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MULTIPLICITY-FREE SKEW SCHUR FUNCTIONS WITH FULL INTERVAL SUPPORT

OLGA AZENHAS, ALESSANDRO CONFLITTI, AND RICARDO MAMEDE

Abstract. It is known that the Schur expansion of a skew Schur function runs over the interval of partitions, equipped with dominance order, defined by the least and the most dominant Littlewood–Richardson filling of the skew shape. We characterise skew Schur functions (and therefore the product of two Schur functions) which are multiplicity-free and the resulting Schur expansion runs over the whole interval of partitions, i.e., skew Schur functions having Littlewood–Richardson coefficients always equal to 1 over the full interval.

1. Introduction and statement of results

The ring of symmetric functions has a linear basis of Schur functions s_ν indexed by partitions ν. Skew Schur functions $s_{\lambda/\mu}$ are symmetric functions indexed by skew partitions λ/μ, and they can be expressed as a linear combination of Schur functions by means of the Littlewood–Richardson coefficients $c^\lambda_{\mu,\nu}$, which are non-negative integers,

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu,\nu} s_\nu.$$

In particular, the product of two Schur functions is governed by these coefficients, $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda$. In representation theory, this basis is important because its elements occur as characters of the general linear group GL_n, and they correspond to characters of the symmetric group via the Frobenius map. Schur functions also have an intersection-theoretic interpretation as representatives of Schubert classes in the cohomology ring of a Grassmannian. Thus Littlewood–Richardson coefficients amount to multiplicities of irreducible representations, as well as to multiplicities in the decomposition of the cup product of Schubert classes.

For any skew shape A, the support of A (or s_A) is defined to be the set of the conjugates of those partitions ν such that the Schur function s_ν appears with a positive integer coefficient in the Schur expansion of s_A. It is well known that the support of A, considered as a subposet of the dominance lattice, has a top element, $\mathbf{n}$, the conjugate of the partition formed by the row lengths of A, and a bottom element, $\mathbf{w}$, the partition formed by the column lengths of A. More precisely, the Schur expansion above with $A = \lambda/\mu$ can be written within the interval $[\mathbf{w}, \mathbf{n}]$ in the dominance lattice, as

$$s_{\lambda/\mu} = \sum_{\nu' \in [\mathbf{w},\mathbf{n}]} c^\lambda_{\mu,\nu} s_\nu. \tag{1.1}$$

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A very general problem is the classification of the shapes A whose support consists of the whole interval $[\mathbf{w},\mathbf{n}]$ in the dominance lattice. In other words, given the triple of partitions (μ,ν,λ), with $\mu\subseteq\lambda$, we address the question under which conditions we have: $c^{\lambda}_{\mu\,\nu}>0$ if and only if $\nu'\in[\mathbf{w},\mathbf{n}]$, or $c^{\lambda}_{\mu\,\nu}>0$ if and only if $\mu\cup\nu\preceq\lambda\preceq\mu+\nu$. Efforts to this classification in the case of ribbon shapes have been in progress [19, 21, 10]. This problem is equivalent to the classification of skew characters of the symmetric group and Schubert products which obey the same properties [12, 13]. Here we answer the question for which skew shapes A the Schur function s_A can be expressed as

$$s_A=\sum_{\nu'\in[\mathbf{w},\mathbf{n}]}s_\nu, \tag{1.2}$$

that is, with the coefficient $c^{\lambda}_{\mu,\nu}=1$ over the whole interval $[\mathbf{w},\mathbf{n}]$, and a corresponding classification for Schur function products $s_\mu s_\nu$.

The main theorem, Theorem 1.1, gives a classification of the multiplicity-free skew Schur functions s_A with full interval support (1.2), up to a block of maximal width or maximal depth, and up to a π-rotation and/or conjugation of the skew shape A. The classification consists of a list of seven different configurations for A with rather delicate conditions on relative part sizes of A in Figure 1.1. The subsequent Corollary 1.2 classifies the multiplicity-free Schur functions products with full interval support. Our proofs combine the classification of multiplicity-free skew Schur functions due to Gutschwager [12] and to Thomas and Yong [26] (which implies the classification of multiplicity-free products of Schur functions due to Stembridge [25]) with a procedure described in [1], here called Algorithm 1 in Subsection 3.2. The algorithm is made of several steps and, given a skew shape, produces all the Littlewood–Richardson fillings (from here on: LR fillings) from the least to the most dominant one. Most of the steps in the algorithm are necessary to understand which skew shapes are prevented to attain the full interval. A key tool towards our classification is the family of skew shapes in (4.1) not attaining the full interval.

1.1. **Schur support and multiplicity-freeness.** The motivation to study full interval multiplicity-free skew shapes A in (1.2) comes naturally when one writes the expansion (1.1), and, in particular, when one imposes in this expansion all the coefficients to be equal to 1. The dominance order "$\preceq$" on partitions has been used before in the study of Schur functions to prove that the monomial $x^\mu=x_1^{\mu_1}x_2^{\mu_2}\cdots$ occurs in s_λ if and only if $\mu\preceq\lambda$ (see [15, 17]), and to deduce necessary conditions on the support of a skew Schur function s_A, namely, that the LR filling contents of the skew shape A vary between those defined by the least and the most dominant LR filling of A (see [1, 19, 28]). The starting point of our study is a procedure described in [1], here Algorithm 1, which, given a skew shape, produces all the LR fillings from the least to the most dominant one. Indeed, the multiplicity-free phenomenon has been extensively studied before by several authors. In [25], the products of Schur functions that are multiplicity-free are completely classified, i.e., products for which every coefficient in the resulting Schur function expansion is either 0 or 1. This is done both for Schur functions in infinitely many variables and for Schur functions in finitely many variables. The latter is equivalent to a classification of all multiplicity-free tensor products of irreducible representations of GL_n or SL_n, or in other words, it is completely determined when the outer products of characters of the symmetric groups have no multiplicity. Afterwards, in [3], the analogous classification for Schur P-functions was achieved, which solved a similar problem for (projective) outer products of spin characters of double covers of the symmetric groups, and finally, in [23], the multiplicity-free problem for the expansion of Schur P-functions in terms of the Schur basis is solved, which in turn yields criteria for when an irreducible spin character of the twisted symmetric groups in the product of a basic spin character with an irreducible character of the symmetric groups is 0 or 1. The characterisation of multiplicity-free skew Schur functions was solved in [12, 26]. Furthermore, using a different combinatorial model, namely the hive model, in [7] results similar to those investigated in [12, 25, 26] are obtained.

Another problem that has received much attention — see for instance [5, 8, 14, 13, 19, 20, 21, 16] — is to determine if the difference $s_A - s_B$ of two skew Schur functions is Schur-positive, i.e., if all of the coefficients are non-negative integers when this difference is expanded as a linear combination of Schur functions. A strong necessary condition for Schur positivity of $s_A - s_B$ is that the support of B is contained in the support of A, and therefore the ordering on skew shapes defined by support containment is related to the Schur-positivity ordering.

1.2. **Statement of the main results.** We may assume our skew shape A without empty rows or empty columns. If $\tilde{A}$ is the skew Young diagram obtained from A by deleting any empty row and any empty column, the corresponding skew Schur functions are equal $s_A = s_{\tilde{A}}$. A skew Schur function without empty rows or empty columns is said to be *basic* (cf. [7]). This identity allows each skew Schur function to be expressed as a basic skew Schur function. Schur functions are also invariant under π-rotation and conjugation. The classification in the main theorem of the full interval multiplicity-free skew shapes in (1.2) consists of a list of seven different configurations comprising restrictions on the relative part sizes as described in Figure 1.1 below.

Theorem 1.1 (Main Theorem). *A basic skew Schur function $s_{\lambda/\mu}$ is multiplicity-free and its support is the whole Schur interval $[\mathbf{w}, \mathbf{n}]$ if and only if, up to a block of maximal width or maximal depth, and up to a π-rotation and/or conjugation, at least one of the following is true:*

(i) *λ/μ is a partition or a π-rotation of a partition;*
(ii) *λ/μ is a two column or a two row diagram ($A1$ configuration);*
(iii) *λ/μ is an $A2$ configuration;*
(iv) *λ/μ is an $A3$ configuration;*
(v) *λ/μ is an $A4$ configuration;*
(vi) *λ/μ is an $A6$ configuration;*
(vii) *λ/μ is an $A7$ configuration,*

as described in Figure 1.1.

As a consequence of Theorem 1.1 and the classification of multiplicity-free Schur function products [25], we get, in the corollary below, the characterisation of the multiplicity-free Schur function products that attain the full interval. It reduces to the product of two Schur functions whose indexing partitions are precisely given by the configurations: partition, $A1$, and appropriate instances of $A2$ and $A4$ in Figure 1.1, as detailed in (c) and (c') of the corollary.

Corollary 1.2. *The Schur function product $s_\mu s_\nu$ is multiplicity-free and its support is the whole Schur interval if and only if at least one of the following is true:*

(a) *μ or ν is the zero partition;*
(b) *μ and ν are both one row partitions or both one column partitions;*
(c) *$\mu = (1^x)$ is a one column partition and $\nu = (z, 1^y)$ is a hook such that either $z = 2$ and $1 \le x \le y + 1$, or $z \ge 3$ and $x = 1$ (or vice versa);*

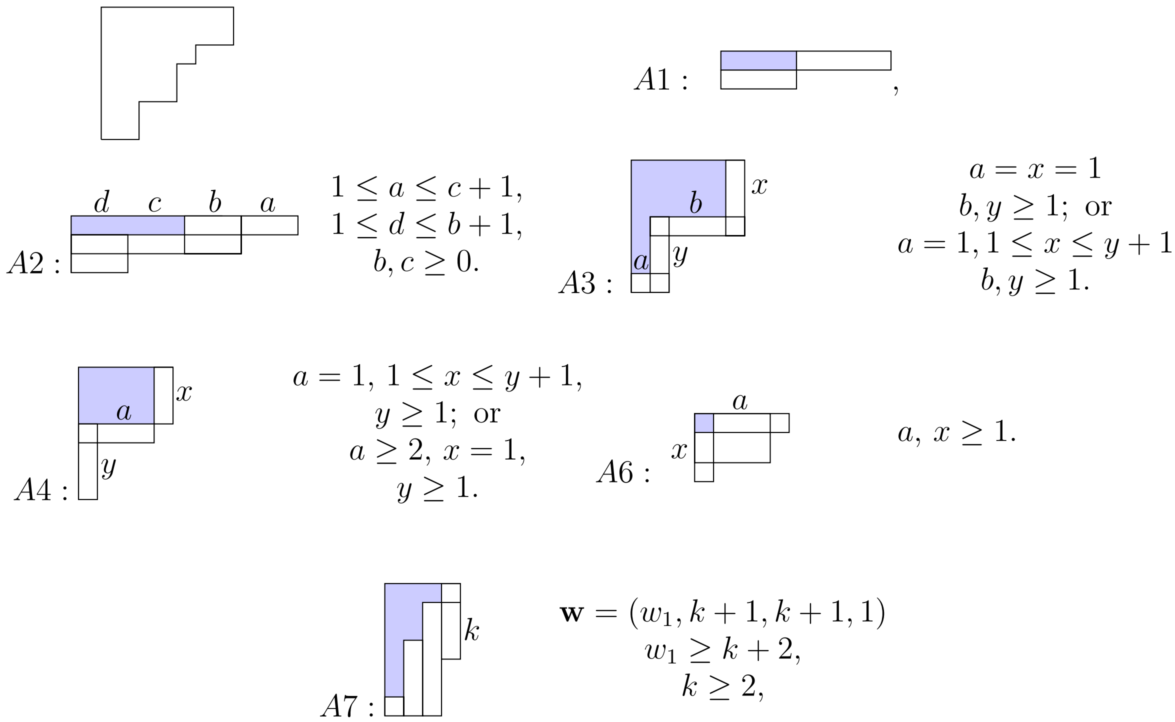


FIGURE 1.1. The seven full interval multiplicity-free skew shapes (i)–(vii) in Theorem 1.1, up to a block of maximal width or maximal depth, and up to a π-rotation and/or conjugation, with the inner shape coloured in blue, where in the $A7$ configuration, $\mathbf{w}$ is the partition formed by the column lengths of the skew diagram.

(c') $\mu = (x)$ *is a one row partition and* $\nu = (z, 1^y)$ *is a hook such that either* $y = 1$ *and* $1 \leq x \leq z$*, or* $y \geq 2$ *and* $x = 1$ *(or vice versa).*

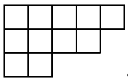
1.3. **Organisation and contents.** The paper is organised in five sections. In the next section, which in turn is divided into four subsections, we give necessary definitions regarding partitions, skew shapes and operations on them; the lattice of integer partitions with dominance order; Littlewood–Richardson tableaux using the notion of complete sequence of strings introduced in [1]; Schur, skew Schur functions, and, following the presentation given in [7], the classification of multiplicity-free skew Schur functions due to Gutschwager [12] and to Thomas and Yong [26], and therefore the classification of multiplicity-free products of Schur functions due to Stembridge [25]. In Section 3, the notion of Schur interval and support of a skew diagram and therefore of skew Schur function, considering the conjugate of the content of an LR tableau, are introduced. Algorithm 1 in [1], one of the main tools in this work, is introduced. Some other related results in [1, 27] are also recalled.

Section 4 is divided into two subsections. In Subsection 4.1, general skew shapes whose support does not achieve the Schur interval are described. The reader should particularly note Lemma 4.5 and Corollary 4.6, which give the family (4.1) of skew shapes with non-full interval support. They will be extensively used in the remainder of the paper to prevent full support. On the other hand, we observe that horizontal (and vertical) strips have full support, and that the support of a disconnected skew shape is equal to the Schur interval only if its components are ribbon shapes. (This is not sufficient, though.)

Moreover, in Proposition 4.7, we conclude that the Schur function product $s_\mu s_\nu$ has full interval support only if μ and ν are either one row or one column partitions, or one of the following holds: both are hooks, or they are a one row partition and a hook, or vice versa. In Subsection 4.2, the support of specific skew shapes, which include all configurations $A2, A3, A4, A6$ and $A7$, are analysed, and used in the last section, which is devoted to the proof of our main theorem, Theorem 1.1, and furthermore to the proof of Corollary 1.2. We remark that the strategy in the proof of Theorem 1.1 follows closely the one used in [7, Lemma 7.1] as, roughly speaking, we have to *shrink* the multiplicity-free skew shapes in order to fit the full Schur interval.

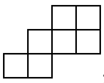
2. Preliminaries

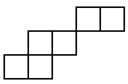
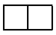
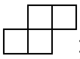
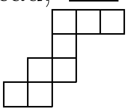
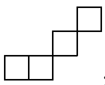
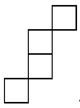
2.1. **Partitions and diagrams.** Let $\mathbb{N}$ denote the set of non-negative integers. A weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)})$ whose sum is n is said to be a *partition* of n, denoted by $\lambda \vdash n$. We say that n is the *size* of λ, denoted by $|\lambda|$, and we call the λ_i's the *parts* of λ and $\ell(\lambda)$ the *length* of λ. It is convenient to set $\lambda_k = 0$ for $k > \ell(\lambda)$. We also let 0 denote the partition with length 0. The set of all partitions of $n \in \mathbb{N}$ is denoted by P_n. If $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+j-1} = a$, we denote the sublist $\lambda_i, \dots, \lambda_{i+j-1}$ by a^j, for $j > 0$. We identify a partition $\lambda \vdash n$ with its *Young diagram*, in "English convention", which we also denote by λ: containing λ_i left justified boxes in the ith row, for $1 \le i \le \ell(\lambda)$, and use the matrix-type coordinates to refer to the boxes. For example, if $\lambda = (5, 4, 2)$, which we often abbreviate to 542, the Young diagram is

 .

A partition with at most one part size is called a *rectangle*, and a partition with exactly two different part sizes is called a *fat hook*. A fat hook is said to be a *near rectangle* if it becomes a rectangle when one suppresses one row or one column, and just a *hook* if it becomes a one row rectangle when we suppress a column. If μ is another partition, we write $\lambda \supseteq \mu$ whenever μ is contained in λ as Young diagrams, or, equivalently, $\mu_i \le \lambda_i$, for all $i \ge 1$. In this case, we define the skew diagram λ/μ which is obtained from λ by removing μ_i boxes from the ith row of λ, for $i = 1, \dots, \ell(\mu)$. In particular, $\lambda/0 = \lambda$. The size of λ/μ is $|\lambda| - |\mu|$, denoted by $|\lambda/\mu|$. A a skew diagram is *connected* if, regarded as a union of solid squares, it has a connected interior; otherwise it is *disconnected.* A *ribbon shape* is a connected skew diagram with no blocks of 2×2 squares. A skew diagram forms a *vertical (respectively horizontal) strip* if it has no two boxes in the same row (respectively column). In particular, they are disconnected skew diagrams or single rows or columns.

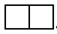
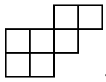
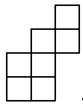
Example 2.1. The skew diagram for $\lambda/\mu = (4, 4, 2)/(2, 1)$ is

 ,

which is connected but it is not a ribbon shape. Instead,  is a disconnected skew diagram with two components  and ; and  is a ribbon shape. The following are horizontal and vertical strips, respectively,  ,  .

The *π-rotation* of a skew diagram λ/μ, denoted by $(\lambda/\mu)^\pi$, is obtained by rotating λ/μ by π radians. Denote by λ' the partition obtained by transposing the diagram of λ, called *conjugate partition* of λ, and set $(\lambda/\mu)' := \lambda'/\mu'$. If $\lambda \subseteq m^n$, then define its *m^n-complement* as the partition λ^*, where $\lambda^*_k = m - \lambda_{n-k+1}$ for $k = 1, 2, \ldots, n$.

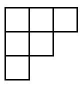
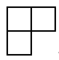
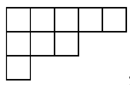
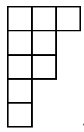
Example 2.2. If $\lambda/\mu = (4,4,2)/(2,1)$, then the 4^3-complement of λ, the π-rotation, and the transposition are

$\lambda^* =$ , $(\lambda/\mu)^\pi =$ , and $(\lambda/\mu)' =$ ,

respectively.

A partition $\lambda \subseteq m^n$ naturally defines a lattice path from the southwest to the northeast corner points of the rectangle. Following Thomas and Yong [26], we let the *m^n-shortness* of λ to be the length of the shortest straight line segment of the path of length $m+n$ from the southwest to northeast corner of m^n that separates λ from the π-rotation of λ^*. For instance, if $\lambda = (4,4,2)$ then the path from the southwest to the northeast corner of the 4^3 rectangle that borders λ is $(2,1,2,2)$, and therefore the 4^3-shortness of λ is 1.

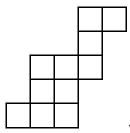
The *sum* $\lambda + \mu$ of two partitions λ and μ is the partition whose parts are equal to $\lambda_i + \mu_i$, with $i = 1, \ldots, \max\{\ell(\lambda), \ell(\mu)\}$. Using conjugation, we define the *union* $\lambda \cup \mu := (\lambda' + \mu')'$. Equivalently, $\lambda \cup \mu$ is obtained by taking all parts of λ jointly with those of μ and rearranging all these parts in descending order.

Example 2.3. Let $\lambda =$  and $\mu =$ . Then, $\lambda + \mu =$ , $\lambda \cup \mu =$ .

Fix a positive integer n, and let λ and μ be two partitions with length $\leq n$. The *product* $\lambda^\pi \bullet_n \mu$ of two partitions λ and μ is defined as

$$(\lambda_1 + \mu_1, \ldots, \lambda_1 + \mu_n)/\lambda^*,$$

where λ^* is the λ_1^n-complement. (When it is clear from the context we shall avoid the subindex n in the notation $\bullet_n$.) Graphically, place λ^π in the southeast corner of the $\lambda_1 \times n$ rectangle, and place μ in the northwest corner of the $\mu_1 \times n$ rectangle. Then, form the $(\lambda_1 + \mu_1) \times n$ rectangle by gluing together the $\lambda_1 \times n$ rectangle and the $\mu_1 \times n$ rectangle, in this order. The outcome is a connected skew diagram if $n < \ell(\lambda) + \ell(\mu)$ and a disconnected one otherwise. As illustrated below with $\lambda = (3, 2^2, 0^2)$ and $\mu = (2, 1^2, 0^2)$, we obtain $\lambda^\pi \bullet_5 \mu = (5, 4^2, 3^2)/(3^2, 1^2)$,

$\lambda^\pi \bullet \mu =$ .

2.2. Dominance order on partitions.

Definition 2.1. The *dominance order* on partitions $\lambda, \mu \vdash n$ is defined by setting $\lambda \preceq \mu$ if

$$\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i,$$

for $i = 1, \ldots, \min\{\ell(\lambda), \ell(\mu)\}$.

$(P_n, \preceq)$ is a lattice with maximum element (n) and minimum element (1^n), and is self-dual under the map which sends each partition to its conjugate. Graphically, $\lambda \preceq \mu$ if and only if the diagram of λ is obtained by "lowering" at least one box in the diagram of μ. Clearly $\lambda \preceq \mu$ if and only if $\mu' \preceq \lambda'$. Moreover, μ *covers* λ, written as $\lambda \lhd \mu$, if and only if μ is obtained from λ by lifting exactly one box in the diagram of λ to the next available position such that the transfer must be from some λ_{i+1} to λ_i , or from λ'_{i-1} to λ'_i. The interval $[\lambda, \mu]$ denotes the set of all partitions ν such that $\lambda \preceq \nu \preceq \mu$. The chain $\lambda = \lambda^0 \preceq \lambda^1 \preceq \cdots \preceq \lambda^k = \mu$, $k \geq 0$, is said to be *saturated* if λ^i covers λ^{i-1}, for $i = 1, \dots, k$ [6, 11].

2.3. **Littlewood–Richardson tableaux.** A *semi-standard Young tableau* (SSYT) T of shape λ/μ is a filling of the boxes in the diagram λ/μ with integers such that: (i) the entries of each row weakly increase when read from left to right, and (ii) the entries of each column strictly increase when read from top to bottom. The *reading word* w of a SSYT T is the word obtained by reading the entries of T from right to left and top to bottom [24]. If, for all positive integers i and j, the first j letters of w includes at least as many i's as $(i+1)'$s, then we say that w is a *lattice word*. If α_i is the number of i's appearing in T, and therefore in w, then the sequence $(\alpha_1, \alpha_2, \dots)$ is called the *content* of T, and of w. Clearly, the content of a lattice word is a partition. A SSYT T whose word is a lattice word is said to be a Littlewood–Richardson tableau (LR tableau for short). Frequently we write LR filling of the shape λ/μ to mean a filling of that shape satisfying (i) and (ii) above, whose reading word is a lattice word. Indeed the resulting SSYT is an LR tableau of shape λ/μ. Given the partition $m = (m_1, \dots, m_s)'$ the set of all lattice words with content m is equal to the set of all shuffles of the s words $12 \cdots m_1$, $12 \cdots m_2, \dots, 12 \cdots m_s$. Recall that a word w is a shuffle of the words u and v if u and v can be embedded as subwords of w that occupy complementary sets of positions within w. A shuffle w of the words $u_1, u_2, \dots, u_q$ is the empty word for $q = 0$, the word u_1 for $q = 1$, and is, otherwise, a shuffle of u_1 with some shuffle of the words $u_2, \dots, u_q$ [2].

Example 2.4. The following is a SSYT of shape $\lambda/\mu = (4,3,3,2,1)/(2,1,1)$, content $m = (4,2,2,1) = (4,3,1,1)'$ and reading word $w = 112132413$:

$$\begin{array}{cccc} & & 1 & 1 \\ & 1 & 2 & \\ & 2 & 3 & \\ 1 & 4 & & \\ 3 & & & \end{array}.$$

The reading word $w = 112132413$ is a lattice word, and it is a shuffle of the four words 1234, 123, 1 and 1. Therefore this SSYT is an LR tableau.

Taking into account the shuffle property of a lattice word, we give another characterisation of a LR tableau that we shall rather use in this work. This is based on [1, § 3], especially Definitions 5 and 6 and Theorem 5, and we refer to it for proofs and further details.

Definition 2.2. Given a semi-standard tableau T, a sequence $S_k = (y_1, y_2, \dots, y_k)$ of k positive integers is a k-*string* (or just string, for short, when there is no ambiguity) of T if $y_1 < \cdots < y_k$ and the rightmost box in row y_j is labelled with j; the corresponding strip, denoted by $\mathrm{st}\,(S_k)$, is the union of all rightmost boxes in rows y_j, for all $j = 1, \dots, k$.

We say that $S_k = (y_1, y_2, \dots, y_k) \leq S_t = (z_1, z_2, \dots, z_t)$ if $k \geq t$ and $y_j \leq z_j$ for all $j = 1, \dots, t$.

We define in a recursive way $(S_{m_1}, S_{m_2}, \dots S_{m_s})$ a *complete sequence of strings* of the tableau T having content $(m_1, m_2, \dots, m_s)'$ (note that we are writing the content in terms of its conjugate) if S_{m_1} is a string of T and $(S_{m_2}, \dots S_{m_s})$ is a complete sequence of strings of the tableau $T \setminus \mathrm{st}\,(S_{m_1})$ (when this set is not empty) having content $(m_2, \dots, m_s)'$.

In other word, $(S_{m_1}, S_{m_2}, \dots S_{m_s})$ is a complete sequence of strings of T if S_{m_j} is a string for $T \setminus \{\bigcup_{k=1}^{j-1} \mathrm{st}\,(S_{m_k})\}$ for all $j = 1, \dots, s$.

Example 2.5.

$$T = \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline & 1 & 2 & \\ \hline & 2 & 3 & \\ \hline 1 & 4 & & \\ \hline 3 & & & \\ \hline \end{array}$$

is a LR tableau with content $(4, 2, 2, 1) = (4, 3, 1, 1)'$ and it admits $S_4 = (1, 2, 3, 4) \leq S_3 = (1, 3, 5) \leq S_1 = (2) \leq S_1 = (4)$ as complete sequence of strings.

In fact, we have

$$\begin{aligned} T &= \begin{array}{cccc} & & 1 & 1 \\ & 1 & 2 & \\ & 2 & 3 & \\ 1 & 4 & & \\ 3 & & & \end{array} \\ T \setminus \mathrm{st}\,(S_4) &= \begin{array}{ccc} & & 1 \\ & 1 & \\ & 2 & \\ 1 & & \\ 3 & & \end{array} \\ (T \setminus \mathrm{st}\,(S_4)) \setminus \mathrm{st}\,(S_3) &= \begin{array}{cc} & 1 \\ 1 & \end{array} \\ ((T \setminus \mathrm{st}\,(S_4)) \setminus \mathrm{st}\,(S_3)) \setminus \mathrm{st}\,(S_1) &= \boxed{1}. \end{aligned}$$

The following result is nothing but [1, Theorem 5].

Proposition 2.1. *A semi-standard tableau with content $m = (m_1, m_2, \dots, m_s)'$ is an LR tableau if and only if it has a complete sequence of strings $S_{m_1}, S_{m_2}, \dots, S_{m_s}$; and, in particular, there is always one satisfying $S_{m_1} \leq S_{m_2} \leq \cdots \leq S_{m_s}$.*

2.4. **Schur functions, skew Schur functions and multiplicity-free classification.** Let Λ denote the ring of symmetric functions in the variables $x = (x_1, x_2, \dots)$ over $\mathbb{Q}$, say. The Schur functions s_λ form an orthonormal basis for Λ [24], with respect to the Hall inner product, and may be defined in terms of SSYT by

$$s_\lambda = \sum_T x^T \in \Lambda, \tag{2.1}$$

where the sum is over all SSYT of shape λ and x^T denotes the monomial

$$x_1^{\#1's\ in\ T} x_2^{\#2's\ in\ T} \cdots .$$

Replacing λ by λ/μ in (2.1) gives the definition of the skew Schur function $s_{\lambda/\mu} \in \Lambda$, where now the sum is over all SSYT of shape λ/μ. For instance, the SSYT shown in the Example 2.4 above contributes the monomial $x_1^4 x_2^2 x_3^2 x_4^1$ to $s_{43321/211}$.

The product of two Schur functions s_μ and s_ν can be written as a positive linear combination of Schur functions by the *Littlewood–Richardson rule* which states

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda,$$

where the *Littlewood–Richardson coefficient* $c^\lambda_{\mu\nu}$ is the number of LR tableaux with shape λ/μ and content ν [18]. The Littlewood–Richardson coefficients can also be used to expand skew Schur functions $s_{\lambda/\mu}$ in terms of Schur functions:

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu.$$

If $c^\lambda_{\mu\nu}$ is 0 or 1 for all λ (respectively all ν), then we say that the product of Schur functions $s_\mu s_\nu$ (respectively the skew Schur function $s_{\lambda/\mu}$) is *multiplicity-free*.

The Littlewood–Richardson coefficients satisfy a number of symmetry properties [24], including:

$$c^\lambda_{\mu\nu} = c^\lambda_{\nu\mu} \quad \text{and} \quad c^\lambda_{\mu\nu} = c^{\lambda'}_{\mu'\nu'}. \tag{2.2}$$

Moreover, we have

$$s_\lambda = s_{\lambda^\pi} \quad \text{and} \quad s_{\lambda/\mu} = s_{(\lambda/\mu)^\pi}. \tag{2.3}$$

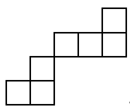
Another useful fact about skew Schur functions is that

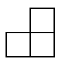
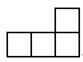
$$s_{\lambda/\mu} = s_{\tilde\lambda/\tilde\mu},$$

where $\tilde\lambda/\tilde\mu$ is the skew Young diagram obtained from λ/μ by deleting any empty row and any empty column. A skew Schur function without empty rows or empty columns is said to be *basic* [7]. Therefore, the previous identity allows each skew Schur function to be expressed as a basic skew Schur function.

If λ/μ is not connected, and consists of two components A and B, which may themselves be either Young diagrams or skew Young diagrams, then the combinatorial definition of (skew) Schur function (2.1) gives (see [24, 7])

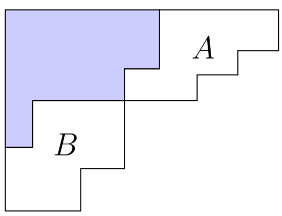
$$s_{\lambda/\mu} = s_A s_B = s_B s_A.$$

Example 2.6. If $\lambda/\mu = 5522/421 =$  , we have the disconnected components

$A =$  and $B =$ .

Therefore, $s_{5522/421} = s_{22/1} s_{33/2}$.

Any product $s_A s_B$ of skew Schur functions s_A and s_B is again a skew Schur function, as the figure below makes evident,



.

In particular, a product of Schur functions $s_\mu s_\nu$ may be seen as a skew Schur function with $A = \mu$ and $B = \nu$ in the previous picture.

For the following characterisation of the basic multiplicity-free skew Schur function, jointly due to Gutschwager and to Thomas and Yong, we follow [7].

Theorem 2.2 (Gutschwager [12], Thomas and Yong [26])**.** *The basic skew Schur function $s_{\lambda/\mu}$ is multiplicity-free if and only if at least one of the following is true:*

R0 μ or λ^* is the zero partition 0;
R1 μ or λ^* is a rectangle of m^n-shortness 1;
R2 μ is a rectangle of m^n-shortness 2 and λ^* is a fat hook (or vice versa);
R3 μ is a rectangle and λ^* is a fat hook of m^n-shortness 1 (or vice versa);
R4 μ and λ^* are rectangles;

where λ^* is the m^n-complement of λ with $m = \lambda_1$ and $n = \lambda'_1$.

In particular, for partitions μ and ν, the product $s_\mu s_\nu$ of Schur functions is a skew Schur function, and we get the following characterisation of the multiplicity-free product of skew Schur functions, due to Stembridge, as a corollary of the above theorem.

Corollary 2.3 (Stembridge [25])**.** *The Schur function product $s_\mu s_\nu$ is multiplicity-free if and only if at least one of the following is true:*

P0 μ or ν is the zero partition 0;
P1 μ or ν is a one-line rectangle;
P2 μ is a two-line rectangle and ν is a fat hook (or vice versa);
P3 μ is a rectangle and ν is a near rectangle (or vice versa);
P4 μ and ν are rectangles.

3. The Schur interval

3.1. **Skew Schur function support.** Given partitions $\mu \subseteq \lambda$, let A denote the skew diagram λ/μ. We associate to the skew diagram A two partitions: $rows(A)$ obtained by sorting the row lengths of A into weakly decreasing order, and similarly $cols(A)$ by sorting column lengths [1, 19]. It is known that $cols(A) \preceq rows(A)'$ [17, 15, 28, 1, 19]. For abbreviation, we write $\mathbf{w} := cols(A)$ and $\mathbf{n} := rows(A)'$. (When there is danger of confusion, we write $\mathbf{w}(A)$ and $\mathbf{n}(A)$, respectively.) If A consists of two disconnected partitions ϕ and θ, then $\mathbf{w} = \phi \cup \theta$ and $\mathbf{n} = \phi + \theta$.

Definition 3.1. The interval $[\mathbf{w}, \mathbf{n}] = \{\nu \in \mathbf{P}_{|\mathbf{A}|} : \mathbf{w} \preceq \nu \preceq \mathbf{n}\}$ is called the *Schur interval* of A.

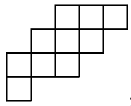
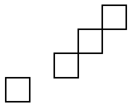
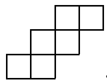
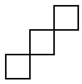
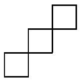
The Schur interval of A and A^π is the same, and due to the equivalence $\mathbf{w} \preceq \nu \preceq \mathbf{n}$ if and only if $\mathbf{n}' \preceq \nu' \preceq \mathbf{w}'$, the Schur interval of A' is $[\mathbf{n}', \mathbf{w}']$.

Suppose that $\mathbf{n} = (n_1, \ldots, n_s)$. We may decompose $A = \lambda/\mu$ into a sequence of n_i-vertical strips V_i, for $i = 1, 2, \ldots, s$, as follows:

1. $i := 0$. Do $V_0 := \emptyset$ and $\lambda^0/\mu := \lambda/\mu = A = A \setminus V_0$.
2. $i := i+1$. Define the n_i-vertical strip V_i formed by the rightmost box of each row in the skew-diagram $\lambda^{i-1}/\mu = A \setminus (V_0 \cup V_1 \cup \cdots \cup V_{i-1})$. Let $\lambda^i/\mu := A \setminus (V_0 \cup V_1 \cup \cdots \cup V_i)$ be the skew diagram obtained by removing that strip.
3. If $i = s$, stop. Else go to 2.

Let $V = (V_1, \ldots, V_s)$ denote the sequence of vertical strips obtained by the previous process, called the *V-sequence of A*. This means that each entry of $\mathbf{n}$ is obtained by successively pushing up some boxes in each entry of $\mathbf{w}$: n_1 is the number of non-empty rows of A; n_2 is the number of rows of A of length at least 2, …, and n_s is the number of rows of length s. Hence, n_s is the number of rows of the strip $V_s = A \setminus (V_1 \cup \cdots \cup V_{s-1})$, consisting of the leftmost boxes in each row of A with longest length $s = \ell(\mathbf{n})$. Each vertical strip V_i intersects the rows of A with longest length s, and, therefore, $\ell(\mathbf{w}) \geq \ell(\mathbf{n})$.

We shall denote the minimum and the maximum of $\mathrm{supp}(\lambda^i/\mu)$ by $\mathbf{w}^i$ and $\mathbf{n}^i$, respectively, for $i = 1, \ldots, s-1$. If $\mathbf{w} = (w_1, \ldots, w_r)$, then $r = \ell(\mathbf{w})$ is the number of non-empty columns of A, and w_1 is the length of the longest column of A.

Example 3.1. Consider the skew diagram $A =$  , with maximal filling $\mathbf{n} = (4, 3, 3)$. The V_1 strip of A is  and $A \setminus V_1 =$  . Next, we consider the V_2 strip  and $A \setminus (V_1 \cup V_2) =$  coincides with the V_3 strip.

Definition 3.2. Given the skew diagram A, the support $\mathrm{supp}(A)$ of A, or of s_A, is the set of those partitions ν' for which s_ν appears with non-zero coefficient when we expand s_A in terms of Schur functions. Equivalently,

$$\mathrm{supp}(A) = \{\nu' : c^\lambda_{\mu\nu} > 0\}.$$

Notice that we have defined the support of A in terms of the conjugate of the contents of the LR fillings of A. From [17, 15, 28, 1, 19], we know that $\nu \in \mathrm{supp}(A)$ only if $cols(A) \preceq \nu \preceq rows(A)'$ and therefore $\mathrm{supp}(A) \subseteq [\mathbf{w}, \mathbf{n}]$. Due to the rotation symmetry (2.3), the support of a skew diagram A equals the support of A^π. Moreover, by the conjugation symmetry of the Littlewood–Richardson coefficients (2.2) and the equivalence $\lambda \preceq \mu \Leftrightarrow \mu' \preceq \lambda'$, we know that $\nu \in \mathrm{supp}(A)$ if and only if $\nu' \in \mathrm{supp}(A')$.

It is furthermore known that $\mathbf{w}$ and $\mathbf{n}$ are in $\mathrm{supp}(A)$ and the coefficients $c^\lambda_{\mu,\mathbf{w}'}$ and $c^\lambda_{\mu,\mathbf{n}'}$ are both equal to 1 (see [1, 19]). The only LR tableau with shape A and content $\mathbf{w}'$ is obtained by filling the boxes of each column, from top to bottom, with the integers $1, 2, \ldots$. To describe the only LR tableau with shape A and content $\mathbf{n}'$, let $V = (V_1, \ldots, V_s)$ be the V-sequence of A. The LR tableau with shape A and content $\mathbf{n}'$ is obtained by filling each vertical strip V_i with the integers $1, \ldots, n_i$, $i = 1, \ldots, s$.

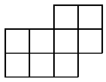
Although $\mathbf{w}'$ and $\mathbf{n}'$ are the most and the least dominant LR filling contents of A, respectively, since $\mathbf{w}$ and $\mathbf{n}$ are the minimum and maximum of $\mathrm{supp}(A)$, we will refer to the corresponding LR fillings of A as the minimum and maximum ones. The reason for this terminology comes from the fact that the lattice words with those contents $\mathbf{w}'$ and $\mathbf{n}'$ are shuffles of the words $12\ldots w_i$ and $12\ldots n_i$, respectively, $i \geq 1$, and the partitions defined by their lengths satisfy $\mathbf{w} = (w_1, \ldots w_r) \preceq \mathbf{n} = (n_1, \ldots, n_s)$.

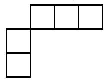
Example 3.2. The LR fillings of $A = 5442211/331$ with the least and most dominant conjugate contents (which are $\mathbf{w} = cols(A)$ and $\mathbf{n} = rows(A)'$, respectively), are

$$\begin{array}{ccccc} & & & 1 & 1 \\ & & & 2 & \\ & 1 & 1 & 3 & \\ 1 & 2 & & & \\ 2 & 3 & & & \\ 3 & & & & \\ 4 & & & & \end{array} \qquad \begin{array}{ccccc} & & & 1 & 1 \\ & & & 2 & \\ & 1 & 2 & 3 & \\ 3 & 4 & & & \\ 4 & 5 & & & \\ 6 & & & & \\ 7 & & & & \end{array},$$

where $\mathbf{w} = (4, 3, 3, 1, 1) \preceq \mathbf{n} = (7, 4, 1)$. The lattice word of content $\mathbf{w}'$ is a shuffle of the words 1234, 123, 123, 1, 1 with lengths given by $\mathbf{w}$; the lattice word of content $\mathbf{n}'$ is a shuffle of the words 1234567, 1234, 1 with lengths given by $\mathbf{n}$.

The example below shows that, in general, we have $\mathrm{supp}(A) \subsetneq [\mathbf{w}, \mathbf{n}]$.

Example 3.3. (1) The support of $A =$  is only the set $\{\mathbf{w} = 3222, \mathbf{n} = 3321\}$, and therefore $s_A = s_{441} + s_{432}$. In this case, $\mathrm{supp}(A) = \{\mathbf{w} = 3222, \mathbf{n} = 3321\} = [\mathbf{w}, \mathbf{n}]$.

(2) The Schur interval of $A =$  is the chain $\mathbf{w} = 2111 \prec 221 \prec \mathbf{n} = 311$ while $\mathrm{supp}(A) = \{21^3, 31^2\} \subsetneqq [\mathbf{w}, \mathbf{n}]$.

3.2. **An algorithm to construct LR tableaux.** The next algorithm provides a procedure to systematically construct all partitions in $\mathrm{supp}(\lambda/\mu) \cap [\mathbf{w}, \mathbf{n}]$. During the process, all LR tableaux of shape λ/μ are also exhibited. We remark that the algorithm is essentially a rephrasing of Lemmas 3 and 4 and Algorithm 4 in [1, § 4], so we refer there for further details and proofs.

Algorithm 1.

Procedure 1.
Input of the procedure: an LR tableau T of shape λ/μ and content $m = (m_1, m_2, \ldots, m_s)'$ (therefore it admits a complete sequence of strings $(S_{m_1}, S_{m_2}, \ldots S_{m_s})$ because of Proposition 2.1).

If, for all $j = 1, \ldots, s$, the strip $\mathrm{st}\left(S_{m_j}\right)$ intersects all rows of the tableau $T \setminus \{\bigcup_{k=1}^{j-1} \mathrm{st}\,(S_{m_k})\}$, then *Output of the procedure:* T (i.e., the procedure does nothing)

else

Begin

$t := \min\{j = 1, \ldots, s$ s. t. $\mathrm{st}\left(S_{m_j}\right)$ does not achieve all rows of the tableau $T \setminus \{\bigcup_{k=1}^{j-1} \mathrm{st}\,(S_{m_k})\}\}$.

$t_1 := \min\{j$ such that the row j in the tableau $T \setminus \{\bigcup_{k=1}^{t-1} \mathrm{st}\,(S_{m_k})\}$ is not achieved by $\mathrm{st}\,(S_{m_t})\}$.

$X :=$ set of boxes made of the rightmost (with respect to the tableau $T \setminus \{\bigcup_{k=1}^{t-1} \mathrm{st}\,(S_{m_k})\}$) box in rows $(t_1 \bigcup \{j > t_1$ such that $j \in S_{m_t}\})$ (i.e., $X :=$ rightmost (with respect to the tableau $T \setminus \{\bigcup_{k=1}^{t-1} \mathrm{st}\,(S_{m_k})\}$) box in row $t_1 \bigcup (\mathrm{st}\,(S_{m_t})$ below row t_1)).

Output of the procedure: $T_1 :=$ tableau obtained by T increasing by one the filling of the set X.

End

Input of the algorithm: $\mu \subseteq \lambda$.

$\mathbf{n} := (\mathbf{rows}(\lambda/\mu))'$.

$\mathbf{w} := \mathbf{cols}(\lambda/\mu)$.

$n :=$ number of columns of λ/μ.

for $i = 0, 1, \ldots, n-1$ do $(\lambda/\mu)^{n-i} :=$ the skew diagram defined by the $n-i, n-i+1, \ldots, n$ columns of λ/μ.

$T^{\{0\}} :=$ the LR tableau with shape λ/μ and content $\mathbf{w}'$.

$T^{[n]} :=$ the LR tableau of shape $(\lambda/\mu)^n$.

$i := 0$.

Repeat

Begin

To each LR tableau $T \in T^{[n-i]}$ adjoin to the leftmost column of T the $(n-i-1)$-st column of $T^{\{0\}}$ such that the LR tableau obtained is of shape $(\lambda/\mu)^{n-i-1}$.

Apply the Procedure to construct all LR tableaux of shape $(\lambda/\mu)^{n-i-1}$ containing $T \in T^{[n-i]}$, and denote this set by $T^{[n-i-1]}$.

Add the remaining columns of $T^{\{0\}}$ to each LR tableau $T^{[n-i-1]}$, obtaining a set, denoted by $T^{\{i+1\}}$, of LR tableaux of shape λ/μ.

Output of the algorithm: set $T^{\{i+1\}}$.

$i := i + 1$.

End

until $i = n$.

This algorithm produces a sequence of sets of LR tableaux of shape λ/μ

$$T^{\{0\}} \subseteq T^{\{1\}} \subseteq T^{\{2\}} \subseteq \cdots \subseteq T^{\{n\}}, \tag{3.1}$$

such that, if G is in $T^{\{i\}}$ with conjugate content γ, and B is in $T^{\{i-1\}}$ with conjugate content β, then $\beta \preceq \gamma$, for all $i = 0, \dots, n$.

Example 3.4. To make things clear, we present here some instances of application of Procedure 1.

$$\begin{array}{ccc} & 1 & 1 \\ & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 4 & \\ 5 & & \end{array} \longrightarrow \begin{array}{ccc} & 1 & 1 \\ & 2 & 2 \\ 1 & 3 & 3 \\ 4 & 4 & \\ 5 & & \end{array}$$

$$\begin{array}{ccc} & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & \\ 3 & 4 & \\ 5 & & \end{array} \longrightarrow \begin{array}{ccc} & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 3 & \\ 4 & 4 & \\ 5 & & \end{array}$$

$$\begin{array}{cccc} & & & 1 \\ & 1 & 1 & \\ 1 & 2 & 2 & \\ 2 & & & \end{array} \longrightarrow \begin{array}{cccc} & & & 1 \\ & 1 & 2 & \\ 1 & 2 & 3 & \\ 2 & & & \end{array}$$

Note that in the first two instances the conjugate content of the output covers the conjugate content of the input in dominance order, whereas in the third instance it does not.

As an easy consequence of the algorithm above, we exhibit a chain in $\mathrm{supp}(A)$, with respect to dominance order, that goes from $\mathbf{w} = (w_1, \dots, w_r)$ to $\mathbf{n} = (n_1, \dots, n_s)$. Start with the minimum LR filling of A, that is, the only filling of A with content $\mathbf{w}'$. If one fills the vertical strip V_1 with $12\dots n_1$ then λ^1/μ has the minimum LR filling, and one gets an LR filling of A with conjugate content

$$\sigma^1 := (n_1) \cup \mathbf{w}^1,$$

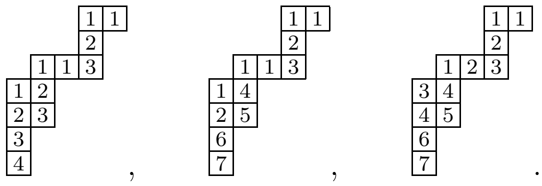
where $\mathbf{w}^1$ is the minimum of the supp(λ^1/μ). From Algorithm 1, one knows that $\mathbf{w} \preceq \sigma^1 \preceq \mathbf{n}$. (It is worth noting that the partition σ^1 is obtained by subtraction from the entries $2, \dots, r$ of $\mathbf{w}$, and adding these non-negative quantities to the first entry. Graphically, these operations correspond to lifting the rightmost box in some of the rows of A to the first row, and thus we have $\mathbf{w} \preceq \sigma^1$.) One may now repeat the above argument with the skew diagram λ^1/μ and we are led to a chain of partitions

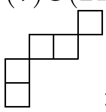
$$\mathbf{w} \preceq \sigma^1 \preceq \sigma^2 \preceq \cdots \preceq \sigma^{s-2} \preceq \sigma^{s-1} = \mathbf{n}, \tag{3.2}$$

where $\sigma^i := (n_1, \cdots, n_i) \cup \mathbf{w}^i \in \text{supp}(\lambda/\mu)$ and $\mathbf{w}^i$ is the minimum of supp(λ^i/μ), for $1 \le i \le s-1$. We have $\mathbf{w}^{i-1} \preceq (n_i) \cup \mathbf{w}^i$, for $1 \le i \le s-1$, with $\mathbf{w}^0 := \mathbf{w}$.

The next example illustrates this construction.

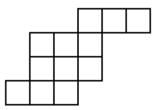
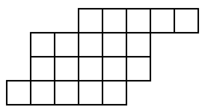
Example 3.5. Consider the skew diagram A in Example 3.2 and its sequence $V = (V_1, V_2, V_3)$ of vertical strips. Start with the minimum LR filling of A where $\mathbf{w} = (4, 3, 3, 1, 1)$, and apply Algorithm 1 to produce LR tableaux such that: the vertical strip V_1 is filled with the word 1234567, and the vertical strips V_1 and V_2 are filled with 1234567 and 1234, respectively (hence V_3 is filled with 1),

$T^{\{0\}} =$ 

The conjugate contents are: $\mathbf{w}$; $\sigma^1 = (72111) = (7) \cup (2111)$, with $\mathbf{w}^1 = 2111$ the conjugate content of the minimum LR filling of $A \setminus V_1 =$  ; and $\mathbf{n} = (741) = (74) \cup (1)$, with $\mathbf{w}^2 = (1)$ the conjugate content of the minimum LR filling of $A \setminus (V_1 \cup V_2) = V_3 = \square$. We have $\mathbf{w} \prec \sigma^1 \prec \sigma^2 = \mathbf{n}$.

Definition 3.3. The skew diagrams A and B are said to be equal up to a block of maximal depth if, for some $x \in \mathbb{N}$ and $n \ge \max\{\ell(A), \ell(B)\}$, we have $A = u^\pi \bullet_n v$ and $B = [u + (x^n)]^\pi \bullet_n v$, with u and v partitions. Similarly, they are said to be equal up to a block of maximal width if $A = (u^\pi \bullet_n v)'$ and $B = ([u + (x^n)]^\pi \bullet_n v)'$.

Example 3.6. Let $u = (3, 2, 2, 0)$, $v = (3, 1, 1, 0)$ and $n = 4$. Then

$A = u^\pi \bullet v =$  and $B = [u + (2^4)]^\pi \bullet v =$ 

are equal up to a block of maximal depth (2^4).

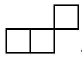
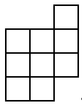
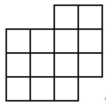
Lemma 3.1. *Let A and B be skew diagrams, and let u and v be partitions with $\max\{\ell(u), \ell(v)\} \le n$. Let $x \in \mathbb{N}$.*

(a) *If $A = u^\pi \bullet_n v$ and $B = [u^\pi + (x^n)] \bullet_n v$, then* $[\mathbf{w}(B), \mathbf{n}(B)] = [n^x \cup \mathbf{w}(A), n^x \cup \mathbf{n}(A)]$. *If $A = (u^\pi \bullet_n v)'$ and $B = ([u^\pi + (x^n)] \bullet_n v)'$, then* $[\mathbf{w}(B), \mathbf{n}(B)] = [x^n + \mathbf{w}(A), x^n + \mathbf{n}(A)]$.

(b) ([1, Lemma 5]) *If $A = u^\pi \bullet_n v$ and $B = u^\pi + (x^n) \bullet_n v$, then $c \in \mathrm{supp}(B)$ if and only if $c = b \cup (n^x)$ with $b \in \mathrm{supp}(A)$. If $A = (u^\pi \bullet_n v)'$ and $B = (u^\pi + (x^n) \bullet_n v)'$, then $c \in \mathrm{supp}(B)$ if and only if $c = b + (x^n)$ with $b \in \mathrm{supp}(A)$.*

Note that this follows from the conjugation symmetry and Algorithm 1. In the application of this algorithm, the rectangle (x^n) of length n is always filled with x words $12 \cdots n$. Therefore, if c is an element of the support of B, then it has the form $b \cup (n^x)$ for some partition b in the support of A.

Example 3.7. The following skew diagrams are equal up to a block of maximal depth and maximal width

$A =$ , $B =$ , $C =$ ,

$\mathbf{w}(A) = 111 \lhd \mathbf{n}(A) = 21$; $\mathbf{w}(B) = 333 = 2^3 + \mathbf{w}(A) \lhd \mathbf{n}(B) = 2^3 + \mathbf{n}(A) = 432$; $\mathbf{w}(C) = 4333 = 4^1 \cup \mathbf{w}(B) \lhd \mathbf{n}(C) = 4 \cup \mathbf{n}(B) = 4432$.

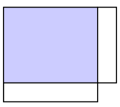
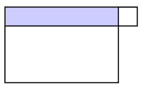
Next, the skew shapes whose support has only one element, that is, $\mathbf{w} = \mathbf{n}$, are characterised, and, therefore, the skew Schur functions which are Schur functions. Note that, in (3.2), we have $\sigma^i = \mathbf{w}$, $1 \leq i \leq s-1$, if and only if A or A^π is a partition.

Proposition 3.2. *Let A be a skew diagram and let u, v and ν be partitions. Then,*

(a) ([1, Theorems 3, 16]; [4, Lemma 4.4]) *$\mathbf{w} = \mathbf{n} = \nu$ if and only if $A = \nu$ or $A = \nu^\pi$. In this case, $\mathrm{supp}(A) = \{\mathbf{w} = \mathbf{n}\} = [\mathbf{w}, \mathbf{n}]$.*

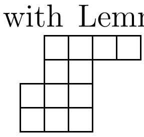
(b) ([27]) *$s_A = s_\nu$ if and only if $A = \nu$ or $A = \nu^\pi$.*

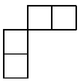
In the next proposition, we characterise the skew diagrams A whose supports have only two elements, that is, $\mathrm{supp}(A) = \{\mathbf{w}, \mathbf{n}\}$, and, in particular, those whose Schur interval has only two elements, $[\mathbf{w}, \mathbf{n}] = \{\mathbf{w}, \mathbf{n}\}$. This was shown in [1] by means of Algorithm 1. Consider the skew diagrams: $F1 = ((a+1)^x, a)/(a^x)$, and $\widetilde{F}1 = (a+1, a^x)/(a)$, a, $x \geq 1$:

(3.3) $F1$  $\widetilde{F}1$  .

Proposition 3.3 ([1, Theorem 16])**.** *Let A be a skew diagram with $\mathbf{w} \precneqq \mathbf{n}$. Then, $\mathrm{supp}(A) = \{\mathbf{w}, \mathbf{n}\}$ if and only if, up to a π-rotation or conjugation, and up to a block of maximal width or maximal depth, A is either an $F1$ or an $\widetilde{F}1$ configuration. In particular, if A is a disconnected two column (row) diagram, with one connected component being a single box, then we have $\mathbf{w} \lhd \mathbf{n}$ and $\mathrm{supp}(A) = [\mathbf{w}, \mathbf{n}] = \{\mathbf{w}, \mathbf{n}\}$.*

Remark 3.1. Note that, when applying Algorithm 1 to $F1$, we find that the only string in the minimum LR filling of $F1$ that we can stretch is the string of length x, which can only be stretched in one way, which gives rise to the maximum LR filling. Therefore, the support of $F1$ is formed by only $\mathbf{w} = (x, 1^a) \preceq \mathbf{n} = (x+1, 1^{a-1})$. If a, $x \geq 2$, the partition $\mathbf{n}$ does not cover $\mathbf{w}$ (for instance $\xi = (x, 2, 1^{a-2}) \in [\mathbf{w}, \mathbf{n}]$), and therefore the support is not the full interval. The proof is similar for $\tilde{F}_1$, considering its π-rotation.

Example 3.8. Proposition 3.3 can be used together with Lemma 3.1 to show that the support of the skew diagram $\lambda/\mu = (5, 3^3)/(1^2) =$  is not the entire Schur interval. By Lemma 3.1, it is enough to consider the support of the simple skew diagram

$\alpha/\beta =$  obtained from λ/μ by removing the block (2^4). Since the resulting diagram is the π-rotation of an $F1$ configuration with $a, x \geq 2$, the support of λ/μ is strictly contained in its Schur interval.

4. Recognition of full and non-full interval supports

In the rest of the paper, the general philosophy of the application of Algorithm 1 to a skew diagram A consists in the prolongation of its strings, starting with the minimum LR filling of A, in any possible way. We remark that in [1] a characterisation is given when, in Algorithm 1, the partition corresponding to the content of the output covers the partition corresponding to the content of the input in dominance order. Thus, when applying a step of the algorithm, we can check whether the partition corresponding to the new content covers the preceding one. If not, then we have a "suspicious" interval that may contain a partition not in the support of A.

4.1. **Bad configurations.** We start this section with the analysis of some particular configurations of boxes such that their appearance in a skew diagram A implies that A has non-full support, $\mathrm{supp}(A) \subsetneqq [\mathbf{w}, \mathbf{n}]$.

Lemma 4.1. *If A is a skew diagram and the support of $A \setminus V_1$ is not the entire Schur interval, then neither is the support of A.*

Proof. Let $[\mathbf{w}^1, \mathbf{n}^1]$ be the Schur interval of $A \setminus V_1$ and $\mathbf{w}^1 \preceq \xi \preceq \mathbf{n}^1$ such that $\xi \notin \mathrm{supp}(A \setminus V_1)$. Then $\mathbf{w} \preceq (n_1) \cup \xi \preceq \mathbf{n}$ and $(n_1) \cup \xi \notin \mathrm{supp}(A)$ since the only way to put the string $n_1 \cdots 21$ in A is to fill the strip V_1 with it and what remains is $A \setminus V_1$. □

We observe that, if $\mathrm{supp}(A \setminus V_1)$ attains the Schur interval, this does not mean that the same happens to A, as one can see in the next example.

Lemma 4.2. *Let A be a skew diagram with two or more connected components. If there is a component containing a two by two block of boxes, then the support of A is not the entire Schur interval.*

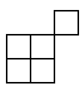
Proof. Let $\mathbf{n} = (n_1, \dots, n_s)$. Recall that $\mathbf{n}^{i-1} = (n_i, \dots, n_s)$ is the maximum of $\mathrm{supp}(A \setminus \bigcup_{k=1}^{i-1} V_k)$, $i = 2, \cdots s$, and that n_i is the number of rows of $A \setminus \bigcup_{k=1}^{i-1} V_k$, for all i. Since there is a 2 by 2 block in one of the connected components of A, there must exist a column in $A \setminus V_1$ whose length is at least 2. Let $\mathbf{w}^1 = (\overline{w}_1, \dots, \overline{w}_\ell, 1^q)$, with $\overline{w}_\ell \geq 2$ for some $\ell \geq 1$ and $q \geq 0$. Clearly $(n_2, \dots, n_s) \succcurlyeq (\overline{w}_1, \dots, \overline{w}_\ell - 1, 1^{q+1})$, and from (3.2) we have

$$\sigma^1 = (n_1, \overline{w}_1, \dots, \overline{w}_\ell, 1^q) \in [\mathbf{w}, \mathbf{n}].$$

Note that $\ell(\mathbf{w}) \geq \ell(\mathbf{w}^1) + o$, where o is the number of components of A. Since A has at least two components, we have $\ell(\mathbf{w}) \geq \ell(\sigma) + 2$. Then, the partition

$$\xi := (n_1, \overline{w}_1, \dots, \overline{w}_\ell - 1, 1^{q+1})$$

clearly satisfies $\mathbf{w} \preceq \xi \preceq \sigma$. Moreover, since $(\overline{w}_1, \dots, \overline{w}_\ell - 1, 1^{q+1}) \not\succeq \mathbf{w}^1$, it follows that $(\overline{w}_1, \dots, \overline{w}_\ell - 1, 1^{q+1}) \notin \mathrm{supp}(A \setminus V_1)$, and therefore we conclude that $\xi \notin \mathrm{supp}(A)$. □

Example 4.1. The support of the skew diagram $A =$  is not the entire Schur interval since it has two connected components, and one of them has a 2 by 2 block. We may follow the proof of the previous lemma to get a partition in the Schur interval that does not belong to the support of A. Note that $\mathbf{w} = (2, 2, 1)$, $\mathbf{n} = (3, 2)$, $\mathbf{w}^1 = (2) = \mathbf{n}^1$, $\mathbf{w} \preceq$

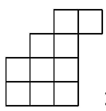
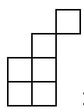
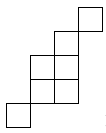
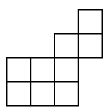
$\sigma^1 = (3,2) = \mathbf{n}$, and $[\mathbf{w},\mathbf{n}] = \{\mathbf{w} = 221, \xi = 311, \mathbf{n} = 32\}$ with $\xi \notin \mathrm{supp}(A) = \{\mathbf{w},\mathbf{n}\}$. Note also that $A \setminus V_1 = V_2$ and V_2 is a column of A.

Corollary 4.3. *If A is a skew diagram with two or more components and the support of A is the whole Schur interval $[\mathbf{w},\mathbf{n}]$, then the components of A are ribbon shapes.*

Corollary 4.4. *Let A be a skew diagram such that $\ell(\mathbf{w}) > \ell(\mathbf{n}) = s$ (equivalently, it has no block of maximal width), and the strip V_s is a column of A of length greater than or equal to 2. Then the support of A is not $[\mathbf{w},\mathbf{n}]$.*

Proof. Since $\ell(\mathbf{w}) > \ell(\mathbf{n}) = s$, A is not a partition, and since V_s is a column, A has precisely $|V_s| \geq 2$ rows of length $s \geq 2$ such that they form a rectangle, hence containing a 2×2 block of boxes. If A is disconnected, then we are done. Otherwise, as all vertical strips V_i, $1 \leq i \leq s$, traverse that rectangle, we may delete the vertical strips $V_1, \ldots, V_k$, for some k with $1 \leq k < s-1$, until we get a disconnected skew diagram. At this point, the conditions of Lemma 4.2 are satisfied, and, due to Lemma 4.1, we are also done in this case. □

Example 4.2. For instance, it follows from Corollary 4.4 that the supports of the skew diagrams

$A =$  ; $B =$  ; $C =$  ; $D =$ 

are strictly contained in the Schur interval $[\mathbf{w},\mathbf{n}]$. For A, for instance, we have $\mathbf{w} = 4321$ and $\mathbf{n} = 442$, where V_3 is a column of A of length two, and $\ell(\mathbf{w}) = \ell(\mathbf{n}) + 1$. The Schur interval is $[\mathbf{w},\mathbf{n}] = \{\mathbf{w} = 4321, 4411, 433, \mathbf{n} = 442\}$, and we have $\xi = 4411 \notin \mathrm{supp}(A) = \{\mathbf{w} = 4321, 433; \mathbf{n} = 442\}$. Therefore $s_A = s_{\mathbf{w}} + s_{433} + s_{\mathbf{n}}$. For D, we have $\xi = 4311 \in [\mathbf{w} = 32^3, \mathbf{n} = 432] \setminus \mathrm{supp}(D)$.

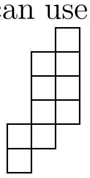
Lemma 4.5. *Let A be a connected skew diagram such that*

$$\mathbf{w} = (w_1, \ldots, w_r) \preceq \sigma^1 = [(n_1) \cup \mathbf{w}^1] = (n_1, \overline{w}_2, \ldots, \overline{w}_\ell, w_{\ell+1}, \ldots, w_r) \preceq \mathbf{n} = (n_1, \ldots, n_s),$$

for some ℓ with $3 \leq \ell \leq r$ such that $\overline{w}_k \leq w_k$ for $k = 2, \ldots, \ell$ and $0 < \overline{w}_\ell < w_\ell$. Moreover, assume the existence of two integers i and j with $2 \leq i < j \leq \ell$ such that $\overline{w}_i \geq \overline{w}_j + 2$ and $w_j > \overline{w}_j$. Then the support of A is not the entire Schur interval.

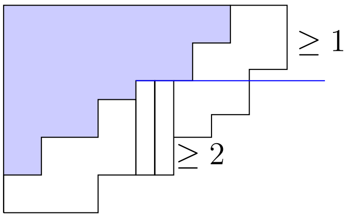
Proof. Consider the partition ξ obtained from σ^1 by replacing the entries $\overline{w}_i$ and $\overline{w}_j$ by $\overline{w}_i - 1$ and $\overline{w}_j + 1$, respectively. It is clear that $\xi \preceq \sigma^1 \preceq \mathbf{n}$. Note also that, while ξ is obtained from σ^1 by lowering one box from one row of length $\overline{w}_i$ to one of length $\overline{w}_j$, $\mathbf{w}$ is obtained from σ^1 by lowering $n_1 - w_1 = w_2 + \cdots + w_\ell - (\overline{w}_2 + \cdots + \overline{w}_\ell \geq 1)$ boxes from the first row to some rows which include the one of length $\overline{w}_j$, since $w_j > \overline{w}_j$. Thus, $\mathbf{w}$ can be obtained from ξ by lowering $k = n_1 - w_1$ boxes, in particular, $w_i - \overline{w}_i + 1$ boxes to row i and $w_j - \overline{w}_j - 1$ to row j. Thus $\mathbf{w} \preceq \xi \preceq \sigma^1$.

Moreover, we have $\xi^1 \prec \mathbf{w}^1$, where ξ^1 denotes the partition obtained from ξ by removing the first entry. Thus $\xi^1 \notin \mathrm{supp}(A \setminus V_1)$. By Lemma 4.1, we conclude that $\xi \notin \mathrm{supp}(A)$. □

Example 4.3. We can use the lemma above to conclude that the support of the connected skew diagram $A =$  is not the entire Schur interval. Indeed, we have $\mathbf{w} = (4,4,2) \preceq$

$\sigma^1 = (6,3,1) \preceq \mathbf{n} = (6,4)$, with $\mathbf{w}^1 = (\overline{w}_2, \overline{w}_3) = (3,1)$, $\ell = r = 3$, and the last two entries of σ^1 differ by 2, and they are strictly smaller than the corresponding entries of $\mathbf{w}$. Thus by lowering one box from row two to row three in σ^1, one obtains $\xi = 622$, and $\mathbf{w}$ can be obtained from ξ by lowering $w_2 - \overline{w_2} + 1 = 2$ boxes to row two, and $w_3 - \overline{w_3} - 1 = 0$ boxes to row three.

As a consequence of the last lemma, we describe below a large group of skew diagrams whose support is strictly contained in the Schur interval. Let $F0$ be a skew diagram having two columns of the same length, starting and ending on the same rows, say x and y, and such that all columns to the right of those two equal columns end at least two rows above row y, with at least one of these columns starting at least one row above row x, as illustrated by

(4.1) $F0$ 

.

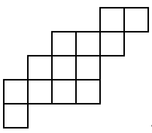
Denote by $F0', F0^\pi$ and $F0^{\pi'}$ the skew diagrams which are, in this order, the conjugate, the π-rotation, and the conjugate of the π-rotation of an $F0$ skew diagram.

Corollary 4.6. *The supports of the skew diagrams $F0$, $F0'$, $F0^\pi$, and $F0^{\pi'}$ are strictly contained in the Schur interval.*

Proof. Due to the conjugate symmetry in (2.2) and to the π-rotation symmetry (2.3), it is enough to consider the $F0$ configuration. Denote by $a+b$ the length of column W_i, which is also the length of column W_{i+1} of $F0$, where $b \geq 0$ is the number of boxes that column W_{i+1} shares with the column to its right, and $a \geq 2$ is the number of boxes of W_{i+1} with no right neighbour.

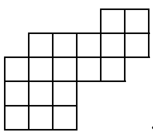
Consider $F0 \setminus V_1$ and $\mathbf{w}^1 = (\overline{w}_2, \ldots, \overline{w}_\ell, w_{\ell+1}, \ldots, w_q)$ with $\overline{w}_f = a+b$ and $\overline{w}_g = b$ satisfying $\overline{w}_f \geq \overline{w}_g + 2$, for some integers f, g, ℓ with $2 \leq f < g \leq \ell \leq q$. By Lemma 4.5, it follows that supp$(F0)$ is strictly contained in the Schur interval. $\square$

Example 4.4. (a) To illustrate the previous corollary, consider the skew diagram

$A =$  ,

with $\mathbf{w} = (3,3,2,2,2,1)$ and $\mathbf{n} = (5,4,3,1)$. Clearly, A is a $F0$ configuration since the third and fourth columns have the same length and they start in the same row, to its right all columns end two rows above the last row of these columns, and there are columns that start one row above the first row of these columns. Thus, by the previous corollary, the support of A is not the entire Schur interval. Moreover, following the proof of Lemma 4.5, we construct the partition $\xi = (5, 2^3, 1^2)$ which belongs to the interval $[\mathbf{w}, \mathbf{n}]$ but is not an element of supp(A).

(b) Consider now the skew diagram

$B =$  .

We have $\mathbf{w} = 443322 \preceq \xi = 533322 \preceq sigma^1 = 543222$ with $r = 6$ and $\ell = 4$, and $\xi \notin \mathrm{supp}(B)$.

We are now in the position to conclude the following statement from Remark 3.1 and Corollary 4.3.

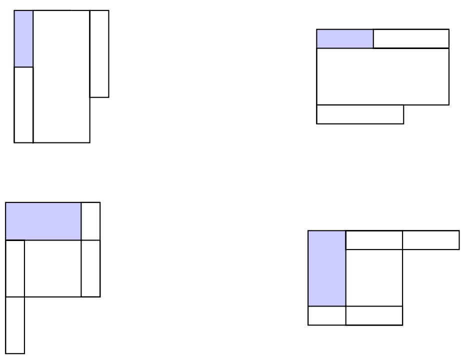
Proposition 4.7. *If A is a skew diagram with two or more components, and the support of A is the whole Schur interval, then the components of A are ribbon shapes. In particular, the Schur function product $s_\mu s_\nu$ has Littlewood–Richardson coefficients always positive over the full interval only if μ and ν are either rows or columns, or one of the following holds: both are hooks, they are a one-line rectangle and a hook, or vice-versa.*

4.2. **Recognition of full configurations.** Next we examine some particular configurations which are needed to characterise the multiplicity-free skew Schur functions that achieve the full interval. We start with the two row (column) skew diagrams. (The one row (column) case is a partition and, therefore, the support is $[\mathbf{w}, \mathbf{n}] = \{\mathbf{w} = \mathbf{n}\}$.)

Proposition 4.8 ([1, Theorem 15])**.** *If λ/μ has exactly two rows, then* $\mathrm{supp}(\lambda/\mu) = [\mathbf{w}, \mathbf{n}]$ *and it is a saturated chain. More generally, if λ/μ is a skew diagram with one of the configurations* (4.2)*, then the support of λ/μ is the entire interval* $[\mathbf{w}, \mathbf{n}]$*, and it is a saturated chain.*

Let $\lambda/\mu = (2^a, 1^b)/(1^c)$ be a two column skew diagram. Lemma 3.1 shows that the support of a skew diagram $A = ((x+2)^{a+b}, (x+1)^c)/(1^a)$, obtained by inserting a block (x^{a+b+c}) of maximal length between the two columns of λ/μ, or a skew diagram $B = ((n+2)^{a+b}, 1^c)/((x+1)^a)$ obtained by adding the partition (n^{a+b}) to λ and (n^a) to μ, is again equal to its entire Schur interval. This construction, together with its conjugates, yields four cases, whose schematic representations are shown below. These diagrams have been arranged so that the ones in the right column are the conjugates of those in the left column below.

(4.2)



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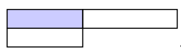
A two column or a two row disconnected skew diagram is called an $A1$ *configuration.*

(4.3)



.

Skew Schur functions whose shapes are strips made either of columns or rows always attain the full interval. Let e_n be the elementary symmetric function of degree n and h_n the complete homogeneous symmetric function of degree n. Then

$$s_{(1^n)} = \sum_{i_1 < \ldots < i_n} x_{i_1} \ldots x_{i_n} = e_n, \quad s_n = \sum_{i_1 \leq \ldots \leq i_n} x_{i_1} \ldots x_{i_n} = h_n.$$

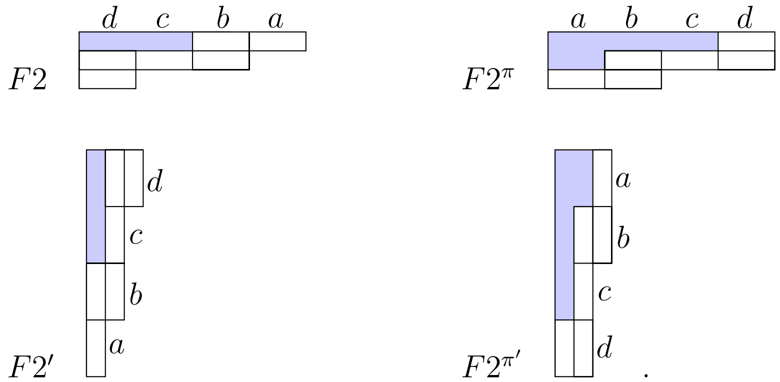
Let $\mu = (\mu_1, \dots, \mu_l)$ be a partition and $A = (1^{\mu_1}) \oplus \cdots \oplus (1^{\mu_l})$. The Schur interval of A is $[\mu, (|\mu|)]$. It is known (see [9, 24]) that

$$s_A = e_\mu := e_{\mu_1} e_{\mu_2} \dots e_{\mu_l} = \sum_\lambda K_{\lambda,\mu} s_{\lambda'}.$$

The Kostka number $K_{\lambda,\mu}$ is non-zero if and only if $\mu \preceq \lambda$, equivalently, $\lambda \in [\mu, (|\mu|)]$. Hence, $s_A = \sum_{\lambda \in [\mu,(1^{|\mu|})]} K_{\lambda,\mu} s_{\lambda'}$, and supp$(A)$ equals its Schur interval. Thus supp(A') or the support of h_μ is its Schur interval $[(1^{|\mu|}), (\mu')]$. Therefore, *if A is a direct sum of either columns or rows, the support of A equals its Schur interval.*

Consider now skew diagrams λ/μ with three rows (columns) where $\mu = (d + c)$ is a one row (column) rectangle and $\lambda^* = (a + b + c, a)$ is a fat hook, or vice versa, for some integers $a, d \geq 1$ and $b, c \geq 0$. There are four cases, as illustrated by

(4.4)



The diagrams have been arranged so that those on the right are the π-rotations of the diagrams on the left, and the diagrams in the second row are the conjugates of the ones in the first row.

If the diagrams in (4.4) *satisfy the additional conditions $a \leq c + 1$ and $d \leq b + 1$, then they are called $A2, A2^\pi, A2'$ and $A2^{\pi'}$ configurations, as illustrated above, replacing the letter F by A.*

Proposition 4.9. *Let λ/μ be one of the skew diagrams in* (4.4)*. Then the support of λ/μ coincides with its Schur interval if and only if it is an $A2, A2^\pi, A2'$ or an $A2^{\pi'}$ configuration.*

Proof. By assumption $a, d \geq 1$ and $b, c \geq 0$. If $a = 0$ or $d = 0$, we are in the case of two columns (rows) that we already studied in Proposition 4.8. Due to the rotation and conjugation symmetry, we only consider case $F2'$. We will start by showing that, if $a > c + 1$ or $d > b + 1$, the support of λ/μ is not the entire Schur interval. For the first case, $a > c + 1$, just note that with $k := a - (c + 1)$ the partition

$$\xi = (d + c + b + k, d + b + c + 1)$$

belongs to the Schur interval of λ/μ, since the first entry of $\mathbf{w}$ is $\max\{d + c + b, b + a\}$ and the first entry of $\mathbf{n}$ is $d + c + b + a$. Moreover, we have $\xi \notin$ supp(λ/μ) since, when placing the string of length $d + c + b + k$ in λ/μ, we must place the integer $d + b + c + 1$ in the first column of the diagram, leaving no room to place the second string.

For the case $d > b + 1$, note that the entries of $\mathbf{w}$ are $b + c + d, a + b$ and d, and that $\sigma^1 = (a + b + c + d, d, b)$ with $d \geq b + 2$. By Lemma 4.5, it follows that the support of λ/μ is not the entire Schur interval.

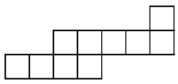
Thus, if $a > c+1$ or $d > b+1$, the support of λ/μ does not coincide with its Schur interval. For the rest of the proof, assume $a \leq c+1$ and $d \leq b+1$. Then we have

$$\mathbf{w} = (d+c+b, b+a, d) \preceq \mathbf{n} = (d+c+b+a, d+b).$$

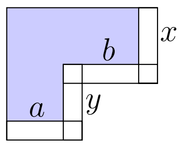
Let $\xi = (\xi_1, \xi_2, \xi_3) \in [\mathbf{w}, \mathbf{n}]$. We will show that $\xi \in \operatorname{supp}(\lambda/\mu)$. Since $\mathbf{w} \preceq \xi \preceq \mathbf{n}$, we must have

$$d+c+b \leq \xi_1 \leq d+c+b+a \quad \text{and} \quad d+c+2b+a \leq \xi_1+\xi_2 \leq 2d+c+2b+a.$$

Then $\xi_1 = d+c+b+k$ and $a+b-k \leq \xi_2 \leq d+a+b-k$, for some $k \in \{0,\dots,a\}$. Due to Algorithm 1, this means that, for each $k \in \{0,\dots,a\}$, after placing (in the unique possible way) the string of length ξ_1 in the diagram λ/μ, we must insert the strings of length ξ_2, ξ_3 in the skew diagram $\tilde{\lambda}/\mu$ obtained by removing the boxes of the string of length ξ_1. The Schur interval of $\tilde{\lambda}/\mu$ is $[\tilde{\mathbf{w}} = (b+a-k, d), \tilde{\mathbf{n}} = (d+b+a-k)]$. Since $\tilde{\lambda}/\mu$ has two columns, by Proposition 4.8, we have $[\tilde{\mathbf{w}} = (b+a-k, d), \tilde{\mathbf{n}} = (d+b+a-k)] = \operatorname{supp}(\tilde{\lambda}/\mu)$. On the other hand, we have $\tilde{\mathbf{w}} \preceq (\xi_2, \xi_3) \preceq \tilde{\mathbf{n}}$, hence $(\xi_2, \xi_3) \in \operatorname{supp}(\tilde{\lambda}/\mu)$. Finally, note that, if $k \geq 0$, it follows from the inequality $a-k < c+1$ that $\xi_2 \leq d+b+a-k < d+c+b+1$. Therefore we also have $\xi \in \operatorname{supp}(\lambda/\mu)$. $\square$

Example 4.5. The skew diagram $\lambda/\mu =$  is an $A2^\pi$ configuration with $d = 1$ and $c = b = a = 2$. Therefore, its Schur interval coincides with its support.

The next configuration that we consider is the ribbon skew diagram λ/μ, with $\mu = ((a+b+1)^x, a^y)$ and $\lambda^* = ((b+1)^{y+1})$, for some integers $a, b, x, y \geq 1$, as illustrated by

(4.5) $F3$ 

.

An $F3$ configuration (4.5) *with $a = x = 1$, or $a = 1$ and $x \leq y+1$, or $a \leq b+1$ and $x = 1$, is called an $A3$ configuration.*

Proposition 4.10. *Let λ/μ be a skew diagram with a configuration* (4.5)*. Then its support equals its Schur interval if and only if it is an $A3$ configuration. Moreover, if $a = x = 1$, the support of $A3$ is*

$$[\mathbf{w}, \mathbf{n}] = \{\mathbf{w}, \xi_2 = (y+2, 2, 2, 1^{b-1}), \xi_3 = (y+2, 3, 1^b), \xi_1 = (y+3, 1^{b+2}), \mathbf{n}\},$$

and the skew Schur function $s_{\lambda/\mu} = s_{\mathbf{w}'} + s_{\xi_1'} + s_{\xi_2'} + s_{\xi_3'} + s_{\mathbf{n}'}$ has exactly five components all with multiplicity 1.

Proof. We start by noticing that, when both integers a and x are strictly greater than 1, then the minimum and maximum of the $\operatorname{supp}(\lambda/\mu)$ are given by

$$\mathbf{w} = (w_1, w_2, 1^{a+b}) \preceq \mathbf{n} = (x+y+2, 2^{\min\{b+1,a\}}, 1^{a+b+1-2\min\{b+1,a\}}),$$

with $w_1 = \max\{y+2, x+1\}$, $w_2 = \min\{y+2, x+1\}$ and $\min\{b+1, a\} \geq 2$. Therefore, we can consider the partition

$$\xi := (w_1, w_2, 3, 1^{a+b-3}).$$

It is straightforward to check that $\mathbf{w} \preceq \xi \preceq \mathbf{n}$ and that ξ is not in the support of λ/μ.

In the case $a = x = 1$, in which the minimum and maximum of the support are given by

$$\mathbf{w} = (y + 2, 2, 1^{b+1}) \preceq \mathbf{n} = (y + 3, 2, 1^b),$$

the Schur interval is $[\mathbf{w}, \mathbf{n}] = \{\mathbf{w}, (y + 3, 1^{b+2}), (y + 2, 2, 2, 1^{b-1}), (y + 2, 3, 1^b), \mathbf{n}\}$, and we can check directly that this interval is equal to $\operatorname{supp}(\lambda/\mu)$.

So we are left with the case $a = 1$ and $x > 1$, since the remaining case is obtained by conjugation symmetry. The minimum and maximum of the support are

$$\mathbf{w} = (w_1, w_2, 1^{b+1}) \preceq \mathbf{n} = (x + y + 2, 2, 1^b),$$

where w_1 and w_2 are defined as in the previous case. Thus, if ξ is a partition in the Schur interval, it must satisfy $\xi = (\xi_1, \xi_2, \xi_3, 1^b)$, for some integers ξ_i. Moreover, we must have

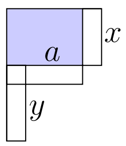
$$\begin{array}{ccccc} \max\{y+2, x+1\} & \leq & \xi_1 & \leq & x+y+2, \\ x+y+3 & \leq & \xi_1+\xi_2 & \leq & x+y+4, \\ & & \text{and} & & \\ x+y+4 & \leq & \xi_1+\xi_2+\xi_3 & \leq & x+y+5. \end{array}$$

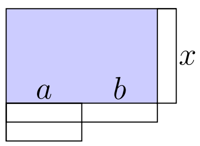
There are two possibilities for the sum $\xi_1 + \xi_2$. As $\xi_3 \geq 1$, when this sum equals $x+y+3$, it follows that ξ_3 must be either 1 or 2, and if $\xi_1 + \xi_2 = x + y + 4$ then $\xi_3 = 1$. Consider $\xi_1 + \xi_2 = x + y + 3$. We have $\tilde{\mathbf{w}}_1 := (w_1, w_2) \preceq (\xi_1, \xi_2) \preceq \tilde{\mathbf{n}}_1 := (x + y + 2, 1)$, where $\tilde{\mathbf{w}}_1$ and $\tilde{\mathbf{n}}_1$ are the minimum and maximum LR fillings of the two column skew diagram $\tilde{A}$ obtained from λ/μ by removing all except the second and last columns. Since, by Proposition 4.8, we have $\operatorname{supp}(\tilde{A}) = [\tilde{\mathbf{w}}_1, \tilde{\mathbf{n}}_1]$, we conclude that we may place strings of length ξ_1 and ξ_2 using only the second and last columns of λ/μ. As the remaining entries of ξ are equal to 1, and, at most one, equal to 2, it follows that $\xi \in \operatorname{supp}(\lambda/\mu)$.

Assume now that $\xi_1 + \xi_2 = x + y + 4$. In this case, we have $(\xi_1, \xi_2) \in [\tilde{\mathbf{w}}_2, \tilde{\mathbf{n}}_2]$, where $\tilde{\mathbf{w}}_2 = (w_1, w_2, 1)$ and $\tilde{\mathbf{n}}_2 = (x + y + 2, 2)$ are the minimum and maximum LR fillings of the skew diagram B obtained from λ/μ removing all columns except the first two and the last one. Note that B is an $F2^{\pi'}$ configuration as in (4.4), and, by Proposition 4.9, we have $\operatorname{supp}(B) = [\tilde{\mathbf{w}}_2, \tilde{\mathbf{n}}_2]$ if and only if $x \leq y + 1$. Thus, if $x \leq y + 1$, we find that $(\xi_1, \xi_2) \in \operatorname{supp}(B)$ and similarly, as before, it follows that $\xi \in \operatorname{supp}(\lambda/\mu)$. If, on the other hand, we have $x > y + 1$, then, by Proposition 4.9, we can consider a partition $(\sigma_1, \sigma_2) \in [\tilde{\mathbf{w}}_2, \tilde{\mathbf{n}}_2]$ which is not in the set $\operatorname{supp}(B)$. It follows that $\sigma := (\sigma_1, \sigma_2, 1^b) \in [\mathbf{w}, \mathbf{n}]$ but $\sigma \notin \operatorname{supp}(\lambda/\mu)$. $\square$

In the next lemmas, we analyse some families of skew diagrams needed in the sequel. We start with skew diagrams λ/μ of types $F4$ and $\widetilde{F}4$ defined by the partitions $\lambda = ((a + 2)^x, a + 1, 1^y)$ and $\mu = ((a + 1)^x)$, respectively, for some a, x, $y \geq 1$ such that not both x and y are equal to 1, and by the partitions $\lambda = ((a + b + 1)^x, a + b, a)$ and $\mu = ((a + b)^x)$, for some integers $b \geq 1$ and $a, x > 1$, as illustrated by

(4.6)

$F4$ 

$\widetilde{F}4$ 

Note that, if we let $x = y = 1$ in an $F4$ configuration, or $x = 1$ in an $\widetilde{F}4$ configuration, then we get an $F2$ configuration.

An $F4$ configuration with $a = 1$ and $x \leq y + 1$, or $a \geq 2$ and $x = 1$, is called an $A4$ configuration.

Proposition 4.11. (i) *If λ/μ is a skew diagram with configuration $F4$, then its support is equal to the Schur interval if and only if it is an $A4$ configuration. Moreover, if $a \geq 2$ and $x = 1$, the support of $A4$ is $[\mathbf{w}, \mathbf{n}] = \{\mathbf{w}, \xi = (y+1, 2, 1^{a-1}), \mathbf{n}\}$, and the skew Schur function $s_{\lambda/\mu} = s_{\mathbf{w}'} + s_{\xi'} + s_{\mathbf{n}'}$ has exactly three components all with multiplicity 1.*

(ii) *The support of $\widetilde{F}4$ is strictly contained in the Schur interval.*

Proof. We start with an $F4$ configuration. If $a \geq 2$, then the minimum and maximum of $\mathrm{supp}(\lambda/\mu)$ are

$$\mathbf{w} = (w_1, w_2, 1^a) \preceq \mathbf{n} = (x+y+1, 1^a),$$

with $w_1 = \max\{x, y+1\}$, $w_2 = \min\{x, y+1\}$ and $\ell(\mathbf{w}) = \ell(\mathbf{n}) + 1$. If $x \geq 2$, the partition $\xi := (w_1, w_2, 2, 1^{a-2})$ satisfies $\ell(\xi) = \ell(\mathbf{n})$, and thus $\mathbf{w} \preceq \xi \preceq \mathbf{n}$, but it is not in the support of λ/μ since the strings of length w_1 and w_2 must fill the first and last columns, leaving no space for the string of length 2. On the other hand, if $x = 1$, then the Schur interval of λ/μ is

$$[\mathbf{w}, \mathbf{n}] = \{\mathbf{w}, (y+1, 2, 1^{a-1}), \mathbf{n}\} = \mathrm{supp}(\lambda/\mu).$$

Assume now that $a = 1$. If $x > y+1$, then $\mathbf{w} = (x, y+1, 1) \preceq \mathbf{n} = (x+y+1, 1)$, and $\xi := (x, y+2) \in [\mathbf{w}, \mathbf{n}]$, but clearly $\xi \notin \mathrm{supp}(\lambda/\mu)$. If otherwise, we have $x \leq y+1$. Then the minimum and the maximum of λ/μ are given by

$$\mathbf{w} = (y+1, x, 1) \preceq \mathbf{n} = (x+y+1, 1).$$

A partition $\xi = (\xi_1, \xi_2, \xi_3) \in [\mathbf{w}, \mathbf{n}]$ must satisfy $y+1 \leq \xi_1 \leq x+y+1$ and $x+y+1 \leq \xi_1 + \xi_2 \leq x+y+2$ with $\xi_3 \in \{0, 1\}$. Let $\xi_1 = y+1+k$, for some $k \in \{0, \ldots, y+1\}$. Then, we get $x - k \leq \xi_2 \leq x+1-k$, and since $x \leq y+1$ it follows that the partition ξ belongs to the support of λ/μ.

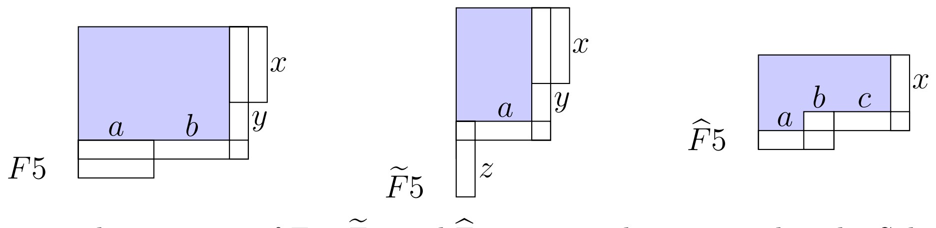
Finally, suppose that λ/μ is an $\widetilde{F}4$ configuration. Then the minimum and maximum of $\mathrm{supp}(\lambda/\mu)$ are

$$\mathbf{w} = (x, 2^a, 1^b) \preceq \mathbf{n} = (x+2, 2^{a-1}, 1^b),$$

and, since $a \geq 2$, the vector $\xi := (x+2, 2^{a-2}, 1^{b+2})$ is a partition and satisfies $\mathbf{w} \preceq \xi \preceq \mathbf{n}$, but $\xi \notin \mathrm{supp}(\lambda/\mu)$. □

The next skew diagrams λ/μ are $F5$, $\widetilde{F}5$ and $\widehat{F}5$, respectively, defined by the partitions: $\mu = ((a+b)^{x+y})$ and $\lambda^* = (b+2, 1^{y+1})$, with $a, x \geq 2$ and $b, y \geq 0$; $\mu = ((a+1)^{x+y})$ and $\lambda^* = ((a+2)^z, 1^{y+1})$, with $a \geq 2, y \geq 0$ and $x, z \geq 1$; and $\mu = ((a+b+c)^x, a)$ and $\lambda^* = (c+1)$, with $x \geq 2$ and either $a, b, c \geq 1$ or $b = 0$ and $a, c \geq 1$ with $a + c \geq 3$, as illustrated by

(4.7)



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Lemma 4.12. *The supports of $F5$, $\widetilde{F}5$ and $\widehat{F}5$ are strictly contained in the Schur interval.*

Proof. The minimum and maximum of the support of $F5$ are

$$\mathbf{w} = (x+y+1, x, 2^a, 1^b) \preceq \mathbf{n} = (x+y+2, x+2, 2^{a-2}, 1^{b+1}),$$

with $\ell(\mathbf{w}) = \ell(\mathbf{n}) + 1$. Since a, $x > 1$, we may consider the partition

$$\xi := (x+y+1, x+1, 3, 2^{a-2}, 1^b),$$

which is an element of the Schur interval, but it is not in the support of $F5$.

In the case of $\widetilde{F}5$, we have

$$\mathbf{w} = (w_1, w_2, w_3, 1^a) \preceq \mathbf{n} = (x+y+z+1, x+1, 1^a),$$

with w_1, w_2, w_3 the lengths of the first and the two last columns of $\widetilde{F}5$ in decreasing order. Take

$$\xi := (x+y+z+1, x, 2, 1^{a-1}) \in [\mathbf{w}, \mathbf{n}].$$

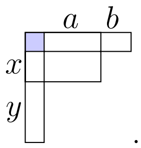
Since, after placing the strings of length $x+y+z+1$ and x in the unique possible positions in $\widetilde{F}5$, we are left with only a single row, it follows that $\xi \notin \operatorname{supp}(\widetilde{F}5)$.

Finally, we consider the case of an $\widehat{F}5$ configuration. If $b \geq 1$, the minimum and the maximum of the support are given by

$$\mathbf{w} = (x+1, 2^b, 1^{a+c}) \preceq \mathbf{n} = (x+2, 2^b, 2^{\min\{a-1,c\}}, 1^{a-1+c-2\min\{a-1,c\}}).$$

It is easy to check that the partition $\xi := (x+1, 3, 2^{b-1}1^{a+c-1})$ is an element of the Schur interval but is not in the support of $\widehat{F}5$. Similarly, if $b = 0$, the partition $\xi = (x+1, 3, 1^{a+c-3})$ belongs to the Schur interval but not to the support of $\widehat{F}5$. □

The next family of skew diagrams λ/μ is denoted by $F6$, and it is defined by partitions $\lambda = (a+b+1, (a+1)^x, 1^y)$ and $\mu = (1)$, for some integers $a, x > 0$ and $b, y \geq 1$,

(4.8) $F6$  .

If $b = y = 1$ an $F6$ configuration is called an $A6$ configuration.

Proposition 4.13. *The support of a skew diagram λ/μ which is an $F6$ configuration equals the Schur interval if and only if it is an $A6$ configuration. Moreover, in this case, the support is*

$$[\mathbf{w}, \mathbf{n}] = \{\mathbf{w} = ((x+1)^{a+1}, 1), \xi = (x+2, (x+1)^{a-1}, x, 1), \mathbf{n} = (x+2, (x+1)^a) : a,\ x \geq 1\},$$

and the skew Schur function $s_{\lambda/\mu} = s_{\mathbf{w}'} + s_{\xi'} + s_{\mathbf{n}'}$ has three components all with multiplicity 1.

Proof. We consider the case $b \geq 2$. Notice that the case $y \geq 2$ is the conjugate of the former. In our case, the minimum and the maximum of the support are

$$\mathbf{w} = (x+y, (x+1)^a, 1^b) \preceq \mathbf{n} = (x+y+1, (x+1)^a, 1^{b-1}),$$

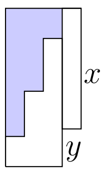
and we can consider the partition $\xi := (x+y, (x+1)^a, 2, 1^{b-2}) \in [\mathbf{w}, \mathbf{n}]$, which clearly does not belong to the support of λ/μ.

For the remaining case $y = b = 1$, notice that the Schur interval is given by

$$[\mathbf{w}, \mathbf{n}] = \{\mathbf{w} = ((x+1)^{a+1}, 1), \xi = (x+2, (x+1)^{a-1}, x, 1), \mathbf{n} = (x+2, (x+1)^a)\}.$$

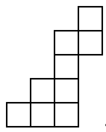
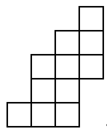
Since $\xi \in \operatorname{supp}(\lambda/\mu)$, the result follows. □

In the next lemma, we analyse the support of a skew diagram λ/μ, with $\ell + 1 \geq 4$ columns, having the form

(4.9) $F7$  , where the first ℓ columns end in the same row and have pairwise distinct lengths, x is the length of the last column, and $\lambda^* = (1^y)$. Moreover, by Lemma 3.1, we assume without loss of generality that the next-to-last column starts at least one row below the topmost box of the last column, and that the first column has length $\leq y$. Denote by $(w_1, \ldots, w_\ell)$ the partition formed by the first ℓ columns of the diagram, and let $k \geq 0$ be the number of rows that the last two columns share.

A skew diagram F7 such that $\ell = 3$, $x = w_2$, $w_3 = 1$ *and* $k = w_2 - 1$ *is called an A7 configuration.*

We distinguish two cases: either (i) $w_2 \leq y$, or (ii) $w_2 > y$. In the first case, the last column shares rows only with column ℓ, and, in the second case, the last column shares rows with at least columns ℓ and $\ell - 1$. Examples of $A7$ configurations of types (i) and (ii) are shown below:

(i)  , (ii)  .

In the next two lemmas we show that the support of an $A7$ configuration is the entire Schur interval.

Lemma 4.14. *Let* λ/μ *be an F7 configuration* (4.9) *such that* $w_2 \leq y$. *Then the support of* λ/μ *equals the Schur interval if and only if* $\ell = 3$, $w_3 = 1$, $x = w_2$ *and* $k = w_2 - 1$.

Proof. The condition $w_2 \leq y$ means that the last column of λ/μ may share rows only with the next-to-last column. Then, the minimum and maximum of the support of λ/μ are given by

$$\mathbf{w} = (w_1, \ldots, w_\ell) \cup \{x\} \preceq \mathbf{n} = (w_1 + x - k, w_2 + k, w_3, \ldots, w_\ell),$$

with $\ell(\mathbf{w}) = \ell(\mathbf{n}) + 1$.

Note that, if $w_\ell \geq 2$, it follows from Corollary 4.4 that $\mathrm{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$, and if $\ell \geq 4$ and $w_\ell = 1$, the partition $\xi := (w_1 + x - k, w_2 + k, w_3, \ldots, w_{\ell-1} - 1, 1, 1)$ shows that $\mathrm{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$. Moreover, if $k = 0$, the skew diagram is disconnected, and, by Lemma 4.2, the support of λ/μ is not the entire Schur interval.

So, assuming $\ell = 3$, $k > 0$ and $w_3 = 1$, we have

$$\mathbf{w} = (w_1, w_2, 1) \cup \{x\} \preceq \mathbf{n} = (w_1 + x - k, w_2 + k, 1).$$

If $x < w_2$ then $1 \leq k < x < w_2$, and, in particular, we get $w_2 \geq k + 2$. Since w_2 and k are the lengths of the second and third columns of λ^1/μ (see page 10 for the definition of λ^1/μ), it follows from Lemma 4.5 that $\mathrm{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$. The same situation happens if $x > w_2$, since in this case the partition $\xi := (w_1, w_2 + 1) \cup (x)$ satisfies $\mathbf{w} \preceq \xi \preceq \mathbf{n}$, but it does not belong to the support of λ/μ.

We are, therefore, left with the case $\ell = 3, k > 0, x = w_2$ and $w_3 = 1$. Note that $1 \leq k \leq w_2 - 1$. If $k < w_2 - 1$, the second and third columns of λ^1/μ have lengths w_2 and k, respectively, and by Lemma 4.5, we find that the support of λ/μ is not the entire Schur interval. So we must also consider $k = w_2 - 1$. In this case, the minimum and the maximum of the support are given by

$$\mathbf{w} = (w_1, w_2, w_2, 1) \preceq \mathbf{n} = (w_1 + 1, w_2 + w_2 - 1, 1).$$

Let $\xi := (\xi_1, \xi_2, \xi_3, \xi_4) \in [\mathbf{w}, \mathbf{n}]$, and note that $w_1 \leq \xi_1 \leq w_1 + 1$.

If $\xi_1 = w_1$, then the inequalities $\mathbf{w} \preceq \xi \preceq \mathbf{n}$ imply that $\alpha \preceq (\xi_2, \xi_3, \xi_4) \preceq \beta$, where α and β are the minimum and the maximum of the support of the skew diagram A obtained from λ/μ by removing the third column. Since A is an $A2^{\pi'}$ configuration, it follows that $(\xi_2, \xi_3, \xi_4) \in \mathrm{supp}(A)$, and therefore $\xi \in \mathrm{supp}(\lambda/\mu)$.

For the remaining case $\xi_1 = w_1 + 1$, the situation is analogous, since in this case we have $\mathbf{w}^1 \preceq (\xi_2, \xi_3, \xi_4) \preceq \mathbf{n}^1$, where $\mathbf{w}^1$ and $\mathbf{n}^1$ are the minimum and the maximum LR filling of $A \setminus V_1$. Since this diagram is also an $A2^{\pi'}$ configuration, we find that $\xi \in \mathrm{supp}(\lambda/\mu)$. $\square$

Lemma 4.15. *Let λ/μ be an $F7$ configuration* (4.9) *such that $w_2 > y$. Then the support of λ/μ equals the Schur interval if and only if $\ell = 3$, $w_3 = 1$, $x = w_2$ and $k = w_2 - 1$.*

Proof. We start the proof by showing that, if the conditions $\ell = 3$, $w_3 = 1$, $x = w_2$ and $k = w_2 - 1$ are not satisfied, then the support of λ/μ is not the entire Schur interval. Consider the minimal and maximal fillings of the diagram,

$$\mathbf{w} = (w_1, w_2, \ldots, w_\ell) \cup (x) \preceq \mathbf{n} = (n_1, n_2, \ldots, n_s),$$

where $n_1 = x + y$, $n_2 = w_1$ and $\ell(\mathbf{w}) > \ell(\mathbf{n})$.

Note that, if $k = 0$, the diagram λ/μ is disconnected, with one of the connected components having a 2 by 2 block. In this case, by Lemma 4.2, $\mathrm{supp}(\lambda/\mu)$ is not the entire Schur interval. So, assume $k > 0$. If $x > w_2$, then the partition $\xi := (w_1, \ldots, w_{\ell-2}, w_{\ell-1} + 1, w_\ell - 1) \cup (x)$ shows that $\mathrm{supp}(\lambda/\mu)$ is strictly contained in the Schur interval $[\mathbf{w}, \mathbf{n}]$. On the other hand, if $x < w_2$, then $1 \leq k < x < w_2$, and this implies $w_2 \geq k + 2$. Since k and w_2 are the lengths of two columns of λ^1/μ, it follows from Lemma 4.5 that also in this case the support of λ/μ is not the entire Schur interval.

So, for the rest of the proof we assume $x = w_2$, and therefore

$$\mathbf{w} = (w_1, w_2, w_2, w_3, \ldots, w_\ell) \preceq \mathbf{n} = (n_1, n_2, \ldots, n_s),$$

with $n_1 = w_2 + y$ and $n_2 = w_1$. Note that this implies $y \geq 2$.

If $\ell \geq 4$, then the partition $\xi := (w_1, w_2, w_2, \ldots, w_{\ell-2}, w_{\ell-1} + 1, w_\ell - 1)$ clearly shows that $\mathrm{supp}(\lambda/\mu)$ is not the entire Schur interval. Thus, consider $\ell = 3$, and note that, since $1 \leq k \leq w_2 - 1$, it follows from Lemma 4.5 that $\mathrm{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$ except if $k = w_2 - 1$.

So, assume now that $\ell = 3, x = w_2$ and $k = w_2 - 1$. Then $n_1 = w_2 + y = w_1 + 1$, and the minimal and maximal fillings are now

$$\mathbf{w} = (w_1, w_2, w_2, w_3) \preceq \mathbf{n} = (w_1 + 1, w_1, w_3 + h),$$

where $h > 0$ is the number of rows that the second and the last columns share. It follows that, if $w_3 \geq 2$, the partition $\xi := (w_1, w_2 + 1, w_2 + 1, w_3 - 2)$ satisfies $\mathbf{w} \preceq \xi \preceq \mathbf{n}$ but is not in the support of λ/μ.

To finish the proof, consider $\ell = 3, w_3 = 1, x = w_2$ and $k = w_2 - 1$, and let $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ be a partition in the Schur interval $[\mathbf{w}, \mathbf{n}]$. Using the same argument used in the proof of Lemma 4.14, it is easy to show that ξ belongs to the support of λ/μ, and it follows that in this case the support of λ/μ is the entire Schur interval. $\square$

From Lemmas 4.14 and 4.15 we deduce the following result.

Corollary 4.16. *If the skew diagram λ/μ is an $F7$ configuration, then its support is the Schur interval if and only if λ/μ is an $A7$ configuration.*

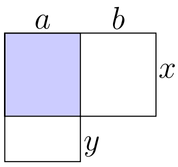
5. Full interval linear expansion of multiplicity-free skew Schur functions

We are now ready to identify the basic multiplicity-free skew Schur functions whose support is the entire interval $[\mathbf{w}, \mathbf{n}]$. Our strategy and terminology follows closely the one used in the proof of [7, Lemma 7.1].

Proof of Theorem 1.1. If λ/μ satisfies one or more of the conditions listed in the theorem, then the corresponding Schur function is multiplicity-free. These conditions imply particular instances of the configurations $R0$–$R4$ described in Theorem 2.2; namely $A1$ is in $R4$; $A2$ is in $R1$, or $R3$; $A3$ and $A4$ are in $R3$; $A6$ is in $R1$ or $R3$; and $A7$ is in $R1$. The strategy for the converse is to use Corollary 4.6 to analyse the support of the basic multiplicity-free skew Schur function $s_{\lambda/\mu}$ listed under cases $R0$–$R4$ in Theorem 2.2. We next consider, in bold, the five cases, $R0$–$R4$, albeit not in numerical order.

R0. If μ or λ^* is the zero partition 0 then either λ/μ or $(\lambda/\mu)^\pi$ is a partition. This means that the minimum and maximum LR fillings of λ/μ coincide and therefore, $\mathrm{supp}(\lambda/\mu) = [\mathbf{w}, \mathbf{n}] = \{\mathbf{n} = \mathbf{w}\}$.

R4. In this case both μ and λ^* are rectangles, and thus λ is a fat hook. Due to Lemma 3.1, we may assume that $\lambda = ((a+b)^x, b^y)$ and $\mu = (a^x, 0^y)$ with $a, b, x, y \geq 1$, as illustrated below:

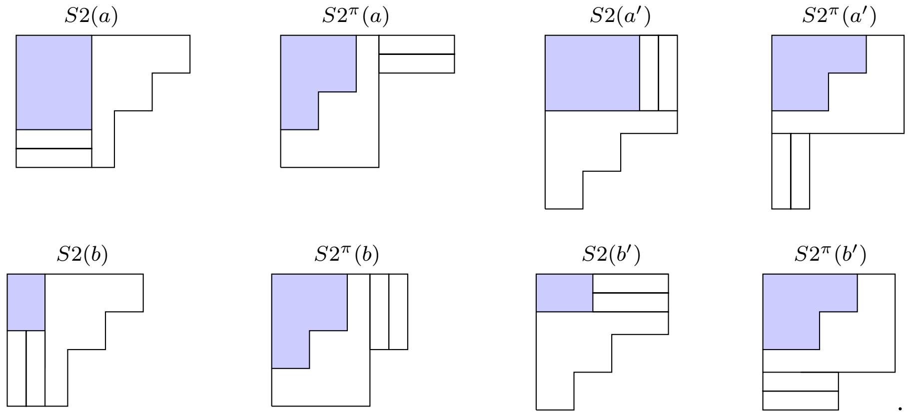
$\lambda/\mu =$ 

.

Since λ/μ has two disconnected components, by Lemma 4.2, it follows that, if any of the components has a 2 by 2 block, then its support is not the entire Schur interval. Thus, we are left with four cases to analyse. If $a = b = 1$ or if $x = y = 1$, we get a two column or a two row diagram. In both cases, by Proposition 4.8, the Schur interval coincides with the support of λ/μ. In any other case, we get either an $F1$ or an $F1^\pi$ configuration, and, by Proposition 3.3, we find that the support of λ/μ is strictly contained in its Schur interval.

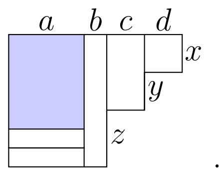
Therefore, by Lemma 3.1, we find that if both μ and λ^* are rectangles, then $\mathrm{supp}(\lambda/\mu) = [\mathbf{w}, \mathbf{n}] = \{\mathbf{n}, \mathbf{w}\}$ if and only if λ/μ satisfies conditions (ii) of the theorem.

R2. The two main subcases, μ a rectangle of m^n-shortness 2 and λ^* a fat hook, and vice versa, are denoted by $\mathbf{S2}$ and $\mathbf{S2}^\pi$, respectively. Each has four subcases, as illustrated by



These skew Young diagrams are arranged so that those of type $S2^{\pi}$ are the π-rotations of those of type $S2$, and the right-hand block of four (3rd and 4th column) consists of the conjugates of the left-hand block of four (1st and 2nd column). Due to the rotation symmetry and the conjugate symmetry, we have only to consider two cases, which we select to be $\mathbf{S2(a)}$ and $\mathbf{S2(b)}$.

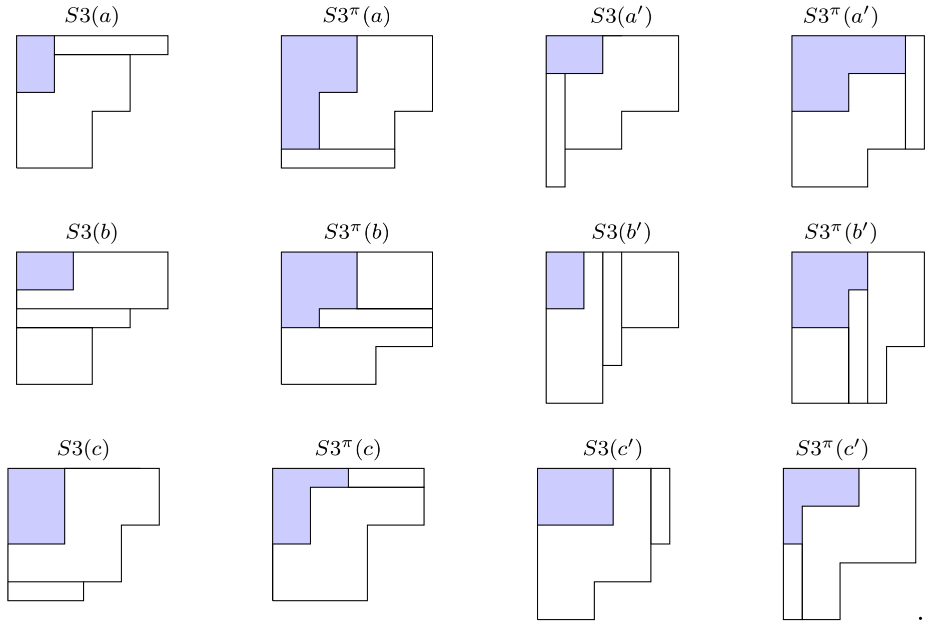
$\mathbf{S2(a)}$. In this case $\lambda = ((a+b+c+d)^x, (a+b+c)^y, (a+b)^z)$ and $\mu = (a^{x+y+z-2})$ with $x+y+z \geq 4$, for some integers a, b, c, d, x, y, z such that $a \geq 2$, $c, d, x, y, z \geq 1$ and $b \geq 0$, as illustrated in the following figure:



By Lemma 3.1, we may assume that $b = 0$. We start by noticing that, if $z = 2$, the skew diagram λ/μ is disconnected with a 2 by 2 block, in which case we have $\mathrm{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$, by Lemma 4.2. Assume now that $z = 1$, and note that, by the hypothesis on λ/μ, the length of the diagram is greater than or equal to 4. Therefore, an $F0^{\pi}$ or an $F0'$ configuration appears if $c \geq 2$ or if x, $d \geq 2$, respectively. Again, in these cases the support of λ/μ is strictly contained in the Schur interval by Corollary 4.6. Now, if $c = x = 1$, we get an $F5$ configuration, and if $c = d = 1$, we get the conjugate of an $F5$ configuration. By Lemma 4.12, we get that $\mathrm{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$. Therefore, in all subcases, the support of the skew diagram $\mathbf{S2(a)}$ is strictly contained in its Schur interval.

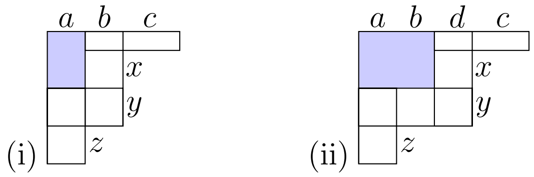
The analysis of case $\mathbf{S2(b)}$ is completely analogous to the previous one.

$\mathbf{R3}$. The two main subcases, μ a rectangle and λ^* a fat hook of m^n-shortness 1, and vice versa, are denoted by $\mathbf{S3}$ and $\mathbf{S3^{\pi}}$, respectively. Each has six subcases, as illustrated by



As before, these skew Young diagrams are arranged so that those of type $\mathbf{S3}^\pi$ are the π-rotations of those of type $\mathbf{S3}$. This time the right-hand block of six consists of the conjugates of the left-hand block of six. Due to the rotation symmetry and the conjugation symmetry, we only have to consider three cases. We choose these to be $\mathbf{S3}(\mathbf{a})$, $\mathbf{S3}(\mathbf{b}')$ and $\mathbf{S3}(\mathbf{c}')$.

$\mathbf{S3}(\mathbf{a})$. There are two subcases which, by Lemma 3.1, may be reduced to (i) $\mu = (a^{x+1})$ and $\lambda^* = ((b+c)^z, c^{x+y})$ for some integers $a, b, c, z \geq 1$ and $x, y \geq 0$ such that $x + y \geq 1$; and (ii) $\mu = ((a+b)^{x+1})$ and $\lambda^* = ((b+c+d)^z, c^{y+x})$ where $a, b, c, z, y \geq 1$ and $d, x \geq 0$, as illustrated below:



In the subcase $\mathbf{S3}(\mathbf{a})$(i), we start by identifying an $F0, F0^\pi$ or an $F0^{\pi'}$ configuration in the diagram whenever $a, z \geq 2$ or if $x \geq 1$ and $b \geq 2$, or even if $a, y \geq 2$. In these cases, the support of the skew diagram is not the entire Schur interval by Corollary 4.6. There remain thus six cases to consider.

If $a = 1$ and $x = 0$ we get an $F6$ configuration, and by Lemma 4.8 we have $\mathrm{supp}(\lambda/\mu) = [\mathbf{w}, \mathbf{n}]$ if and only if λ/μ is an $A6$ configuration. If $a = b = 1$ and $x > 0$, we get the conjugate of the π-rotation of an $\widehat{F}5$ configuration if $c \geq 2$, and an $F2'$ configuration if $c = 1$. By Proposition 4.9 and Lemma 4.12, we find that in these cases, $\mathrm{supp}(\lambda/\mu) = [\mathbf{w}, \mathbf{n}]$ if and only if λ/μ is an $A2$ configuration.

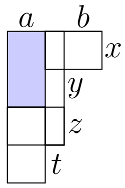
Assume now that $a \geq 2$ and $z = 1$. Then, if $y = x = 0$ we get a two row skew diagram, the configuration (ii) of the statement under proof, and thus its support equals the Schur interval. If $y = 0$ and $b = 1$, we get the conjugate of an $F4$ configuration, and, by Lemma 4.11, its support is equal to the Schur interval if and only if λ/μ is an $A4$ configuration. Finally, if $y = 1$ and $x = 0$ we get an $F2$ configuration, and if $y = 1$ and $b = 1$ we get the transpose of an $\widetilde{F}5$ configuration. By Proposition 4.9 and

Lemma 4.12, we find that the support is equal to the Schur interval if and only if λ/μ is an $A2$ configuration.

Consider now the subcase **S3**(**a**)(ii). If $a, z \geq 2$, we have an $F0$ configuration, and if $x \geq 1$ and $d \geq 2$, we have an $F0^{\pi}$ configuration. Note also that, if $y \geq 2$, we have an $F0'^{\pi}$ configuration. So we are left with six cases to analyse, all having $y = 1$.

If $d = 0$ and $z = 1$, we get an $F2$ configuration (recall that we restrict ourselves to basic skew Schur functions); if $d = 0$ and $a = 1$, we get either an $F2$ or an $\widehat{F}5$ configuration; if $d = 1$ and $z = 1$, we get the conjugate of an $\widetilde{F}5$ configuration; if $d = 1$ and $a = 1$, we get the π-rotation of an $F3$ configuration; if $x = 0$ and $z = 1$, we also get an $F2$ configuration; and finally, if $x = 0$ and $a = 1$, we get the π-rotation of an $\widehat{F}5$ configuration. Using Propositions 4.9, 4.10 and Lemma 4.12, it follows that the support of λ/μ is the full Schur interval if and only if λ/μ is an $A2$ or an $A3$ configuration.

S3(**b**$'$). Using Lemma 3.1, we may assume that $\lambda = ((a + b + 1)^x, (a + 1)^{y+z}, a^t)$ and $\mu = (a^{x+y})$, for some integers $a, b, x, t \geq 1$, and $y, z \geq 0$ such that $y + z \geq 1$, as illustrated by

$\lambda/\mu =$ 

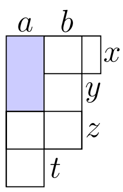
.

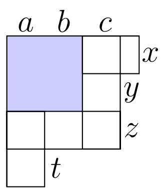
If $a, t \geq 2$ or $a, z \geq 2$, we get an $F0$ or an $F0^{\pi'}$ configuration. In these cases we know from Corollary 4.6 that the support of the configuration is not the entire Schur interval. An $F0'$ configuration also appears whenever $b, x \geq 2$, and again the support of λ/μ is strictly contained in its Schur interval. So we are left with six cases to analyse.

If $a = b = 1$, we get an $F2'$ configuration, and if $a = x = 1$, we get either an $F2'$ configuration (if $b = 1$), or the conjugate of the π-rotation of an $\widehat{F}5$ configuration (otherwise). By Proposition 4.9 and Lemma 4.12, it follows that in these cases the support of λ/μ is equal to the entire Schur interval if and only if λ/μ is an $A2$ configuration.

Assume now that $a \geq 2$ and $t = z = 1$. If also $b = 1$, we get an $F5$, and if $x = 1$, we get the conjugate of an $\widetilde{F}5$ configuration. Again, by Lemma 4.12, it follows that in these cases the support of λ/μ is strictly contained in the Schur interval. For the remaining two cases, assume that $a \geq 2$, $t = 1$ and $z = 0$. If $b = 1$, we get the conjugate of an $\widetilde{F}4$ configuration, and if $x = 1$, we get the conjugate of an $F4$ configuration. In these cases, by Lemma 4.11, the support of λ/μ is the full Schur interval if and only if λ/μ is the conjugate of an $A4$ configuration.

S3(**c**$'$). Due to Lemma 3.1, there are only two subcases to study: (i) $\mu = (a^{x+y})$ and $\lambda^* = ((b + 1)^t, 1^{z+y})$ for some integers $a, b, x, t \geq 1$ and $y, z \geq 0$ such that $y + z \geq 1$; and (ii) $\mu = ((a + b)^{x+y})$ and $\lambda^* = ((b + c + 1)^t, 1^{z+y})$ with $a, b, x, t, z \geq 1$ and $y, c \geq 0$, as illustrated below:

(i) 

(ii) 

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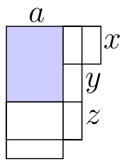
In subcase **S3**(**c**$'$)(i), we have $F0$ and $F0^{\pi}$ configurations whenever $a, t \geq 2$ or $b, x + y \geq 2$, respectively. In these cases, we have $\mathrm{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$ by Corollary 4.6. It remains to consider four cases. If $a = b = 1$, we get an $F2'$ configuration, and if $a = x + y = 1$, we get an $F6$ configuration. Then, Proposition 4.9 and Lemma 4.8 show that, in each case, the support of λ/μ is equal to the Schur interval if and only if λ/μ is an $A2'$ or an $A6$ configuration.

Assume now that $a \geq 2$ and $t = 1$. If $b = 1$, we get the diagram



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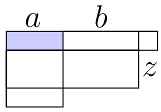
Now, an $F0'$ configuration appears if $z \geq 2$, and, if $z = 1$, we get the conjugate of an $\widetilde{F}5$ configuration. By Theorem 4.6 and Lemma 4.12, it follows that in these cases the Schur interval contains the support of λ/μ strictly. On the other hand, if $x + y = 1$, then we must have $x = 1$ and $y = 0$, and thus λ/μ has the form



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As before, an $F0'$ configuration appears if $z \geq 2$, and, if $z = 1$, we get an $F2$ configuration, in which case, by Corollary 4.6 and Proposition 4.9, respectively, we find that the Schur interval equals the support of λ/μ if and only it is an $A2$ configuration.

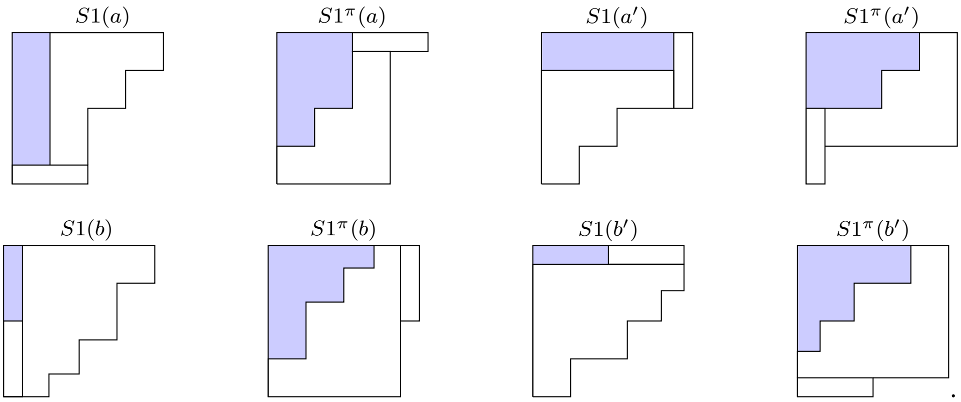
Consider now the subcase $\mathbf{S3}(\mathbf{c}')$(ii). If $a, t \geq 2$ or $z \geq 2$ or $c, x + y \geq 2$, we get $F0$, $F0^{\pi'}$ or $F0^{\pi}$ configurations on λ/μ. In all these cases, using Corollary 4.6, we find that $\mathrm{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$. So we may assume $z = 1$.

If we have $c = 0$, we get an $F4$ or an $\widetilde{F}4$ configuration if $a = 1$ and $t = 1$. In this case, by Lemma 4.11, the support of λ/μ is equal to its entire Schur interval if and only if the λ/μ is an $A4$ configuration.

If $c = 1$, then we get either an $F5$, an $\widetilde{F}5$ or an $F3$ configuration if $a, x \geq 2$ and $t = 1$, or if $a = 1$ and $t, x \geq 2$, or if $a = t = x = 1$. In these cases, by Proposition 4.10 and Lemma 4.12, the support of the skew diagram is equal to the Schur interval if and only if λ/μ is an $A3$ configuration.

Finally, assume that $c \geq 2$ and $x + y = 1$. If $t = 1$, we get an $F3$ configuration, and, if $a = 1$ and $t \geq 2$, we get the π-rotation of an $\widehat{F}5$ configuration. By Proposition 4.9 and Lemma 4.12, we find that, in these cases, $\mathrm{supp}(\lambda/\mu) = [\mathbf{w}, \mathbf{n}]$ if and only if λ/μ is an $A3$ configuration.

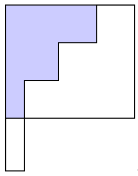
R1. There are two main subcases. We denote them by $S1$ and $S1^{\pi}$ in which μ and λ^* are rectangles of m^n-shortness 1. Each has four subcases, as illustrated by



.

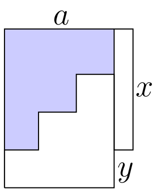
These skew Young diagrams are arranged so that those of type $S1^{\pi}$ are the π-rotations of their left neighbour of type $S1$. Moreover, the right-hand block of four consists of the conjugates of the left-hand block of four. Due to the rotation symmetry and the conjugation symmetry, it is therefore only necessary to consider two cases. We select these to be $\mathbf{S1}^{\pi}(\mathbf{a}')$ and $\mathbf{S1}^{\pi}(\mathbf{b})$.

$\mathbf{S1}^{\pi}(\mathbf{a}')$. Using Lemma 3.1, we may assume that

$\lambda/\mu =$  .

Moreover, we may assume that the diagram has at least 4 columns, three of them but the first have different sizes, otherwise we are in case **R3** or **R4**. In this case, λ/μ is disconnected and the largest component has a 2 by 2 block. By Lemma 4.2, it follows that the support of λ/μ is not the full Schur interval.

$\mathbf{S1}^{\pi}(\mathbf{b})$. We may use again Lemma 3.1 to reduce our study to the skew diagrams of the form

$\lambda/\mu =$ 

,

where $a+1$ is the number of columns of the diagram, for which we assume that among the first a there are at least three distinct lengths (otherwise we are in one of the previous cases), x is the length of the last column and $\lambda^* = (y)$. Moreover, we assume without loss of generality that the next-to-last column starts at least one row below the topmost box of the last column, and that the first column has length $\leq y$. Note that, if there are at least two columns among the first a ones having the same length and $y \geq 2$, then λ/μ is an $F0$ configuration. We are left with two cases: either (i) $y = 1$, or (ii) the first a columns have pairwise distinct lengths. In the first case, the minimal and maximal fillings of λ/μ are

$$\mathbf{w} = (x, n_1^{s_1}, \ldots, n_r^{s_r}, 1^{s_{r+1}}) \preceq \mathbf{n} = (x+1, n_1^{s_1}, \ldots, n_r^{s_r}, 1^{s_{r+1}-1}),$$

with $x \geq n_1 > \cdots > n_r > 1$ and $r \geq 2$. Therefore, the partition

$$\xi := (x, n_1^{s_1}, \ldots, n_r + 1, n_r^{s_r - 1}, 1^{s_{r+1}-1})$$

shows that the support of λ/μ is not the entire admissible interval.

Finally, assume that among the first a columns of λ/μ there are no two with the same length, and that $y \geq 2$. By Corollary 4.16, it follows that the support of λ/μ is equal to the Schur interval if and only if λ/μ is an $A7$ configuration. □

Example 5.1. Let $\lambda/\mu = ((b+3)^2, 2^{y+1})/(b+2, 1^{y+1})$ be an $A3$ configuration. Here we have $\mathbf{w} = (y+2, 2, 1^{b+1}) \preceq \mathbf{n} = (y+3, 2, 1^b)$. Then

$$s_{\lambda/\mu} = s_{(y+2,2,1^{b+1})'} + s_{(y+2,2,2,1^{b-1})'} + s_{(y+2,3,1^b)'} + s_{(y+3,1^{b+2})'} + s_{(y+3,2,1^b)'}.$$

Example 5.2. Let $\lambda/\mu = (4^4, 3^2)/(3, 2, 1^3)$ be an $A7$ configuration. Here we have $\mathbf{w} = (5, 4, 4, 1) \preceq \mathbf{n} = (6, 5, 3)$. Then

$$s_{\lambda/\mu} = s_{(5,4,4,1)'} + s_{(5,5,3,1)'} + s_{(6,4,3,1)'} + s_{(5,5,4)'} + s_{(6,5,2,1)'} + s_{(6,4,4)'} + s_{(6,5,3)'}.$$

The characterisation of the multiplicity-free Schur function products that attain the full interval, given in Corollary 1.2, is now a consequence of Theorem 1.1, and of Corollary 2.3.

We list now explicitly the partitions (μ, ν, λ) for which $c^{\lambda}_{\mu\,\nu} = 1$ for all $\lambda \in [\mu \cup \nu,\ \mu + \nu]$. Recall that the Pieri rule expresses the product of a Schur function and a single row (column) Schur function in terms of Schur functions [22, 9, 24]. These are precisely the cases where the Hasse diagram of the interval $[\mu \cup \nu,\ \mu + \nu]$ is given by the Pieri rule.

Corollary 5.1. *Let $(\mu,\ \nu, \lambda)$ be a triple of partitions.*

(a) *If μ or ν is the zero partition, then $c^{\lambda}_{\mu,0} = 1$ if and only if $\mu = \lambda$.*

(b) *If $\mu = (1^x)$ and $\nu = (1^y)$ (or vice versa), with $x \geq y \geq 1$, then $c^{\lambda}_{\mu,\nu} = 1$ if and only if $\lambda \in [(1^{x+y});\ (2^y,\ 1^{x-y})]$.*

(b') *If $\mu = (x)$ and $\nu = (y)$ (or vice versa), with $x \geq y \geq 1$, then $c^{\lambda}_{\mu,\nu} = 1$ if and only if $\lambda \in [(x,\ y);\ (x+y)]$.*

(c) *If $\mu = (1^x)$ and $\nu = (2, 1^y)$ are partitions such that $1 \leq x \leq y+1$ (or vice versa), then $c^{\lambda}_{\mu,\nu} = 1$ if and only if $\lambda \in [(2,\ 1^{x+y});\ (3,\ 2^{x-1},\ 1^{y-x+1})]$.*

(c') *If $\mu = (x)$ and $\nu = (z, 1)$ are partitions such that $1 \leq x \leq z$, (or vice versa), then $c^{\lambda}_{\mu,\nu} = 1$ if and only if $\lambda \in [(z,\ x,\ 1);\ (z+x+1, 1)]$.*

(d) *If $\mu = (1)$ and $\nu = (a, 1^y)$ are partitions such that $a \geq 3$, $y \geq 1$ (or vice versa), then $c^{\lambda}_{\mu,\nu} = 1$ if and only if $\lambda \in [(a,\ 1^{y+1});\ (a+1,\ 1^y)]$.*

(d') *If $\mu = (1)$ and $\nu = (z, 1^a)$ are partitions such that $a \geq 2$, $z \geq 1$ (or vice versa), then $c^{\lambda}_{\mu,\nu} = 1$ if and only if $\lambda \in [(z,\ 1^{a+1});\ (z+1,\ 1^a)]$.*

Remark 5.1. We may also explicitly list the multiplicity-free Schur function products whose support is an interval, that is, those whose number of components (summands) is the cardinality of the Schur interval.

(a) $s_0 s_\nu = s_\nu$ has 1 component.

(b) The conjugate Schur interval $[(1^{x+y});\ (2^y,\ 1^{x-y})]$ is a saturated chain. The decomposition

$$s_{(1^x)} s_{(1^y)} = s_{(2^y,\,1^{x-y})} + s_{(2^{y-1},\,1^{x-y+2})} + \cdots + s_{(2^1,\,1^{x+y-2})} + s_{(1^{y+x})}, \quad x \geq y \geq 1,$$

has $y+1$ components. In particular, if $y = 1$, then $s_{(1^x)} s_{(1)} = s_{(2,\,1^{x-1})} + s_{(1^{1+x})}$ has 2 components.

(b') This case arises as the conjugate of (b).

(c) There are two cases for the conjugate Schur interval $[(2,\, 1^{x+y}),\, (3,\, 2^{x-1},\, 1^{y-x+1})]$, with $1 \leq x \leq y+1$. If $y = x-1$, then

$$s_{1^x} s_{2\,1^{x-1}} = s_{(x\,x\,1)'} + s_{(x+1\,x-1\,1)'} + s_{(x+1\,x)'} + s_{(x+2\,x-2\,1)'} + s_{(x+2\,x-1)'} + s_{(x+3\,x-3\,1)'} + \cdots$$
$$+ s_{(2x-1\,1\,1)'} + s_{(2x-1\,2)'} + s_{(2x\,1)'},\ \ x \geq 1;\ \text{and}$$

and, if $y > x-1$, then

$$s_{(2,\,1^{x-1+k})} s_{(1^x)} = s_{(x+k,\,x,\,1)'} + s_{(x+k,\,x+1)'} + s_{(x+1+k,\,x-1,\,1)'} + s_{(x+1+k,\,x)'} + s_{(x+2+k,\,x-2,\,1)'} +$$
$$+ s_{(x+2+k,\,x-1)'} + s_{(x+3+k,\,x-3,\,1)'} + \cdots + s_{(2x-1+k,\,1,\,1)'} + s_{(2x-1+k,\,2)'} + s_{(2x+k,\,1)'},\quad x \geq 1,\ k > 0.$$

(c') This case arises as the conjugate of (c).

(d) The conjugate Schur interval $[(a,\, 1^{y+1});\, (a+1,\, 1^y)]$, $a \geq 3$, $y \geq 1$, has 3 elements,

$$s_{(a,\,1^y)} s_1 = s_{(a,\,1^{y+1})} + s_{(a,\,2,\,1^{y-1})} + s_{(a+1,\,1^y)},\quad a \geq 3,\ y \geq 1.$$

(d') It is the conjugate of (d).

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CMUC, Department of Mathematics, University of Coimbra, Apartado 3008, 3001–454 Coimbra, Portugal
Email address: oazenhas@mat.uc.pt

Email address: alessandro.conflitti@gmail.com

CMUC, Department of Mathematics, University of Coimbra, Apartado 3008, 3001–454 Coimbra, Portugal
Email address: mamede@mat.uc.pt