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Surprising Relations Between Sums-Of-Squares of Characters of the Symmetric Group Over Two-Rowed Shapes and Over Hook Shapes

By Amitai REGEV and Doron ZEILBERGER

Abstract. In a recent article, we noted (and proved) that the sum of the squares of the characters of the symmetric group, $\chi^\lambda(\mu)$, over all shapes λ with two rows and n cells and $\mu = 31^{n-3}$, equals, surprisingly, to 1/2 of that sum-of-squares taken over all hook shapes with $n+2$ cells and with $\mu = 321^{n-3}$. In the present note, we show that this is only the tip of a huge iceberg! We will prove that, if μ consists of odd parts and (a possibly empty) string of *consecutive* powers of 2, namely $2, 4, \dots, 2^{t-1}$ for $t \geq 1$, then the sum of $\chi^\lambda(\mu)^2$ over all two-rowed shapes λ with n cells equals exactly $\frac{1}{2}$ times the analogous sum of $\chi^\lambda(\mu')^2$ over all shapes λ of *hook shape* with $n+2$ cells, where μ' is the partition obtained from μ by retaining all odd parts but replacing the string $2, 4, \dots, 2^{t-1}$ by 2^t.

Recall that the *constant term* of a *Laurent polynomial* in $(x_1, \dots, x_m)$ is the free term, i.e., the coefficient of $x_1^0 \cdots x_m^0$. For example,

$$\mathrm{CT}_{x_1,x_2}(x_1^{-3}x_2 + x_1x_2^{-2} + 5) = 5.$$

Recall that a *partition* (alias *shape*) of an integer n, with m *parts* (alias *rows*), is a non-increasing sequence of positive integers

$$\lambda = (\lambda_1, \dots, \lambda_m),$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$ and $\lambda_1 + \cdots + \lambda_m = n$.

If $\lambda = (\lambda_1, \dots, \lambda_m)$ and $\mu = (\mu_1, \dots, \mu_r)$ are partitions of n with m and r parts, respectively, then it easily follows from [M, p. 114, Eq. (7.8)], that the *characters*, $\chi^\lambda(\mu)$, of the *symmetric group*, S_n, may be obtained via the *constant term* expression

$$\chi^\lambda(\mu) \; = \; \mathrm{CT}_{x_1,\dots,x_m} \frac{\prod_{1\leq i<j\leq m}(1-\frac{x_j}{x_i}) \prod_{j=1}^r \left(\sum_{i=1}^m x_i^{\mu_j}\right)}{\prod_{i=1}^m x_i^{\lambda_i}}. \qquad (Chi)$$

As usual, for a partition μ, $|\mu|$ denotes the sum of its parts, in other words, the integer that is being partitioned.

In [RRZ] we considered two quantities. Let μ_0 be a partition with smallest part ≥ 2. The first quantity, that we will call henceforth $A(\mu_0)(n)$, is the following sum-of-squares over two-rowed shapes λ:

$$A(\mu_0)(n) := \sum_{j=0}^{\lfloor n/2 \rfloor} \chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})^2.$$

(Note that in [RRZ] this quantity was denoted by $\psi^{(2)}(\mu_0 1^{n-|\mu_0|})$.)

The second quantity was the sum-of-squares over *hook-shapes*

$$B(\mu_0)(n) := \sum_{j=1}^{n} \chi^{(j,1^{n-j})}(\mu_0 1^{n-|\mu_0|})^2.$$

(Note that in [RRZ] this quantity was denoted by $\phi^{(2)}(\mu_0 1^{n-|\mu_0|})$.)

In [RRZ] we developed algorithms for discovering (and then proving) closed-form expressions for these quantities, for a *given* (specific) finite partition μ_0 with smallest part larger than one. In fact we proved that each such expression is *always* a multiple of $\binom{2n}{n}$ by a certain rational function of n that depends on μ_0.

Unless μ_0 is very small, these rational functions turn out to be very complicated, but, inspired by the One-Line Encyclopedia of Integer Sequences [S], Alon Regev noted (and then it was proved in [RRZ]) the *remarkable* identity

$$A(3)(n) = \frac{1}{2} B(3,2)(n+2).$$

This led to the following natural question:

> *Are there other partitions, μ_0, such that there exists a partition, μ_0' with $|\mu_0'| = |\mu_0| + 2$, such that the ratio $A(\mu_0)(n)/B(\mu_0')(n+2)$ is a constant?*

This led us to write a new procedure in the Maple package

`http://www.math.rutgers.edu/~zeilberg/tokhniot/Sn.txt`

that accompanies [RRZ], called `SeferNisim(K,N0)`, which searched for such pairs $[\mu_0, \mu_0']$. We then used our *human* ability for *pattern recognition* to notice that all the successful pairs (we went up to $|\mu_0| \leq 20$) turned out to be such that μ_0 either consisted of only odd parts, and then μ_0' was μ_0 with 2 appended, or, more generally, μ_0 consisted of odd parts together with a string of *consecutive* powers of 2 (starting with 2), and μ_0' was obtained from μ_0 by retaining all the odd parts but replacing the string of powers of 2 by a single power of 2, one higher than the highest in μ_0. In symbols, we conjectured (and later proved [see below], *alas*, by purely human means) the following theorem.

Theorem. *Let μ_0 be a partition of the form*

$$\mu_0 = \mathrm{Sort}([a_1, \ldots, a_s, 2, 2^2, \ldots, 2^{t-1}]),$$

where

$$a_1 \geq a_2 \geq \cdots \geq a_s \geq 3$$

are all **odd** *and* $t \geq 1$. *(If $t = 1$ then μ_0 only consists of odd parts.) Define*

$$\mu_0' = \mathrm{Sort}([a_1, \ldots, a_s, 2^t]).$$

Then, for every $n \geq |\mu_0|$, we have

$$A(\mu_0)(n) = \frac{1}{2} B(\mu_0')(n+2).$$

(For a sequence of integers S, the symbol $\mathrm{Sort}(S)$ denotes that sequence sorted in non-increasing order.)

In order to prove our theorem, we need to first recall the following **constant-term** expression for $B(\mu_0)(n)$ from [RRZ].

Lemma 1. *If $\mu_0 = (a_1, \ldots, a_r)$, we have*

$$B(\mu_0)(n) = \mathrm{Coeff}_{x^0} \left[\frac{(1+x)^{2n-2-2(a_1+\cdots+a_r)}}{x^{n-1}} \cdot \prod_{i=1}^{r} (x^{a_i} - (-1)^{a_i})(1 - (-1)^{a_i} x^{a_i}) \right].$$

We need an analogous constant-term expression for $A(\mu_0)(n)$. To that end, let us first spell out Equation (Chi) for the two-rowed case, $m = 2$. In that case, we may write $\lambda = (n-j, j)$. With $\mu_0 = (a_1, \ldots, a_r)$, we have

$$\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|}) \;=\; \mathrm{CT}_{x_1,x_2} \frac{(1 - \frac{x_2}{x_1})(x_1 + x_2)^{n-a_1-\cdots-a_r} \prod_{i=1}^{r} (x_1^{a_i} + x_2^{a_i})}{x_1^{n-j} x_2^{j}}. \qquad (Chi2)$$

This can be rewritten as

$$\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|}) \;=\; \mathrm{CT}_{x_1,x_2} \frac{(1 - \frac{x_2}{x_1})(1 + \frac{x_2}{x_1})^{n-a_1-\cdots-a_r} \prod_{i=1}^{r} \left(1 + (\frac{x_2}{x_1})^{a_j}\right)}{(\frac{x_2}{x_1})^{j}}. \qquad (Chi2')$$

Since the *constant-termand* is of the form $P(\frac{x_2}{x_1})/(\frac{x_2}{x_1})^j$, for some *single-variable* polynomial $P(x)$, the above can be equivalently expressed in the form

$$\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|}) \;=\; \mathrm{Coeff}_{x^0} \frac{(1-x)(1+x)^{n-a_1-\cdots-a_r} \prod_{i=1}^{r} (1 + x^{a_i})}{x^j}. \qquad (Chi2'')$$

Note that the left-hand side is *utter nonsense* if $j > \frac{n}{2}$, but the right-hand side makes perfect sense. It is easy to see that, defining $\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})$ by the right-hand side for $j > \frac{n}{2}$, we get

$$\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|}) \;=\; -\chi^{(j,n-j)}(\mu_0 1^{n-|\mu_0|}).$$

Let us denote the numerator of the constant-termand of (Chi''), namely

$$(1-x)(1+x)^{n-a_1-\cdots-a_r}\prod_{i=1}^{r}(1+x^{a_i})\,,$$

by $P(x)$. Then Equation $(Chi2'')$ can be also rewritten as a *generating function*,

$$P(x)=\sum_{j=0}^{n}\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})\,x^j.$$

Since for any polynomial of a single variable, $P(x)=\sum_{j=0}^{n}c_jx^j$, we have

$$\sum_{j=0}^{n}c_j^2=\mathrm{Coeff}_{x^0}\big[P(x)P(x^{-1})\big]\,,$$

we get

$$\begin{aligned}\sum_{j=0}^{n}\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})^2 &= \mathrm{Coeff}_{x^0}\Bigg[\left((1-x)(1+x)^{n-a_1-\cdots-a_r}\prod_{j=1}^{r}(1+x^{a_j})\right)\\ &\qquad\cdot\left((1-x^{-1})(1+x^{-1})^{n-a_1-\cdots-a_r}\prod_{j=1}^{r}\left(1+x^{-a_j}\right)\right)\Bigg]\\ &=-\mathrm{Coeff}_{x^0}\left[\frac{(1-x)^2(1+x)^{2(n-a_1-\cdots-a_r)}\prod_{j=1}^{r}(1+x^{a_j})^2}{x^{n+1}}\right].\end{aligned}$$

But since, by symmetry,

$$\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})^2=\frac{1}{2}\sum_{j=0}^{n}\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})^2,$$

we have the following auxiliary result.

Lemma 2. *Let $\mu_0=(a_1,\dots,a_r)$ be a partition with smallest part larger than one. Then*

$$A(\mu_0)(n)=-\frac{1}{2}\,\mathrm{Coeff}_{x^0}\left[\frac{(1-x)^2(1+x)^{2(n-a_1-\cdots-a_r)}\prod_{j=1}^{r}(1+x^{a_j})^2}{x^{n+1}}\right].$$

We are now ready to prove the theorem. If $\mu_0=\mathrm{Sort}(a_1,\dots,a_r,2,\dots,2^{t-1})$, then

$$\begin{aligned}A(\mu_0)(n)=-\frac{1}{2}\,\mathrm{Coeff}_{x^0}\Bigg[&\frac{(1-x)^2(1+x)^{2(n-a_1-\cdots-a_r-2-2^2-\cdots 2^{t-1})}}{x^{n+1}}\\ &\cdot\prod_{j=1}^{t-1}\left(1+x^{2^j}\right)^2\prod_{j=1}^{r}(1+x^{a_j})^2\Bigg].\end{aligned}$$

But, transferring a factor of $(1+x)^2$ from the second factor to the product, $\prod_{j=1}^{t-1}\left(1+x^{2^j}\right)^2$, we have

$$(1+x)^{2(n-a_1-\cdots-a_r-2-2^2-\cdots 2^{t-1})}\prod_{j=1}^{t-1}\left(1+x^{2^j}\right)^2$$
$$=(1+x)^{2(n-a_1-\cdots-a_r-1-2-2^2-\cdots 2^{t-1})}\prod_{j=0}^{t-1}\left(1+x^{2^j}\right)^2.$$

Hence,

$$A(\mu_0)(n)=-\frac{1}{2}\,\mathrm{Coeff}_{x^0}\left[\frac{(1-x)^2(1+x)^{2(n-a_1-\cdots-a_r-1-2-2^2-\cdots 2^{t-1})}}{x^{n+1}}\cdot\prod_{j=0}^{t-1}\left(1+x^{2^j}\right)^2\prod_{j=1}^{r}(1+x^{a_j})^2\right].$$

By Euler's good-old $(1-x)\prod_{j=0}^{t-1}(1+x^{2^j})=1-x^{2^t}$, we conclude

$$A(\mu_0)(n)$$
$$=-\frac{1}{2}\,\mathrm{Coeff}_{x^0}\left[\frac{(1-x^{2^t})^2(1+x)^{2(n-a_1-\cdots-a_r-1-2-2^2-\cdots 2^{t-1})}\prod_{j=1}^{r}(1+x^{a_j})^2}{x^{n+1}}\right].$$

On the other hand, since $\mu_0'=\mathrm{Sort}(a_1,\ldots,a_r,2^t)$, and all the a_i's are odd, we have

$$B(\mu_0')(n+2)=-\mathrm{Coeff}_{x^0}\left[\frac{(1+x)^{2n+2-2(a_1+\cdots+a_r+2^t)}}{x^{n+1}}\cdot(x^{2^t}-1)^2\cdot\prod_{j=1}^{r}(x^{a_j}+1)^2\right].$$

This completes the proof, since $-(1+2+2^2+\cdots+2^{t-1})=1-2^t$. □

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AMITAI REGEV, DEPARTMENT OF PURE MATHEMATICS, WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL. EMAIL: `amitai dot regev at weizmann dot ac dot il`.

DORON ZEILBERGER, DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY (NEW BRUNSWICK), HILL CENTER-BUSCH CAMPUS, 110 FRELINGHUYSEN RD., PISCATAWAY, NJ 08854-8019, USA. EMAIL: `zeilberg at math dot rutgers dot edu`.