conjecture 5 is simply the evaluation at $\boldsymbol{x}+y$ of a similar formula conjectured in [12] in relation with (m,n)-parking functions. More precisely, it is conjectured in the mentioned paper, that

Conjecture 6 (Negut). *We have*

$$\begin{aligned}
&\mathcal{C}_{m,n}(\boldsymbol{x};q,t)\\
&= \mathrm{CT}_{z_m,\dots,z_0}\left(\frac{1}{\boldsymbol{z}_{m,n}}\prod_{i=1}^{m}\frac{z_i}{z_i-q\,z_{i+1}}\Omega'(\boldsymbol{x};z_i)\prod_{j=i+1}^{m}\frac{(z_i-z_j)(z_i-qt\,z_j)}{(z_i-qz_j)(z_i-tz_j)}\right).
\end{aligned}$$

From this identity it is clear that Conjecture 5 follows, using the equality $\Omega'(\boldsymbol{x}+y;z_i)=(1+y\,z_i)\,\Omega'(\boldsymbol{x};z_i)$. One may readily show that the specialization at $t=1$ of the right-hand side of Conjecture 5 does indeed give back our previous $\mathcal{C}_{m,n}(\boldsymbol{x}+y;q)=S_{m,n}(\boldsymbol{x};y,q)$, since the relevant constant term formula is shown to hold in [6].

This, together with the results and conjectures that appear in [6], opens up many new avenues of exploration. In particular, we may obtain explicit candidates for the (q,t)-enumeration of special families of (m,n)-Schröder paths (say with return conditions to the diagonal), by the simple device of evaluating analogous symmetric function formulas for (m,n)-Dyck paths at $\boldsymbol{x}+y$. Several questions regarding this are explored in [4].

4. Schröder parking functions

An **(m,n)-Schröder parking function** is a bijective labeling of the up steps of an (m,n)-Schröder path α by the elements of $\{1,2,\dots,n-\mathrm{diag}(\alpha)\}$. One further imposes the condition that consecutive up steps of same x-coordinate have decreasing labels reading them from top to bottom. The path involved in this description is said to be the **shape** of the parking function. For α an (m,n)-Schröder path, we denote by $\mathbb{P}(\alpha)$ the set of parking functions having shape α. When $\mathrm{diag}(\alpha)=0$,

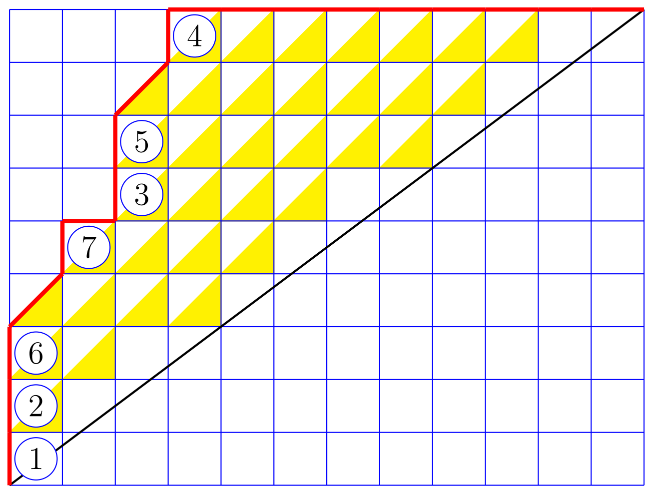

FIGURE 2. A Schröder parking function.

we get the "usual" notion of parking functions of shape α (an (m,n)-Dyck path). The (m,n)-Schröder parking functions may be understood as preference functions, with some of the parking places being closed to parking (these correspond to diagonal steps). For $f \in \mathbb{P}(\alpha)$, we denote the row area for the i^{th} row of the shape of f by $\text{area}_i(f)$. Figure 2 gives an example of a parking function of shape $000\overline{0}1\overline{1}223$.

As we did for paths, we consider the (m,n)-Schröder parking function polynomial

$$P_{m,n}(y,q) = \sum_k P_{m,n}^{(k)}\, y^k := \sum_\alpha |\mathbb{P}(\alpha)|\, q^{\text{area}(\alpha)} y^{\text{diag}(\alpha)}.$$

It is easy to derive the following fact from Corollary 4.

Corollary 7. *For all m and n, we have*

$$P_{m,n}(y,q) = \left\langle \mathcal{C}_{m,n}(\boldsymbol{x}+y;q), \frac{1}{1-p_1(\boldsymbol{x})} \right\rangle.$$

Equivalently, for all k, we have

$$P_{m,n}^{(k)}(q) = \left\langle \mathcal{C}_{m,n}(\boldsymbol{x};q), p_1(\boldsymbol{x})^{n-k} h_k(\boldsymbol{x}) \right\rangle,$$

It follows from this, and the observation preceding (15), that

$$P_{rn,n}^{(k)}(q) = \left\langle \nabla^r(e_n)\big|_{t=1}, p_1(\boldsymbol{x})^{n-k} h_k(\boldsymbol{x}) \right\rangle.$$

Also, for a and b coprime, we have $P_{a,b}^{(k)} = \binom{a}{k} a^{b-k-1}$, which in the case $k=0$ reduces to a formula obtained in [1] for the number of parking functions in the special case of Dyck paths, and a and b coprime.

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