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THE PUSH-THE-BUTTON ALGORITHM FOR CONTRAGREDIENT LIE SUPERALGEBRAS

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Abstract. We apply the idea of the "push-the-button" algorithm, introduced by Chuah and Hu in [*J. Algebra* **279** (2004), 22–37], to the Vogan superdiagram associated to a contragredient Lie superalgebra. We obtain the Borel–de Siebenthal Theorem in the supersetting, leading to the classification of the real forms of contragredient Lie superalgebras (previously discussed in [V. G. Kac, *Adv. Math.* **26** (1977), 8–96; V. Serganova, *Funkt. Anal. Prilozhen.* **17** (1983), 46–54; M. Parker, *J. Math. Phys.* **21** (1980), 689–798]).

1. Introduction

The "push-the-button" algorithm was originally introduced by Chuah et al. in [4] to give an alternative proof of the Borel–de Siebenthal Theorem, a central result in the problem of classification of the real forms of a given complex simple Lie algebra $\mathfrak{s}$.

The purpose of the present paper is to explain how the "push-the-button" algorithm can be successfully applied to the Vogan superdiagram associated to a contragredient Lie superalgebra, so to obtain the equivalent super version of the Borel–de Siebenthal Theorem. Since in the supersetting black and grey vertices have an established meaning, we will *circle* the non-compact roots, instead of coloring them.

The idea of the push-the-button algorithm is not novel in the supersetting. In fact in [13], Hsin has used it to show how one can reduce the number of dark vertices of a Dynkin diagram of a given contragredient Lie superalgebra, however with no mention of the real forms of $\mathfrak{g}$. On the other hand, the problem of the classification of the real forms of contragredient Lie superalgebras was successfully treated in the works by Kac [14], Serganova [17], Parker [16].

We believe that our purely combinatorial approach can help to elucidate the question whether or not two real forms of the same contragredient Lie superalgebra $\mathfrak{g}$ are isomorphic, since it reduces the question to examine the push-the-button algorithm on the Vogan diagram of $\mathfrak{g}_0$. Some care must of course be exerted, because there is not a unique Dynkin diagram associated with $\mathfrak{g}$. Hence we prefer to work with extended Dynkin diagrams and to single out the preferred one.

For clarity of exposition, we limit ourselves to the case

$$\mathfrak{h} \subset \mathfrak{k}, \qquad \mathrm{rk}(\mathfrak{k}) = \mathrm{rk}(\mathfrak{g}),$$

where no arrows appear in the Vogan superdiagrams (see Section 2 and (5) for more details on our hypothesis). This case is very relevant for the applications (see [2], [3]).

Our paper is organized as follows. In Section 2, we recall few known facts about real forms of contragredient Lie superalgebras. In Section 3, we introduce Vogan diagrams and superdiagrams. In Section 4, we show how to adapt the "push-the-button" algorithm to Vogan superdiagrams. In the end, we examine some examples to show how effectively the algorithm allows to decide whether or not two real forms of the same contragredient Lie superalgebra are isomorphic.

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2. Preliminaries

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a contragredient Lie superalgebra, which is not a Lie algebra. $\mathfrak{g}$ is one of the following Lie superalgebras (see [14]):

$$\mathfrak{sl}(m,n),\ B(m,n),\ C(n),\ D(m,n),\ D(2,1;\alpha),\ F(4),\ G(3).$$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with root system $\Delta \subset \mathfrak{h}^*$ and root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha.$$

Let us fix a simple system Π. We can associate to $\mathfrak{g}$ an *extended Dynkin diagram.* Its vertices represent $\Pi \cup \varphi$, φ the lowest root, with colors white, grey or black, together with edges drawn according to [14, pp. 54–55]. As usual, with a common abuse of language, we say "roots" also to refer to vertices of the Dynkin diagram. Let D_0 be the subdiagram of D consisting of the white vertices, and let D_1 be the subdiagram of dark (i.e., grey or black) vertices. There are distinct D due to the choice of Π, but we can pick out a *preferred one*, characterized by the following theorem.

Theorem 2.1 ([5, Theorem 1.1])**.** *There exists an extended Dynkin diagram D such that*

(a) *D_0 is the Dynkin diagram of $\mathfrak{g}_0^{ss}$ (the semisimple part of $\mathfrak{g}_0$);*
(b) *$|D_1| - 1 = \dim \mathfrak{z}(\mathfrak{g}_0)$ (the center of $\mathfrak{g}_0$);*
(c) *D_1 consists of the lowest roots of the adjoint $\mathfrak{g}_0$-representation on $\mathfrak{g}_1$.*

Furthermore, there are unique positive integers $\{a_\alpha\}_D$ without nontrivial common factor such that

$$\sum_{\alpha\in D} a_\alpha \alpha = 0. \tag{1}$$

From now on, we will choose D as the preferred Dynkin diagram.

The real forms $\mathfrak{g}_\mathbb{R}$ and their symmetric spaces have been classified and studied by Parker [16] and Serganova [17]. We have a bijective correspondence:

$$\{\text{real forms}\, \mathfrak{g}_\mathbb{R} \subset \mathfrak{g}\} \leftrightarrow \{\theta \in \mathrm{aut}_{\{2,4\}}(\mathfrak{g})\}. \tag{2}$$

In this correspondence θ stabilizes $\mathfrak{g}_\mathbb{R}$, and the restriction of $\theta_\mathbb{R}$ to $\mathfrak{g}_\mathbb{R}$ is a Cartan automorphism. Hence we have the Cartan decomposition

$$\mathfrak{g}_{0,\mathbb{R}} = \mathfrak{k}_\mathbb{R} + \mathfrak{p}_{0,\mathbb{R}}, \tag{3}$$

where $\mathfrak{k}_\mathbb{R}$ and $\mathfrak{p}_{0,\mathbb{R}}$ are the ±1-eigenspaces of $\theta_\mathbb{R}$ on $\mathfrak{g}_{0,\mathbb{R}}$. Since θ has order 2 on $\mathfrak{g}_0$ and order 4 on $\mathfrak{g}_1$, we have the corresponding complex Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \tag{4}$$

where $\mathfrak{p} = \mathfrak{p}_0 + \mathfrak{p}_1$, $\mathfrak{p}_1 = \mathfrak{g}_1$ and we drop the index $\mathbb{R}$ to mean the complexification. So we immediately have

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}_0, \mathfrak{p}_0] \subset \mathfrak{k}, \quad [\mathfrak{p}_0, \mathfrak{p}_1] \subset \mathfrak{p}.$$

We assume that

$$\mathfrak{h} \subset \mathfrak{k}, \qquad \mathrm{rk}(\mathfrak{k}) = \mathrm{rk}(\mathfrak{g}). \tag{5}$$

Hence $\mathfrak{k}$ and $\mathfrak{p}$ are sums of root spaces and we call a root *compact* or *non-compact* depending on whether its root space sits in $\mathfrak{k}$ or $\mathfrak{p}$.

3. Vogan diagrams and superdiagrams

In the ordinary setting, if $\mathfrak{s}$ is a complex simple Lie algebra and $D_\mathfrak{s}$ its Dynkin diagram, we can associate to a real form $\mathfrak{s}_\mathbb{R}$ a Vogan diagram $(D_\mathfrak{s}, C)$, which corresponds (under our assumption (5)) to a circling C of the non-compact vertices. Vice-versa, every circling on the vertices of $D_\mathfrak{s}$ gives a Vogan diagram corresponding to a real form of $\mathfrak{s}$.

Unlike the Dynkin diagram $D_\mathfrak{s}$, that identifies uniquely the Lie algebra $\mathfrak{s}$, the Vogan diagrams $(D_\mathfrak{s}, C)$ do not correspond bijectively to the real forms of $\mathfrak{s}$; however we have the following important result.

Theorem 3.1 (Borel–de Siebenthal; [15, Thms. 6.88 and 6.96])**.** *Let $\mathfrak{s}$ be a complex simple Lie algebra. Any circling on the Dynkin diagram $D_\mathfrak{s}$ is the Vogan diagram of a real form of $\mathfrak{s}$. Furthermore, any real form $\mathfrak{s}_\mathbb{R}$ of $\mathfrak{s}$ is associated to a Vogan diagram with at most one circled vertex.*

This theorem allows us to associate to a real form a Vogan diagram with just one circled vertex; we call such diagrams *reduced*. Two real forms of $\mathfrak{s}$ are isomorphic if and only if there is a diagram symmetry between their reduced Vogan diagrams (see [6]).

We now turn to examining the supersetting.

Definition 3.2. Let $\mathfrak{g}$ be a complex contragredient Lie superalgebra. A *Vogan superdiagram* is a pair (D, C), where D is the preferred Dynkin diagram of $\mathfrak{g}$, with vertices $\Pi \cup \varphi$, and the *circling* C is a subset of the even roots in $\Pi \cup \varphi$.

If $\mathfrak{g}_\mathbb{R}$ is a real form of of $\mathfrak{g}$, we can associate to it the Vogan superdiagram obtained by taking as circling C the subset of the non-compact even roots in $\Pi \cup \varphi$ (see [7]). Since the odd roots are always non-compact, we omit the circling on them. However, more than one Vogan superdiagram may correspond to the same real form $\mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}$: this depends on the choice of the simple system of $\mathfrak{g}$, which may give a different circling of the even simple roots. For example, the following two Vogan superdiagrams correspond to the same real form $\mathfrak{su}(2,1|1,1)$ of $\mathfrak{sl}(3|2)$:
We will see in the next section, how the push-the-button algorithm allows us to see immediately that these two Vogan superdiagrams correspond to isomorphic superalgebras.

We have however an important difference with the ordinary setting: not all the circlings on the preferred Dynkin D are associated with a real form of $\mathfrak{g}$, but only the *admissible* ones. We have the following theorem.

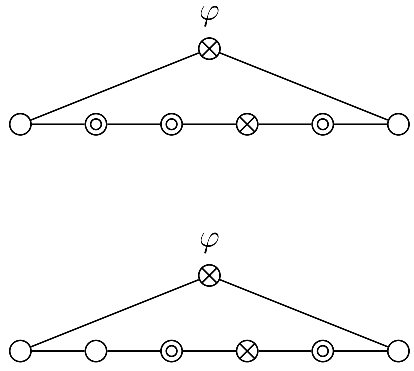


FIGURE 1

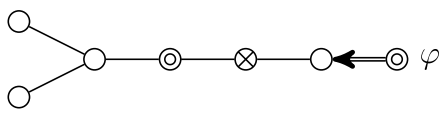


FIGURE 2

Theorem 3.3 ([7, Prop. 2.22]). *A circling C on the preferred Dynkin diagram D of $\mathfrak{g}$ corresponds to a real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$ if and only if*

$$\sum_{\alpha \in C} a_\alpha \alpha = \begin{cases} \text{even}, & \text{if} \quad \mathfrak{g}_0 = \mathfrak{sl}_n(\mathbb{C}), \\ \text{odd}, & \text{if} \quad \mathfrak{g}_0 \neq \mathfrak{sl}_n(\mathbb{C}), \end{cases} \tag{6}$$

where the a_α's are defined in (1).

From now on we will consider only *admissible* circlings, that is circlings satisfying the condition (6). We end the section with an example to clarify the condition (6), which essentially gives necessary and sufficient conditions to extend a real form of $\mathfrak{g}_0$ to a real form of $\mathfrak{g}$.

Example 3.4. Let us consider $\mathfrak{g} = D(4, 2)$ with the circling shown in Figure 2.

We notice that $\mathfrak{g}_0 = D_4 \oplus C_2$ and that the circled vertex has label $a_\alpha = 2$. Hence this circling is not admissible and this abstract Vogan superdiagram does not correspond to any real form of $\mathfrak{g}$, despite the fact that there is a real form of $\mathfrak{g}_0$ corresponding to the (disconnected) Vogan diagram on $\mathfrak{g}_0$:

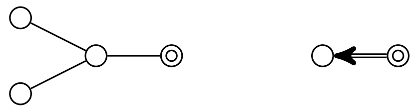


FIGURE 3

Hence the real form of $\mathfrak{g}_0$ described by the Vogan diagram in Figure 3 will not extend to give a real form of the whole $\mathfrak{g}$.

4. The push-the-button algorithm

We now define the operation F_i that will lead us to the push-the-button algorithm (see [9]). Our purpose is to obtain the equivalent of Theorem 3.1 in the super setting.

Definition 4.1. Let (D, C) be a Vogan superdiagram. If $i \in C$ is an even vertex, we define $F_i(D, C)$ as a new superdiagram (D, C'), where all the vertices j adjacent to i have reversed their circling (i.e., they become circled if they were not and become not circled if they were circled), except when j is a longer root joined to i by a double edge or j is odd.

In other words, if we define the neighborhood of vertex i by

$$N(i) = \{\text{vertices adjacent to}\, i \,\text{excluding}\, i\}, \tag{7}$$

then $F_i(D, C) = (D, C')$, where we reverse the circling of all $j \in N(i)$, except when j is a longer root joined to i by a double edge or j is odd.

We also say that the Vogan superdiagram (D, C') is obtained from (D, C) through the operation F_i. The reader can see that an operation F_i can be visually understood as "pressing" on the vertex i: the vertex itself will not change the circling, while the adjacent vertices, if linked by a single edge, will.

We say that two Vogan superdiagrams are *F-related*, if there is a sequence of operations F_i's transforming one into the other.

For example, consider the two diagrams in Figure 4.

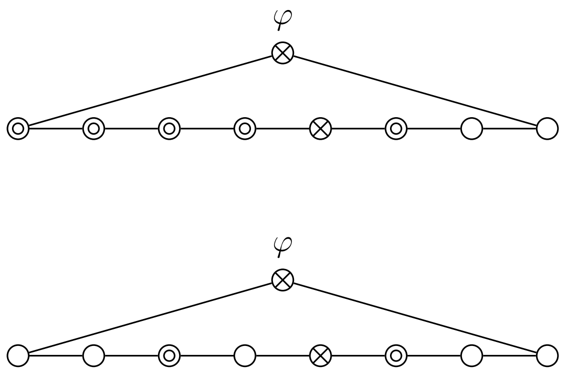


Figure 4

It is immediate to verify that the above diagrams are F-related. In fact F_2 followed by F_4 and F_3 will transform one into the other (assuming the horizontal vertices are labelled with consecutive integers $1, 2, \ldots, 8$). Similarly one can verify that the two diagrams in Figure 1 are F-related: apply the operation F_2.

Proposition 4.2. *If $F_i(D, C) = (D, C')$, then (D, C) and (D, C') correspond to the same real form.*

Proof. The operation F_i corresponds to the reflection s_i for the even vertex i. In fact, if $\alpha = i$ is a circled root and β is adjacent to α we have

$$s_\alpha(\beta) = \beta - n_{\beta\alpha}\alpha,$$

where $n_{\beta\alpha} = -1, -2, -3$. Hence in the Dynkin diagram, the adjacent pair $\{\alpha, \beta\}$ is replaced by the pair $\{-\alpha, \beta - n_{\beta\alpha}\alpha\}$. Since α is even, the pair $\{-\alpha, \beta - n_{\beta\alpha}\alpha\}$ will have the same parity as the pair $\{\alpha, \beta\}$. Hence if β is odd, $\beta - n_{\beta\alpha}\alpha$ is also odd hence it will not change its circling (recall odd roots are always non-compact, hence we omit their circling). If β is even, the root $\beta - n_{\beta\alpha}\alpha$ will have same circling as β if and only if $n_{\beta\alpha}$ is even, hence the result. □

Remark 4.3. The sequence of F_i operations we used in the previous proposition to transform (D, C) into (D, C') corresponds to the action of an element of the Weyl group of $\mathfrak{g}_0$. We cannot however claim that the simple system Γ, associated with (D, C) is transformed by such element into the simple system Γ' associated with (D, C'). This is because simple systems, even associated with the same Dynkin diagram, may not be conjugated by the action of the Weyl group. However, with the push-the-button algorithm, we bypass this difficulty, thus showing another advantage of this purely combinatorial approach to the theory of real forms of contragredient Lie superalgebras.

In [4], Chuah has developed an algorithm (*the push-the-button algorithm*) to prove that, starting from any Vogan diagram associated with the real form of a simple Lie algebra, one can obtain, through F_i operations, a Vogan diagram with just one circled vertex. All the Vogan diagrams obtained in this procedure correspond to the isomorphism class of the same real Lie algebra. It is our purpose to generalize this statement to the super setting.

We are ready to prove the super version of Theorem 3.1.

Theorem 4.4 (Borel–de Siebenthal). *Let $\mathfrak{g}$ be a contragredient Lie algebra, D its preferred Dynkin diagram. Any admissible circling on D is the Vogan superdiagram of a real form of $\mathfrak{g}$. Furthermore, any real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$ is associated to a Vogan superdiagram with an admissible circling having at most as many circled vertices as the number of connected components of $D \setminus D_1$.*

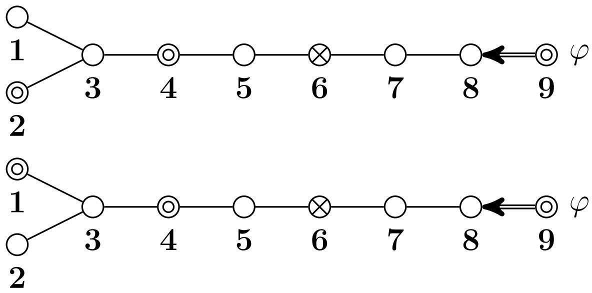
Proof. The first statement is a consequence of Proposition 2.22 in [7]. As for the second statement, Proposition 4.2 says that two Vogan superdiagrams correspond to the same real form if one can be transformed into the other by a sequence of F_i operations. By Corollary 5.2 in [4], the push-the-button algorithm, for ordinary Lie algebras, after a sequence of F-operations, we can obtain a Vogan diagram for each connected component of D_0, with at most one circled vertex. □

We call a Vogan superdiagram with at most as many circled vertices as the number of connected components of $D \setminus D_1$, *reduced*. This theorem allows us to determine immediately whether two real forms of the same contragredient Lie superalgebra are isomorphic. In fact, two real forms are isomorphic if and only if their reduced Vogan superdiagrams are related by a diagram symmetry. Hence, given two real forms, we first draw their Vogan superdiagrams and proceed with the push-the-button algorithm so to obtain two reduced Vogan superdiagrams. Then, we verify if there is a diagram symmetry sending one superdiagram to the other: if there is, the two real forms are isomorphic, otherwise, they are not isomorphic.

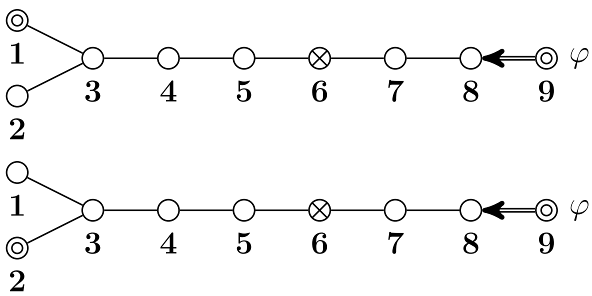
Before we proceed to give examples to illustrate the above procedure, we give a quick summary of the strategy to follow to obtain a reduced Vogan diagram in the ordinary setting. The reader can find the details in [4].

The ordinary push-the-button algorithm consists of two different steps that have to be repeated until there is just one circled vertex left (i.e., a non-compact root). With the first step, through repeated F_i operations, it is possible to bring a pair of circled vertices to the right (respectively left) side of the Vogan diagram. Then the second step consists in pushing the rightmost (respectively leftmost) vertex, so that the number of circled vertices reduces by one. By repeating these two steps a number of times, we can reach a reduced Vogan diagram equivalent to the previous one. We will see that this strategy works also for Vogan superdiagrams, since the push-the-button algorithm operates on the even part of the diagram as detailed in our previous propositions.

Example 4.5. Consider the two real forms of $\mathfrak{g} = D(5,3)$ corresponding to the following two admissible circlings:

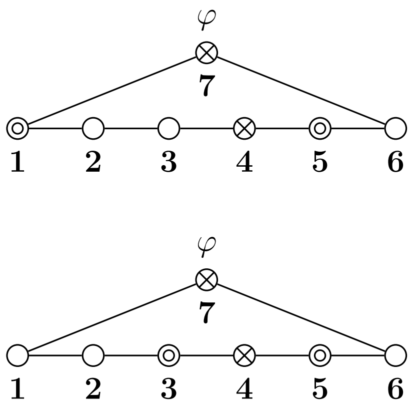


Applying F_i operations, we can reach the two following diagrams with only one circled vertex:



For the first diagram we have to apply F_2, F_3, and finally F_1. For the second one F_1, F_3, and finally F_2. We can easily see that two final diagrams are isomorphic by applying a diagram symmetry.

Example 4.6. As before, let us consider the two following real forms of $\mathfrak{sl}(3|2)$.



We observe that the corresponding Vogan diagrams of the real forms of $\mathfrak{g}_0$ are related by a diagram symmetry; however here we cannot use this symmetry, because of the presence of odd vertices. These Vogan superdiagrams nevertheless correspond to isomorphic real forms and in fact we can reach the second diagram, starting from the first one, with a combination of F_i operations, namely F_1, F_2, F_3.

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