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HOMOGENEOUS REPRESENTATIONS OF TYPE A KLR-ALGEBRAS AND DYCK PATHS

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ABSTRACT. The Khovanov–Lauda–Rouquier (KLR) algebra arose out of attempts to categorify quantum groups. Kleshchev and Ram proved a result reducing the representation theory of these algebras of finite type to the study of irreducible cuspidal representations. In type A, these cuspidal representations are included in the class of homogeneous representations, which are related to fully commutative elements of the corresponding Coxeter groups. In this paper, we study fully commutative elements using combinatorics of Dyck paths. Thereby we classify and enumerate the homogeneous representations for KLR algebras of types A and obtain a dimension formula for some of these representations from combinatorics of Dyck paths.

INTRODUCTION

Introduced by Khovanov and Lauda [11] and independently by Rouquier [16], the Khovanov–Lauda–Rouquier (KLR) algebras (also known as quiver Hecke algebras) have been the focus of many recent studies. In particular, these algebras categorify the lower (or upper) half of a quantum group. More precisely, the Cartan datum associated with a Kac–Moody algebra $\mathfrak{g}$ gives rise to a KLR algebra R. The category of finitely generated projective graded modules of this algebra can be given a bialgebra structure by taking the Grothendieck group, and taking the induction and restriction functors as multiplication and co-multiplication. To say that the KLR algebra R categorifies the negative part $U_q^-(\mathfrak{g})$ of the quantum group, is to say that this bialgebra is isomorphic to Lusztig's integral form of $U_q^-(\mathfrak{g})$.

In the paper [13], Kleshchev and Ram significantly reduce the problem of describing the irreducible representations of the finite type KLR algebras. They defined a class of *cuspidal* representations for these algebras, and showed that every irreducible representation appears as the head of some induction of these cuspidals, and constructed almost all cuspidal representations. Hill, Melvin, and Mondragon in [8] completed the construction of cuspidals in all finite types, and re-frame them in a more unified manner.

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Furthermore, Lauda and Vazirani imposed a crystal structure on the isomorphism classes of irreducible representations of a KLR algebra. They showed in [14] that this crystal is isomorphic to the crystal $B(\infty)$ of the quantum group $U_q(\mathfrak{g})$. Crystals are also used by Benkart, Kang, Oh, and Park in [1] to give a new approach towards the construction of irreducible representations. For more background and other developments, see [4] and [10].

In the process of constructing the cuspidal representations, Kleshchev and Ram defined a class of representations known as *homogeneous representations* [12], those that are concentrated in a single degree. Homogeneous representations include most of the cuspidal representations for finite types with a suitable choice of ordering on words. Therefore it is important to completely understand these representations. As shown in [12], homogeneous representations can be constructed from the sets of reduced words of *fully commutative* elements in the corresponding Coxeter group. These elements were studied by Fan [7] and Stembridge [17, 18], and are closely related to Temperley–Lieb algebras [9].

Motivated by this connection to the homogeneous representations of KLR algebras, we study, in this paper, fully commutative elements of the Coxeter groups of type A_n. We decompose the set of fully commutative elements into natural subsets according to the lengths of fully commutative elements, and study combinatorial properties of these subsets. Our main result (Theorem 2.1) shows that the fully commutative elements of a given length k can be parameterized by the Dyck paths of semi-length n with the property that (*sum of peak heights*) $-$ (*number of peaks*) $= k$. The main idea of the proof is to investigate a canonical form of reduced words for fully commutative elements.[1]

After the parameterization is obtained, we classify and enumerate the homogeneous representations of KLR algebras of type A according to the decomposition of the set of fully commutative elements (Corollary 2.2). In their paper [12], Kleshchev and Ram gave a parameterization of homogeneous representations using *skew shapes*. Our result uses different combinatorial objects, i.e., Dyck paths, and gives a refinement of the classification. Furthermore, we obtain a dimension formula for some homogeneous representations using combinatorics of Dyck paths (Proposition 3.2), which is a reformulation of the Peterson–Proctor formula. The precise relationship between skew shapes and Dyck paths is not

[1]After this paper was accepted, the authors were informed by C. Krattenthaler that the problem of enumerating fully commutative elements by length was also studied by Biagioli, Jouhet and Nadeau in [2]. Interestingly, they give a bijection to Motzkin paths with two types of horizontal steps. Since there is a well-known bijection of these Motzkin paths to Dyck paths, one obtains a different bijection from fully commutative elements to Dyck paths, where the length of a fully commutative element is given by half of the sum of the heights of points of the corresponding Dyck path at even positions. Then, by combining our result with their result and by going through the fully commutative elements, we obtain a bijection from Dyck paths to Dyck paths, which is compatible with the two different ways to parameterize fully commutative elements with respect to length.

clear at the present; our Dyck path realization is more directly related to a canonical form of fully commutative elements.

The outline of this paper is as follows. In Section 1, we fix notations, briefly review the representations of KLR algebras, and explain the relationship between homogeneous representations and fully commutative elements of a Coxeter group. In Section 2, we introduce Dyck paths and study a canonical form of reduced words of fully commutative elements and obtain the main results of this paper. In Section 3, we prove a dimension formula for homogeneous representations when the corresponding Dyck paths satisfy a certain condition.

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1. KLR Algebras and Homogeneous Representations

1.1. **Definitions.** To define a KLR algebra, we begin with a quiver Γ. In this paper, we will focus mainly on quivers of Dynkin types A_n, but for the definition, any finite quiver with no double bonds will suffice. Let I be the set indexing the vertices of Γ, and for indices $i \neq j$, we will say that i and j are neighbors if $i \to j$ or $i \leftarrow j$. Define $Q_+ = \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\, \alpha_i$ as the non-negative lattice with basis $\{\alpha_i : i \in I\}$. The set of all words in the alphabet I is denoted by $\langle I \rangle$, and for a fixed $\alpha = \sum_{i\in I} c_i \alpha_i \in Q_+$, let $\langle I \rangle_\alpha$ be the set of words $\boldsymbol{w}$ on the alphabet I such that each $i \in I$ occurs exactly c_i times in $\boldsymbol{w}$. We define the *height* of α to be $\sum_{i\in I} c_i$. We will write $\boldsymbol{w} = [w_1, w_2, \ldots, w_d]$, $w_j \in I$.

Now, fix an arbitrary ground field $\mathbb{F}$ and choose an element $\alpha \in Q_+$. Then the *Khovanov–Lauda–Rouquier algebra* R_α is the associative $\mathbb{F}$-algebra generated by:

- idempotents $\{e(\boldsymbol{w}) \mid \boldsymbol{w} \in \langle I \rangle_\alpha\}$,
- symmetric generators $\{\psi_1, \ldots, \psi_{d-1}\}$ where d is the height of α,
- polynomial generators $\{y_1, \ldots, y_d\}$,

subject to the relations

$$e(\boldsymbol{w})e(\boldsymbol{v}) = \delta_{\boldsymbol{w}\boldsymbol{v}} e(\boldsymbol{w}), \qquad \sum_{\boldsymbol{w}\in\langle I\rangle_\alpha} e(\boldsymbol{w}) = 1; \tag{1.1}$$

$$y_k e(\boldsymbol{w}) = e(\boldsymbol{w}) y_k; \tag{1.2}$$

$$\psi_k e(\boldsymbol{w}) = e(s_k \boldsymbol{w}) \psi_k; \tag{1.3}$$

$$y_k y_\ell = y_\ell y_k; \tag{1.4}$$

$$y_k \psi_\ell = \psi_\ell y_k \quad \text{for } k \neq \ell, \ell+1; \tag{1.5}$$

$$(y_{k+1}\psi_k - \psi_k y_k)e(\boldsymbol{w}) = \begin{cases} e(\boldsymbol{w}), & \text{if } w_k = w_{k+1}, \\ 0, & \text{otherwise;} \end{cases} \tag{1.6}$$

$$(\psi_k y_{k+1} - y_k \psi_k)e(\boldsymbol{w}) = \begin{cases} e(\boldsymbol{w}), & \text{if } w_k = w_{k+1}, \\ 0, & \text{otherwise;} \end{cases} \tag{1.7}$$

$$\psi_k^2 e(\boldsymbol{w}) = \begin{cases} 0, & \text{if } w_k = w_{k+1}, \\ (y_k - y_{k+1})e(\boldsymbol{w}), & \text{if } w_k \to w_{k+1}, \\ (y_{k+1} - y_k)e(\boldsymbol{w}), & \text{if } w_k \leftarrow w_{k+1}, \\ e(\boldsymbol{w}), & \text{otherwise;} \end{cases} \tag{1.8}$$

$$\psi_k \psi_\ell = \psi_\ell \psi_k \quad \text{for } |k - \ell| > 1; \tag{1.9}$$

$$(\psi_{k+1}\psi_k\psi_{k+1} - \psi_k\psi_{k+1}\psi_k)e(\boldsymbol{w}) = \begin{cases} e(\boldsymbol{w}), & \text{if } w_{k+2} = w_k \to w_{k+1}, \\ -e(\boldsymbol{w}), & \text{if } w_{k+2} = w_k \leftarrow w_{k+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.10}$$

Here $\delta_{\boldsymbol{wv}}$ in (1.1) is the Kronecker delta and, in (1.3), s_k is the k^{th} simple transposition in the symmetric group S_d, acting on the word $\boldsymbol{w}$ by swapping the letters in the k^{th} and $(k+1)^{\text{st}}$ positions. If Γ is a Dynkin-type quiver, we will say that R_α is a KLR algebra of that type.

We impose a $\mathbb{Z}$-grading on R_α by

$$\deg(e(\boldsymbol{w})) = 0, \quad \deg(y_i) = 2, \tag{1.11}$$

$$\deg(\psi_i e(\boldsymbol{w})) = \begin{cases} -2, & \text{if } w_i = w_{i+1}, \\ 1, & \text{if } w_i, w_{i+1} \text{ are neighbors in } \Gamma, \\ 0, & \text{if } w_i, w_{i+1} \text{ are not neighbors in } \Gamma. \end{cases} \tag{1.12}$$

Set $R = \bigoplus_{\alpha \in Q_+} R_\alpha$, and let $\mathrm{Rep}(R)$ be the category of finite dimensional graded R-modules, and denote its Grothendieck group by $[\mathrm{Rep}(R)]$. Then $\mathrm{Rep}(R)$ categorifies one half of the quantum group. More precisely, let $\mathbf{f}$ and $'\mathbf{f}$ be Lusztig's algebras defined in [15, Section 1.2] attached to the Cartan datum encoded in the quiver Γ over the field $\mathbb{Q}(v)$. We put $q = v^{-1}$ and $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$, and let $'\mathbf{f}_\mathcal{A}$ and $\mathbf{f}_\mathcal{A}$ be the $\mathcal{A}$-forms of $'\mathbf{f}$ and $\mathbf{f}$, respectively. Consider the graded duals $'\mathbf{f}^*$ and $\mathbf{f}^*$, and their $\mathcal{A}$-forms

$$'\mathbf{f}^*_\mathcal{A} := \{x \in {'\mathbf{f}^*} : x('\mathbf{f}_\mathcal{A}) \subset \mathcal{A}\} \quad \text{and} \quad \mathbf{f}^*_\mathcal{A} := \{x \in \mathbf{f}^* : x(\mathbf{f}_\mathcal{A}) \subset \mathcal{A}\}.$$

Then Khovanov and Lauda [11] prove that there is an $\mathcal{A}$-linear (bialgebra) isomorphism $[\mathrm{Rep}(R)] \xrightarrow{\sim} \mathbf{f}^*_\mathcal{A}$. More details can be found in [11, 13].

A word $\boldsymbol{w} \in \langle I\rangle_\alpha$ is naturally considered as an element of ${}'\mathbf{f}^*_{\mathcal{A}}$ which is dual to the corresponding monomial in ${}'\mathbf{f}_{\mathcal{A}}$. Let M be a finite dimensional graded R_α-module. Define the q-character of M by

$$\mathrm{ch}_q M := \sum_{\boldsymbol{w}\in\langle I\rangle_\alpha} (\dim_q M_{\boldsymbol{w}})\, \boldsymbol{w} \quad \in {}'\mathbf{f}^*_{\mathcal{A}},$$

where $M_{\boldsymbol{w}} = e(\boldsymbol{w})M$ and $\dim_q V := \sum_{n\in\mathbb{Z}}(\dim V_n)\, q^n \in \mathcal{A}$ for $V = \bigoplus_{n\in\mathbb{Z}} V_n$. A non-empty word $\boldsymbol{w}$ is called *Lyndon* if it is lexicographically smaller than all its proper right factors with respect to a fixed total ordering on I. For $x \in {}'\mathbf{f}^*$ we denote by $\max(x)$ the largest word appearing in x. A word $\boldsymbol{w} \in \langle I\rangle$ is called *good* if there is $x \in \mathbf{f}^*$ such that $\boldsymbol{w} = \max(x)$. Given a module $L \in \mathrm{Rep}(R_\alpha)$, we say that $\boldsymbol{w} \in \langle I\rangle$ is the *highest weight* of L if $\boldsymbol{w} = \max(\mathrm{ch}_q L)$. An irreducible module $L \in \mathrm{Rep}(R_\alpha)$ is called *cuspidal* if its highest weight is a good Lyndon word.

The following theorem explains the importance of cuspidal representations as building blocks for all irreducible representations of R_α.

Theorem 1.1 ([13]; [8], 4.1.1)**.** *Assume that Γ is of finite Dynkin type. Then any irreducible graded R_α-module for $\alpha \in Q_+$ is given by an irreducible head of a standard representation induced from cuspidal representations up to isomorphism and degree shift.*

1.2. **Homogeneous representations.** We define a *homogeneous representation* of a KLR algebra to be an irreducible, graded representation fixed in a single degree (with respect to the $\mathbb{Z}$-grading described in (1.11) and (1.12)). Homogeneous representations form an important class of irreducible modules since most of the cuspidal representations are homogeneous with a suitable choice of ordering on $\langle I\rangle$ ([13, 8]). After introducing some terminology, we will describe these representations in a combinatorial way. We continue to assume that Γ is a simply-laced quiver.

Fix an $\alpha \in Q_+$ and let d be the height of α. For any word $\boldsymbol{w} \in \langle I\rangle_\alpha$, we say that the simple transposition $s_r \in S_d$ is an *admissible transposition for $\boldsymbol{w}$* if the letters w_r and w_{r+1} are neither equal nor neighbors in the quiver Γ. Following Kleshchev and Ram [12], we define the *weight graph* G_α with vertices given by $\langle I\rangle_\alpha$. Two words $\boldsymbol{w}$, $\boldsymbol{v} \in \langle I\rangle_\alpha$ are connected by an edge if there is an admissible transposition s_r such that $s_r\boldsymbol{w} = \boldsymbol{v}$.

We say that a connected component C of the weight graph G_α is *homogeneous* if the following property holds for every $\boldsymbol{w} \in C$:

$$\begin{aligned}(1.13)\qquad &\text{If } w_r = w_s \text{ for some } 1 \le r < s \le d, \text{ then there exist } t, u\\ &\qquad\text{with } r < t < u < s \text{ such that } w_r \text{ is a neighbor of both } w_t \text{ and } w_u.\end{aligned}$$

A word satisfying condition (1.13) will be called a *homogeneous word*.

A main theorem of [12] shows that the homogeneous components of G_α exactly parameterize the homogeneous representations of the KLR algebra R_α.

Theorem 1.2 ([12], Theorem 3.4). *Let C be a homogeneous component of the weight graph G_α. Define an $\mathbb{F}$-vector space $S(C)$ with basis $\{v_{\boldsymbol{w}} : \boldsymbol{w} \in C\}$ labeled by the vertices in C. Then we have an R_α-action on $S(C)$ given by*

$$\begin{aligned} e(\boldsymbol{w}')v_{\boldsymbol{w}} &= \delta_{\boldsymbol{w},\boldsymbol{w}'}v_{\boldsymbol{w}}, && \boldsymbol{w}' \in \langle I \rangle_\alpha, \boldsymbol{w} \in C, \\ y_r v_{\boldsymbol{w}} &= 0, && 1 \le r \le d, \boldsymbol{w} \in C, \\ \psi_r v_{\boldsymbol{w}} &= \begin{cases} v_{s_r\boldsymbol{w}}, & \text{if } s_r\boldsymbol{w} \in C, \\ 0, & \text{otherwise}, \end{cases} && 1 \le r \le d-1, \boldsymbol{w} \in C, \end{aligned}$$

which gives $S(C)$ the structure of a homogeneous, irreducible R_α-module. Further $S(C) \not\cong S(C')$ if $C \neq C'$, and this construction gives all of the irreducible homogeneous modules, up to isomorphism.

As a result, the task of identifying homogeneous representations of a KLR algebra is reduced to identifying homogeneous components in a weight graph. This is simplified further by the following lemma.

Lemma 1.3 ([12], Lemma 3.3). *A connected component C of the weight graph G_α is homogeneous if and only if an element $\boldsymbol{w} \in C$ satisfies the condition* (1.13).

Recall that we call a word satisfying condition (1.13) a homogeneous word. The homogeneous words have other combinatorial characterizations, which we explore in the next subsection.

1.3. **Fully commutative elements of Coxeter groups.** Since the homogeneity of $\boldsymbol{w} \in \langle I \rangle$ does not depend on the orientation of a quiver, it is enough to consider Dynkin diagrams and the corresponding Coxeter groups. Given a simply laced Dynkin diagram, the corresponding Coxeter group will be denoted by W and the generators by s_i, $i \in I$. A reduced expression $s_{i_1} \cdots s_{i_r}$ will be identified with the word $[i_1, \dots, i_r]$ in $\langle I \rangle$. The identity element will be identified with the empty word [ ].

An element $\boldsymbol{w} \in W$ is said to be *fully commutative* if any reduced word for $\boldsymbol{w}$ can be obtained from any other by interchanges of adjacent commuting generators, or equivalently if no reduced word for $\boldsymbol{w}$ has $[i, i', i]$ as a subword where i and i' are neighbors in the Dynkin diagram. Now we have the following lemma, which was first observed by Kleshchev and Ram.

Lemma 1.4 ([12]).

1. *A homogeneous component of the weight graph G_α contains as its vertices exactly the set of reduced expressions for a fully commutative element in W.*
2. *The set of homogeneous components is in bijection with the set of fully commutative elements in W.*

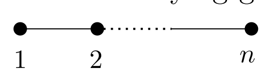
Stembridge [17] classified all of the Coxeter groups that have finitely many fully commutative elements, completing the work of Fan [7], who had done this for the simply-laced types. Fan and Stembridge also enumerated the set of fully commutative elements. In particular, they showed that the number of fully commutative elements in the Coxeter group of type A_n is C_{n+1}, where C_n is the n^{th} Catalan number, i.e., $C_n = \frac{1}{n+1}\binom{2n}{n}$. This fact has an immediate implication on homogeneous representations by Lemma 1.4.

Corollary 1.5. *A KLR algebra $R = \bigoplus_{\alpha \in Q_+} R_\alpha$ of type A_n has C_{n+1} irreducible homogeneous representations.*

In [12], Kleshchev and Ram parameterized homogeneous representations using skew shapes. In this paper, we will decompose the set of fully commutative elements to give a finer enumeration of homogeneous representations in type A_n. More precisely, in the next section, our main result is a fine bijection between the family of irreducible homogeneous representations and the set of *Dyck paths* in accordance with the decomposition of the set of fully commutative elements. This bijection can be used to quickly enumerate the fully commutative elements of a given length and the attached homogeneous representations.

2. Homogeneous Representations of Type A_n KLR Algebras

In this section, we describe all of the homogeneous representations of a KLR algebra of type A_n, associated with a quiver whose underlying graph is



We begin by introducing the main combinatorial tool for our study.

2.1. **Dyck paths.** As in [5], we define a *Dyck path* as a lattice path in the first quadrant consisting of steps $\langle 1, 1\rangle$ (north-east) and $\langle 1, -1\rangle$ (south-east), beginning at the origin and ending at the point $(2n, 0)$. We refer to n as the *semi-length* of the path. By a *peak* we shall mean a rise $\langle 1, 1\rangle$ followed by a fall $\langle 1, -1\rangle$, while a *valley* is a fall, followed by a rise.

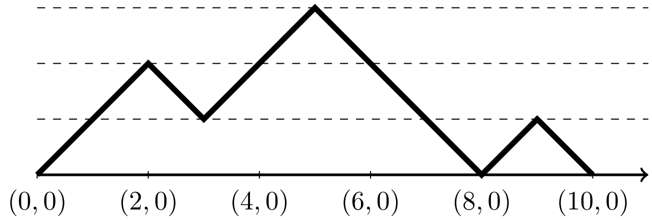


Figure 2.1. An example of a Dyck path of semilength 5.

We will turn our focus to a particular statistic, k, on these Dyck paths. Denote by $\mathcal{D}_{n,k}$ the set of all Dyck paths of semi-length n with the property that

$$(\textit{sum of peak heights}) - (\textit{number of peaks}) = k.$$

For the example path shown in figure 2.1 we have $k = (2+3+1) - 3 = 3$.

Let $T(n,k)$ be the cardinality of the set $\mathcal{D}_{n,k}$, as defined in [6]. It is known that $T(n,k) = 0$ when $k > 1 + \lfloor \frac{n^2}{4} \rfloor$. It is convenient to display the sequence of non-zero values as an array with the entry $T(n,k)$ in the n^{th} row from the top (starting with $n = 0$) and the k^{th} column (beginning with $k = 0$). The top of the array is shown below.

$$\begin{array}{cccccccccc}
1 \\
1 \\
1 & 1 \\
1 & 2 & 2 \\
1 & 3 & 5 & 4 & 1 \\
1 & 4 & 9 & 12 & 10 & 4 & 2 \\
1 & 5 & 14 & 25 & 31 & 26 & 16 & 9 & 4 & 1
\end{array} \tag{2.1}$$

It is well known that the number of Dyck paths of semi-length n is equal to the n^{th} Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$, so we have

$$C_n = \sum_k T(n,k), \tag{2.2}$$

i.e., the sum of entries on the n^{th} row is equal to C_n.

Now we consider fully commutative elements in the Coxeter group S_{n+1} and state our main result. Recall that the *length* of an element of S_{n+1} is defined with respect to the generators (simple transpositions $s_1, \ldots, s_n$) of S_{n+1}.

Theorem 2.1. *Let $\mathcal{C}_{n,k}$ be the set of fully commutative elements of length k in S_{n+1} for $k \geq 0$. Then there is a natural bijection $\Phi : \mathcal{C}_{n,k} \to \mathcal{D}_{n+1,k}$. In particular, we have, for $k \geq 0$,*

$$|\mathcal{C}_{n,k}| = T(n+1,k).$$

By Lemma 1.4, we will identify $\mathcal{C}_{n,k}$ with the set of homogeneous components of weight graphs G_α with α having height k. It follows from Theorem 1.2 that a homogeneous representation is completely determined by a homogeneous component of a weight graph. Thus Theorem 2.1 implies the following results regarding the homogeneous representations.

Corollary 2.2.

(1) *There exists a bijection between the irreducible homogeneous representations of the KLR algebra R of type A_n and the Dyck paths of semi-length $n+1$. Under this bijection, a representation given by a homogeneous component of words with length k corresponds to a Dyck path with*

$$(\textit{sum of peak heights}) - (\textit{number of peaks}) = k.$$

(2) *The total number of homogeneous representations of the KLR algebra R of type A_n, given by homogeneous words of length k is $T(n+1,k)$.*

We will prove Theorem 2.1 in Section 2.3, after we construct canonical words for fully commutative elements in the next subsection.

2.2. **Canonical reduced words.** We define the decreasing segments

$$T_i^j = \begin{cases} [j, j-1, \dots, i+1, i], & \text{for } i \leq j, \\ [\,], & \text{for } i > j. \end{cases}$$

The word T_i^j will also be considered as the element $s_j s_{j-1} \cdots s_{i+1} s_i \in S_{n+1}$. In particular, the product $T_i^j T_{i'}^{j'}$ given by concatenation is well defined.

These segments will be fundamental, so we record some facts here that we will use freely.

Lemma 2.3. *Let T_i^j be a segment, as defined above. Then we have, for $i, i', j, j' \in I$:*

(1) *T_i^j is a homogeneous word;*
(2) *if $i-1 = j' \geq i'$ then $T_i^j T_{i'}^{j'} = T_{i'}^j$;*
(3) *if $j' < i-1$ then $T_i^j T_{i'}^{j'} = T_{i'}^{j'} T_i^j$.*

Proof. These statements follow directly from the definitions. □

We can use these segments to obtain a canonical form for the elements in the Coxeter group S_{n+1}:

Lemma 2.4. *Every element in S_{n+1} can be written in the form*

$$T_{i_1}^1 T_{i_2}^2 \cdots T_{i_n}^n, \tag{2.3}$$

where $1 \leq i_j \leq j+1$ for all $1 \leq j \leq n$.

Remark 2.5. As a check, notice that there are $(n+1)!$ choices for the i_j's in this form, and hence $(n+1)!$ elements in S_{n+1}. The above lemma is standard. One can find a proof in Lemma 3.2 of [3], which uses a Gröbner–Shirshov basis.

Using the canonical form (2.3), we can describe canonical representatives of homogeneous components or fully commutative elements of S_{n+1} in a coherent way.

Proposition 2.6. *Every homogeneous component of a weight graph contains a unique word of the form*

$$T_{i_1}^{m_1} T_{i_2}^{m_2} \cdots T_{i_\ell}^{m_\ell} \tag{2.4}$$

where $i_j \le m_j$ for each j, $1 \le i_1 < i_2 < \cdots < i_\ell \le n$ and $m_1 < m_2 < \cdots < m_\ell$. Equivalently, a fully commutative element of S_{n+1} can be uniquely written in the form (2.4).

Proof. Clearly, every homogeneous component has a unique word of the form (2.3). After omitting, if any, segments of the form T_{j+1}^{j}, we obtain $\boldsymbol{w} = T_{i_1}^{m_1} T_{i_2}^{m_2} \cdots T_{i_\ell}^{m_\ell}$ with $i_j \le m_j$ for each j and $m_1 < m_2 < \cdots < m_\ell$. We only need to prove $1 \le i_1 < i_2 < \cdots < i_\ell \le n$. For the sake of contradiction, assume that $i_r \ge i_s$ for some $r < s$. Without loss of generality, suppose that $i_1 \ge i_2$. Then $\boldsymbol{w}$ has as a subword $[m_1, \dots, i_1, m_2, \dots, i_1, \dots, i_2]$. But this subword has two occurrences of the letter i_1 separated by only one neighbor i_1+1, therefore violating the homogeneity assumption. The equivalence of the second assertion follows from Lemma 1.4. $\square$

2.3. **A bijection—proof of Theorem 2.1.** Recall that $\mathcal{C}_{n,k}$ is the set of fully commutative elements of length k in S_{n+1} for $k \ge 0$. By Lemma 1.4, we will also consider $\mathcal{C}_{n,k}$ as the set of homogeneous components from all weight graphs G_α with α having height k. We need to establish a bijection Φ: $\mathcal{C}_{n,k} \to \mathcal{D}_{n+1,k}$ to prove Theorem 2.1. We first construct a lattice as shown in Figure 2.2, ranging (horizontally) from $(0,0)$ to $(2n+2,0)$. Notice that each square block corresponds to $T_i^j = [j, j-1, \dots, i+1, i]$ for some $i \le j$, and a Dyck path can have peaks at squares T_i^j or at bottom triangles. Now suppose that we have a homogeneous component $C \in \mathcal{C}_{n,k}$. By Proposition 2.6, we can choose a canonical representative $\boldsymbol{w}_C = T_{i_1}^{m_1} T_{i_2}^{m_2} \cdots T_{i_\ell}^{m_\ell}$ with $i_j \le m_j$ for each j, where $i_1 < i_2 < \cdots < i_\ell$ and $m_1 < m_2 < \cdots < m_\ell$. We write $\boldsymbol{w} = \boldsymbol{w}_C$ if there is no peril of confusion.

Definition 2.7. Suppose that $C \in \mathcal{C}_{n,k}$ and $\boldsymbol{w} = T_{i_1}^{m_1} T_{i_2}^{m_2} \cdots T_{i_\ell}^{m_\ell}$ are as above. Then the Dyck path $\Phi(C)$ is defined to be the path with peaks only at the square blocks (see Figure 2.2) containing $T_{i_j}^{m_j}$ $(j = 1, 2, \dots, \ell)$ and possibly, at bottom triangles.

Before we check that the map Φ is well-defined, i.e., $\Phi(C) \in \mathcal{D}_{n+1,k}$, we consider an example to see how the definition works.

Example 2.8. Suppose the quiver Γ is of type A_4, and the homogeneous component C is

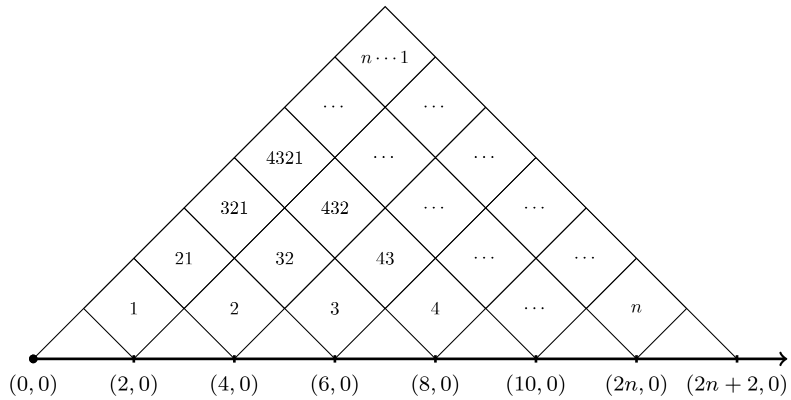
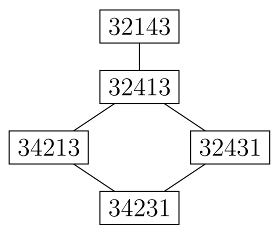
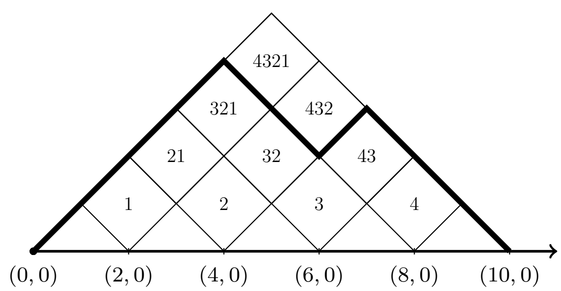


FIGURE 2.2. The triangular lattice for tracing Dyck paths



Then the canonical representative of this component is $\boldsymbol{w} = [3,2,1,4,3] = T_1^3 T_3^4$, and the Dyck path $\Phi(C)$ is given by:



Here the sum of peak heights of the Dyck path is $4 + 3 = 7$, while the number of peaks is 2. Then we see that $k = 7 - 2 = 5$ is equal to the length of the corresponding word $\boldsymbol{w} = [3,2,1,4,3]$.

Lemma 2.9. *The map Φ is well-defined.*

Proof. Let $C \in \mathcal{C}_{n,k}$ be a homogeneous component with canonical representative

$$\boldsymbol{w} = T_{i_1}^{m_1} T_{i_2}^{m_2} \cdots T_{i_\ell}^{m_\ell}.$$

Since $i_1 < i_2 < \cdots < i_\ell$ and $m_1 < m_2 < \cdots < m_\ell$, there is no redundancy among the peaks and the corresponding Dyck path D is uniquely determined. Note that each segment $T_{i_j}^{m_j}$ contains $m_j - i_j + 1$ letters. Since $\boldsymbol{w}$ has k letters by assumption, we have

$$k = \sum_{j=1}^{\ell} [(m_j - i_j) + 1] = \ell + \sum_{j=1}^{\ell} (m_j - i_j).$$

On the other hand, in the path $\Phi(C)$, each of the ℓ segments $T_{i_j}^{m_j}$ corresponds to a peak with height $(m_j - i_j) + 2$. We then have

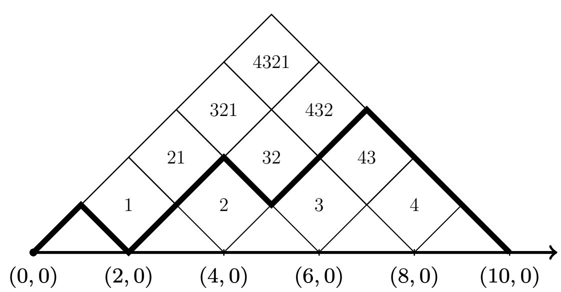
$$(\text{sum of peak heights}) - (\#\text{ of peaks}) = \sum_{j=1}^{\ell} [(m_j - i_j) + 2] - \ell = \ell + \sum_{j=1}^{\ell} (m_j - i_j) = k.$$

Thus $\Phi(C) \in \mathcal{D}_{n+1,k}$ as desired. $\square$

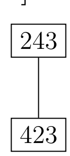
Definition 2.10. To define the inverse, $\Psi : \mathcal{D}_{n+1,k} \to \mathcal{C}_{n,k}$, we simply read the words contained in the square blocks of the peaks of the Dyck path D from left to right, ignoring peaks at bottom triangles. Then we obtain $\boldsymbol{w} = T_{i_1}^{m_1} T_{i_2}^{m_2} \cdots T_{i_\ell}^{m_\ell}$, and $\boldsymbol{w}$ determines the corresponding homogeneous component $\Psi(D)$.

A similar argument as in Lemma 2.9 shows that the map Ψ is well-defined.

Example 2.11. Suppose that we have the Dyck path D:



Reading, from left to right, the segments contained in the peaks, we see that the component $\Psi(D)$ is represented by the word $[2, 4, 3] = T_2^2 T_3^4$. This is the homogeneous component



Note that the sum of peak heights is $1 + 2 + 3 = 6$ and the number of peaks is 3. The value of $k = 6 - 3 = 3$ equals the length of $\boldsymbol{w} = [2, 4, 3]$.

Now we complete the proof that the map Φ is a bijection with inverse Ψ. It is clear from the construction that the blocks in the lattice that form the peaks of $\Phi(C)$ contain the words, $T_{i_1}^{m_1}, T_{i_2}^{m_2}, \dots, T_{i_j}^{m_\ell}$, respectively. Thus, we see that $\Psi(\Phi(C)) = C$. Conversely, suppose that $D \in \mathcal{D}$ is a Dyck path that has been superimposed on the triangular lattice, and assume that D has ℓ peaks ($\ell > 0$) that correspond to square blocks. Reading the words occurring at each of these peaks, we obtain $T_{i_1}^{m_1}, T_{i_2}^{m_2}, \dots, T_{i_\ell}^{m_\ell}$. We notice that, by construction, $i_1 < i_2 < \cdots < i_\ell$ and $m_1 < m_2 < \cdots < m_\ell$. By Proposition 2.6, the word $T_{i_1}^{m_1} T_{i_2}^{m_2} \cdots T_{i_\ell}^{m_\ell}$ represents a homogeneous component. We also see that $\Phi(\Psi(D)) = D$, and so Ψ is a two-sided inverse of Φ, proving the bijection. This completes the proof of Theorem 2.1.

3. Dimensions of Homogeneous Representations

In [12], Kleshchev and Ram explain how each fully commutative element $\boldsymbol{w}$ of the Weyl group A_n can be associated with an abacus diagram, which gives rise to a skew tableau λ. Further, if $\boldsymbol{w}$ is a dominant minuscule element, the Peterson–Proctor hook formula applied to this tableau will count the number of reduced expressions for the fully commutative element, and thus count the dimension of the corresponding R_α-module.

In this section, we will adopt our parameterization of the homogeneous modules and obtain a dimension formula only using combinatorics of Dyck paths. We begin by extending the ascents on the Dyck path to connect peaks of the Dyck path with the corresponding points on the x-axis, and highlighting any square block that appears directly under one of these extended ascents. For example, we have

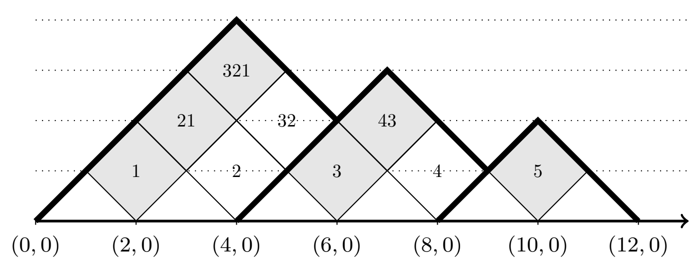


Figure 3.1. A Dyck path with extended ascents

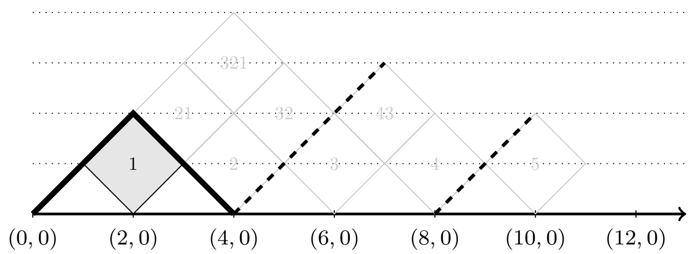
Now, for any block T_i^j that appears on an extended ascent, we draw a subpath $P_D(i, j)$ according to the following instructions:

1. (1) Draw a path from the x-axis past blocks $T_i^i, T_i^{i+1}, \dots$ up to the peak of block T_i^j.
2. (2) From there, the path descends until it hits another extended ascent. If the path does not hit any extended ascent, it returns to the x-axis.

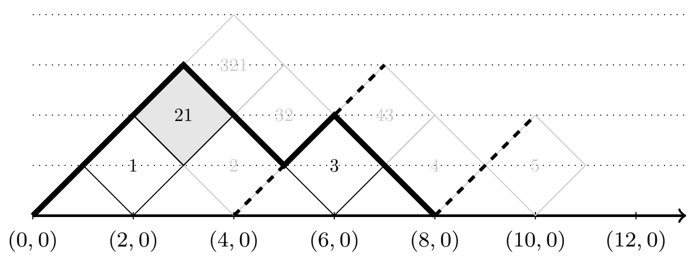
(3) If the path hits an extended ascent (before it returns to the x-axis), take one step up, and then go back to step (2).
(4) When the path returns to the x-axis, the process is complete.

Example 3.1. If D is the Dyck path in Figure 3.1, then we obtain:

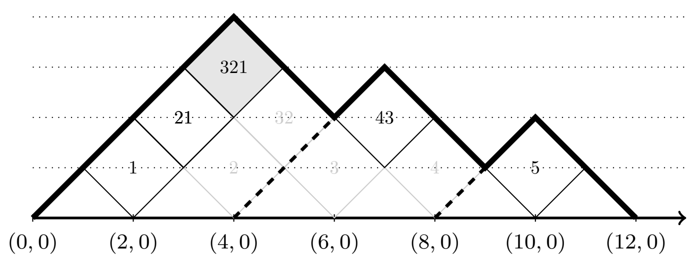
$P_D(1,1)$



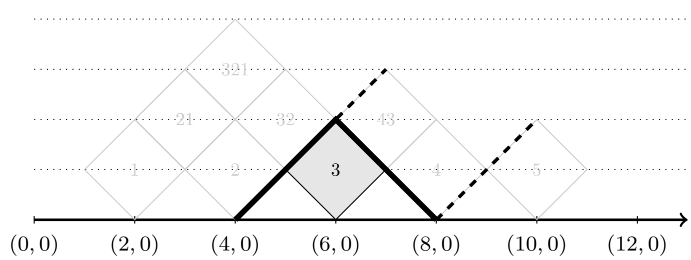
$P_D(1,2)$



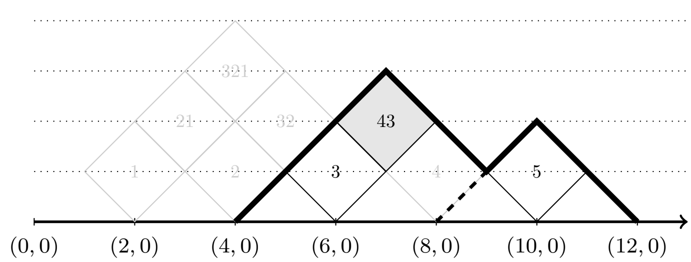
$P_D(1,3)$



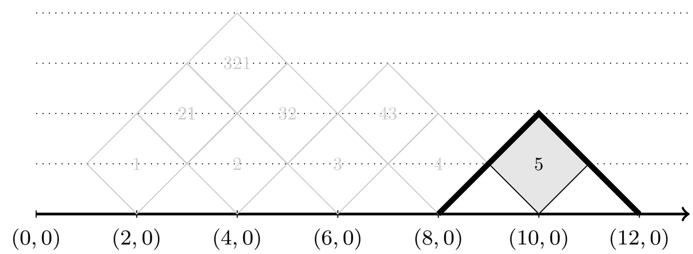
$P_D(3,3)$



$P_D(3,4)$



$P_D(5,5)$



Now we define the number $p_D(i,j)$ by

$$p_D(i,j) := (\#\text{ of steps in the ascents of } P_D(i,j)) - 1. \tag{3.1}$$

Then we observe

$$\begin{aligned} p_D(i,j) &= (\#\text{ of blocks in the ascents}) \\ &= (\#\text{ of peaks}) + (\text{height of the first peak}) - 2. \end{aligned}$$

Recall that we have the map Ψ from the set of Dyck paths into the set of fully commutative elements. Now we state the main result of this section.

Proposition 3.2. *Assume that a Dyck path D does not have an ascent longer than 1 step except for an ascent beginning on the x-axis. Then the dimension d_D of the homogeneous module $S(\Psi(D))$ is given by the formula*

$$d_D = \prod \frac{k!}{p_D(i,j)} \tag{3.2}$$

where $k =$ (sum of peak heights) $-$ (# of peaks) for the path D and the product runs over all blocks T_i^j on the extended ascents of D.

A proof of the above proposition will be given in the rest of this section. Let us see an example before we begin the proof.

Example 3.3. For the path D in Figure 3.1, the numbers $p_D(i,j)$ are shown here:

(i,j)	$(1,1)$	$(1,2)$	$(1,3)$	$(3,3)$	$(3,4)$	$(5,5)$
$p_D(i,j)$	1	3	5	1	3	1

Then the dimension of the homogeneous module corresponding to the fully commutative element $\Psi(D) = 321435$ is

$$d_D = \frac{6!}{1 \cdot 3 \cdot 5 \cdot 1 \cdot 3 \cdot 1} = 16.$$

Recall that an element $\boldsymbol{w} \in S_{n+1}$ is called *dominant minuscule* if there is a dominant integral weight Λ and a reduced expression $\boldsymbol{w} = s_{i_1} s_{i_2} \cdots s_{i_d}$ such that

$$s_{i_k} s_{i_{k+1}} \cdots s_{i_d} \Lambda = \Lambda - \alpha_{i_k} - \alpha_{i_{k+1}} - \cdots - \alpha_{i_d}, \qquad 1 \le k \le d.$$

It is known that dominant minuscule elements are fully commutative. We have the following characterization of dominant minuscule elements.

Proposition 3.4 ([19, Proposition 2.5]). *If $\boldsymbol{w} = s_{i_1}s_{i_2}\cdots s_{i_d} \in S_{n+1}$ is a reduced expression, then $\boldsymbol{w}$ is dominant minuscule if and only if the following two conditions are satisfied:*

(1) *between every pair of occurrences of a generator s_i (with no other occurrences of s_i in between) there are exactly two generators (possibly equal to each other) that do not commute with s_i;*
(2) *the last occurrence of each generator s_i is followed by at most one generator that does not commute with s_i.*

For a Dyck path D, it is clear that $\Psi(D)^{-1}$ is also a fully commutative element. The following corollary characterizes dominant minuscule elements using shapes of Dyck paths. One can compare it with Lemma 3.9 in [12], where *straight shapes* are used.

Lemma 3.5. *Let D be a Dyck path. Then $\Psi(D)^{-1}$ is dominant minuscule if and only if any ascent in D not beginning on the x-axis has a length of* 1.

Proof. Assume that D has no ascents longer than 1 besides those that begin on the x-axis. Since we must have a descent and then an ascent to get from one peak to the next, we see that the condition (i) of Proposition 3.4 is satisfied by $\Psi(D)$ and $\Psi(D)^{-1}$. Write $\Psi(D) = T_{i_1}^{m_1}T_{i_2}^{m_2}\cdots T_{i_\ell}^{m_\ell}$ as before. Then every generator in $T_{i_1}^{m_1}$ first appears in $\Psi(D)$ and there is at most one generator before its occurrence that does not commute with it. Thus $\Psi(D)^{-1}$ satisfies the condition (ii) of Proposition 3.4 with the generators in $T_{i_1}^{m_1}$.

Consider now the peak corresponding to the segment $T_{i_2}^{m_2}$. If we arrive there after an ascent of length 1, then $m_2 = m_1 + 1$ and $i_1 < i_2$. Thus the only new generator appearing in $T_{i_2}^{m_2}$ is $s_{m_2} = s_{m_1+1}$ and it commutes with all the generators preceding it except s_{m_1}. On the other hand, if we arrive at this peak after following an ascent longer than 1 step then, by assumption, this ascent begins on the x-axis. Then, we necessarily find that $i_2 > m_1+1$. So every generator in $T_{i_2}^{m_2}$ appears here for the first time, but commutes with all generators appearing previously. We can continue inductively, analyzing the generators appearing for the first time in each segment $T_{i_j}^{m_j}$, and see that the condition (ii) of Proposition 3.4 is satisfied by $\Psi(D)^{-1}$. Therefore, the element $\Psi(D)^{-1}$ is dominant minuscule.

Conversely, if $\Psi(D)^{-1}$ is dominant minuscule, the condition (ii) of Proposition 3.4 implies that any generator appearing for the first time in a segment $T_{i_j}^{m_j}$ will either commute with all previously appearing generators (thus the ascent begins on the x-axis), or that it does not commute with exactly one previously appearing generator (thus $m_{j-1} = m_j - 1$, and the ascent is of length 1). □

Proof of Proposition 3.2. We will obtain the formula (3.2) as a reformulation of the Peterson–Proctor formula [12, Theorem 3.10]. Write $\boldsymbol{w} = \Psi(D)$. It follows from Lemma 3.5 that $\boldsymbol{w}^{-1}$ is dominant minuscule. Then we only need to establish two things: first, a bijective correspondence between

$$\{\beta \in \Delta^+ : \boldsymbol{w}(\beta) < 0\} \text{ and } \{P_D(i,j) : T_i^j \text{ is on the extended ascents of } D\},$$

where Δ^+ is the set of positive roots; second, the equality $\mathrm{ht}(\beta) = p_D(i,j)$ when β corresponds to the path $P_D(i,j)$.

We write $\Psi(D) = T_{i_1}^{m_1} T_{i_2}^{m_2} \cdots T_{i_\ell}^{m_\ell}$. Each $\beta \in \Delta^+$ with $\boldsymbol{w}(\beta) < 0$ determines a unique (i_k, n_k), $i_k \le n_k \le m_k$, such that

$$\beta = \alpha_{n_k} T_{i_k}^{n_k-1} T_{i_{k+1}}^{m_{k+1}} \cdots T_{i_l}^{m_l} = \alpha_{n_k} T_{i_k}^{n_k-1} T_{i_{k+1}}^{n_k+1} T_{i_{k+2}}^{n_k+2} \cdots T_{i_l}^{n_k+l-k},$$

where the action on α_{n_k} is from the right. On the other hand, each block $T_{i_k}^{n_k}$, $i_k \le n_k \le m_k$, is on an extended ascent and

$$\Psi(P_D(i_k, n_k)) = T_{i_k}^{n_k} T_{i_{k+1}}^{n_k+1} T_{i_{k+2}}^{n_k+2} \cdots T_{i_l}^{n_k+l-k}.$$

Then the correspondence $\beta \mapsto P_D(i_k, n_k)$ is clearly one-to-one and onto.

Furthermore, we see that

$$\beta = \alpha_{n_k} T_{i_k}^{n_k-1} T_{i_{k+1}}^{n_k+1} T_{i_{k+2}}^{n_k+2} \cdots T_{i_l}^{n_k+l-k} = (\alpha_{i_k} + \cdots + \alpha_{n_k}) + \alpha_{n_k+1} + \cdots + \alpha_{n_k+l-k},$$

and $\mathrm{ht}(\beta) = n_k - i_k + 1 + l - k = (\#\text{ of steps in the ascents}) - 1 = p_D(i_k, n_k)$ from (3.1). This completes the proof. □

Even when Proposition 3.2 does not apply directly, we may still find the dimension of the corresponding module: we can

- consider the reverse path (reflected left to right), or
- invert the corresponding fully commutative element, and consider the associated Dyck path.

Note that reversing a path corresponds to the graph automorphism of the Dynkin diagram. The two options would give distinct paths, but if either satisfies the condition of Proposition 3.2, then we can obtain the correct dimension using the formula.

Example 3.6. The path D in Figure 3.2 below does not satisfy the condition of Proposition 3.2. However, we note that the reverse of the path D is nothing but the path in Figure 3.1, for which we computed the dimension in Example 3.3. Thus we obtain the same dimension, 16, for the homogeneous representation corresponding to D.

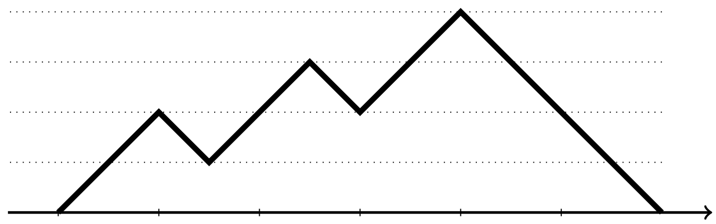


FIGURE 3.2. A Dyck Path for which the formula does not work directly

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