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André Permutation Calculus: a Twin Seidel Matrix Sequence

Dominique Foata and Guo-Niu Han

Abstract. Entringer numbers occur in the André permutation combinatorial set-up under several forms. This leads to the construction of a matrix refinement of the tangent (respectively secant) numbers. Furthermore, closed expressions for the three-variate exponential generating functions for pairs of so-called Entringerian statistics are derived.

Key words and phrases. Entringer numbers, tangent and secant numbers, alternating permutations, André permutations, Seidel matrix sequence, increasing binary trees, greater neighbor of maximum, spike, pit, tight permutations, hooked permutations, Seidel triangle sequence, formal Laplace transform.

Mathematics Subject Classifications. 05A15, 05A30, 11B68, 33B10.

1. Introduction

The notion of *alternating* or *zigzag* permutation devised by Désiré André, back in 1881 [An1881], for interpreting the coefficients $E(n)$ $(n \geq 0)$ of the Taylor expansion of $\tan u + \sec u$, the so-called *tangent* and *secant numbers*, has remained some sort of a curiosity for a long time, until it was realized that the geometry of these alternating permutations could be exploited to obtain further arithmetic refinements of these numbers. Classifying alternating permutations according to the number of inversions directly leads to the constructions of their q-analogs (see [AF80, AG78, St76]). Sorting them according to their first letters led Entringer [En66] to obtain a fruitful refinement $E_n = \sum_m E_n(m)$ that has been described under several forms [OEIS, GHZ11, KPP94, MSY96, St10], the entries $E_n(m)$ satisfying a simple finite difference equation (see (1.1) below).

In fact, these numbers $E_n(m)$, called *Entringer numbers* in the sequel, appear in other contexts, in particular when dealing with analytical properties of the *André permutations*, of the two kinds I and II, introduced by Schützenberger and the first author ([FSch73, FSch71]). For each $n \geq 1$ let And_n^{I} (respectively And_n^{II}) be the set of all André permutations of $12\cdots n$ (see §1.2). It was shown that $\#\,\mathrm{And}_n^{I} = \#\,\mathrm{And}_n^{II} = E_n$. The first purpose of this paper is to show that there are several natural statistics "stat," defined on And_n^{I} (respectively And_n^{II}), whose distributions are *Entringerian*, that is, integer-valued mappings "stat," satisfying

$$\#\{w \in \mathrm{And}_n^{I} \text{ (respectively } \mathrm{And}_n^{II}) : \mathrm{stat}(w) = m\} = E_n(m).$$

The second purpose is to work out a matrix refinement $E_n = \sum_{m,k} a_n(m,k)$ of the tangent and secant numbers, whose row and column sums $\sum_k a_n(m,k)$ and $\sum_m a_n(m,k)$ are themselves refinements of the Entringer numbers. This will be achieved, first by inductively defining the so-called *twin Seidel matrix sequence* (A_n, B_n) $(n \geq 2)$ (see §1.5), then by proving that the entries of these matrices provide the joint distributions of pairs of Entringerian statistics defined on André permutations of each kind (Theorem 1.2). See §1.6 for the plan of action.

The third purpose is to obtain analytical expressions for the joint exponential generating functions for pairs of these Entringerian statistics. See §1.7 and the contents of Section 7 and 8. Let us give more details on the notions introduced so far.

1.1. *Entringer numbers.* According to Désiré André [An1879, An1881] each permutation $w = x_1x_2\cdots x_n$ of $12\cdots n$ is said to be *(increasing) alternating* if $x_1 < x_2$, $x_2 > x_3$, $x_3 < x_4$, etc. in an alternating way. Let Alt_n be the set of all alternating permutations of $12\cdots n$. He then proved that $\#\,\mathrm{Alt}_n = E_n$, where E_n is the *tangent number* (respectively *secant* *number*) when n is odd (respectively even), these numbers appearing in the Taylor expansions of $\sec u$ and $\tan u$:

$$\tan u = \sum_{n\ge 1} \frac{u^{2n-1}}{(2n-1)!} E_{2n-1} = \frac{u}{1!}1 + \frac{u^3}{3!}2 + \frac{u^5}{5!}16 + \frac{u^7}{7!}272 + \frac{u^9}{9!}7936 + \cdots,$$

$$\sec u = \sum_{n\ge 0} \frac{u^{2n}}{(2n)!} E_{2n} = 1 + \frac{u^2}{2!}1 + \frac{u^4}{4!}5 + \frac{u^6}{6!}61 + \frac{u^8}{8!}1385 + \frac{u^{10}}{10!}50521 + \cdots.$$

(See, e.g., [Ni23, pp. 177–178], [Co74, pp. 258–259]).

Let $\mathbf{F}\, w := x_1$ be the *first* letter of a permutation $w = x_1 x_2 \cdots x_n$ of $12\cdots n$. For each $m = 1, \ldots, n$, the *Entringer numbers* are defined by $E_n(m) := \#\{w \in \mathrm{Alt}_n : \mathbf{F}\, w = m\}$, as was introduced by Entringer [En66]. In particular, $E_n(n) = 0$ for $n \ge 2$. He showed that these numbers satisfy the recurrence

$$E_1(1) := 1; \quad E_n(n) := 0 \text{ for all } n \ge 2; \tag{1.1}$$
$$\Delta E_n(m) + E_{n-1}(n-m) = 0 \quad (n \ge 2; m = n-1, \ldots, 2, 1);$$

where Δ stands for the classical finite difference operator (see, e.g. [Jo39]) $\Delta E_n(m) := E_n(m+1) - E_n(m)$. See Fig. 1.1 for the table of their first values. These numbers are registered as the sequence $A008282$ in Sloane's On-Line Encyclopedia of Integer Sequences, together with an abundant bibliography [OEIS]. They naturally constitute a refinement of the tangent and secant numbers:

$$\sum_m E_n(m) = E_n = \begin{cases} \text{tangent number}, & \text{if } n \text{ is odd;} \\ \text{secant number}, & \text{if } n \text{ is even.} \end{cases} \tag{1.2}$$

$m=$	1	2	3	4	5	6	7	8	9	Sum
$n=1$	1									1
2	1	0								1
3	1	1	0							2
4	2	2	1	0						5
5	5	5	4	2	0					16
6	16	16	14	10	5	0				61
7	61	61	56	46	32	16	0			272
8	272	272	256	224	178	122	61	0		1385
9	1385	1385	1324	1202	1024	800	544	272	0	7936

Fig. 1.1. The Entringer Numbers $E_n(m)$

Now, let $\mathbf{L}\, w := x_n$ denote the *last* letter of a permutation $w = x_1 x_2 \cdots x_n$ of $12\cdots n$. In our previous paper [FH14] we made a full study of the so-called *Bi-Entringer numbers* defined by

$$E_n(m,k) := \#\{w \in \mathrm{Alt}_n : \mathbf{F}\, w = m,\, \mathbf{L}\, w = k\},$$

and showed that the sequence of the matrices $(E_n(m,k)_{1\le m,k\le n})$ $(n\ge 1)$ was fully determined by a partial difference equation system and the three-variable exponential generating function for these matrices could be calculated. As the latter analytical derivation essentially depends on the *geometry* of alternating permutations, it is natural to ask whether other combinatorial models, counted by tangent and secant numbers, are likely to have a parallel development.

Let $E(u) := \tan u + \sec u = \sum_{n\ge 0}(u^n/n!)\,E_n$. Then the first and second derivatives of $E(u)$ are equal to $E'(u) = E(u)\sec u$ and $E''(u) = E(u)\,E'(u)$, two identities equivalent to the two recurrence relations

$$(*)\qquad E_{n+1} = \sum_{0\le 2j\le n}\binom{n}{2j} E_{n-2j}E_{2j}\quad (n\ge 0),\qquad E_0 = 1;$$

$$(**)\qquad E_{n+2} = \sum_{0\le j\le n}\binom{n}{j} E_j\,E_{n+1-j}\quad (n\ge 0),\qquad E_0 = E_1 = 1.$$

The first of these relations can be readily interpreted in terms of *alternating* permutations, or in terms of the so-called *Jacobi* permutations introduced by Viennot [Vi80]. The second one leads naturally to the model of *André* permutations, whose geometry will appear to be rich and involves several analytic developments.

1.2. *André permutations.* These permutations were introduced in [FSch73, FSch71], and further studied in [Str74, FSt74, FSt76]. Other properties have been developed in work by Purtill [Pu93], Hetyei [He96], Hetyei and Reiner [HR98], the present authors [FH01], Stanley [St94], in particular in the study of the *cd*-index in a Boolean algebra. More recently, Disanto [Di14] has been able to calculate the joint distribution of the right-to-left minima and left-to-right minima in these permutations.

In the sequel, permutations of a finite set $Y = \{y_1 < y_2 < \cdots < y_n\}$ of positive integers will be written as *words* $w = x_1x_2\cdots x_n$, where the *letters* x_i are the elements of Y in some order. The *minimum* (respectively *maximum*) letter of w, in fact, y_1 (respectively y_n), will be denoted by $\min(w)$ (respectively $\max(w)$). When writing $w = v\,\min(w)\,v'$ it is meant that the word w is the juxtaposition product of the left factor v, followed by the letter $\min(w)$, then by the right factor v'.

Definition. Say that the empty word e and each one-letter word are both *André I* and *André II permutations.* Next, if $w = x_1x_2\cdots x_n$ $(n\ge 2)$ is a permutation of a set of positive integers $Y = \{y_1 < y_2 < \cdots < y_n\}$, write $w = v\,\min(w)\,v'$. Then, w is said to be an *André I* (respectively *André II*) *permutation* if both v and v' are themselves André I (respectively André II) permutations, and furthermore if $\max(vv')$ (respectively $\min(vv')$) is a letter of v'.

The set of all André I (respectively André II) permutations of Y is denoted by And_Y^I (respectively And_Y^{II}), and simply by And_n^I (respectively And_n^{II}) when $Y = \{1, 2, \ldots, n\}$. In the sequel, an André I (respectively André II) permutation, with no reference to a set Y, is meant to be an element of And_n^I (respectively And_n^{II}).

Using such an inductive definition, we can immediately see that $E_n = \#\,\mathrm{And}_n^I = \#\,\mathrm{And}_n^{II}$, the term $\binom{n}{j} E_j\, E_{n+1-j}$ in $(**)$ being the number of all André I (respectively André II) permutations of $x_1 x_2 \cdots x_{n+2}$ such that $x_j = 1$. Further equivalent definitions will be given in the beginning of Section 2. The first André permutations from And_n^I and And_n^{II} are listed in Table 1.2.

André permutations of the first kind:
$n=1$: 1; $n=2$: 12; $n=3$: 123, 213;
$n=4$: 1234, 1324, 2314, 2134, 3124;
$n=5$: 12345, 12435, 13425, 23415, 13245, 14235, 34125, 24135, 23145, 21345, 41235, 31245, 21435, 32415, 41325, 31425.

André permutations of the second kind:
$n=1$: 1; $n=2$: 12; $n=3$: 123, 312;
$n=4$: 1234, 1423, 3412, 4123, 3124;
$n=5$: 12345, 12534, 14523, 34512, 15234, 14235, 34125, 45123, 35124, 51234, 41235, 31245, 51423, 53412, 41523, 31524.

Table 1.2: the first André permutations of both kinds

1.3. *Statistics on André permutations.* The statistics "$\mathbf{F}$" and "$\mathbf{L}$" have been previously introduced. Two further ones are now defined: the *next-to-last* (or the penultimate) *letter* "**NL**" and **gr***eater* **n***eighbor of the maximum* "**grn**": for $w = x_1 x_2 \cdots x_n$ and $n \geq 2$ let $\mathbf{NL}\, w := x_{n-1}$; next, let $x_i = n$ for a certain i $(1 \leq i \leq n)$ with the convention that $x_0 = x_{n+1} := 0$. Then, $\mathbf{grn}\, w := \max\{x_{i-1}, x_{i+1}\}$.

Let (Ens_n) $(n \geq 1)$ be a sequence of non-empty finite sets and "stat" an integer-valued mapping $w \mapsto \mathrm{stat}(w)$ defined on each Ens_n. The pair $(\mathrm{Ens}_n, \mathrm{stat})$ is said to be *Entringerian*, if $\#\mathrm{Ens}_n = E_n$ and $\#\{w \in \mathrm{Ens}_n : \mathrm{stat}(w) = m\} = E_n(m)$ holds for each $m = 0, 1, \ldots, n$. We also say that "stat" is an *Entringerian statistic*. The pair $(\mathrm{Alt}_n, \mathbf{F})$ is Entringerian, *par excellence*, for all $n \geq 1$.

Theorem 1.1. *For each $n \geq 2$ the mappings*
(i) $\mathbf{F}$ *defined on* And_n^I,
(ii) $n - \mathbf{NL}$ *defined on* And_n^I,
(iii) $(n+1) - \mathbf{L}$ *defined on* And_n^{II},
(iv) $n - \mathbf{grn}$ *defined on* And_n^{II},
are all Entringerian statistics.

Statements (i) and (ii) will be proved in Section 2 by constructing two bijections η and θ having the property

$$
\begin{array}{ccccc}
\mathrm{Alt}_n & \xrightarrow{\eta} & \mathrm{And}_n^I & \xrightarrow{\theta} & \mathrm{And}_n^I \\
w & \mapsto & w' & \mapsto & w'' \\
\mathbf{F}\,w & = & \mathbf{F}\,w' & = & (n-\mathbf{NL})w''
\end{array}
\tag{1.3}
$$

For proving (iii) and (iv) we use the properties of two new bijections $\phi : \mathrm{And}_n^I \to \mathrm{And}_n^{II}$ and $g : \mathrm{And}_n^I \to \mathrm{And}_n^I$, whose constructions are described in Sections 3 and 4. By means of these two bijections, as well as the bijection θ mentioned in (1.3), it will be shown in Section 5 that the following properties hold:

$$
\begin{array}{ccc}
\mathrm{And}_n^I & \xrightarrow{\phi\circ g} & \mathrm{And}_n^{II} \\
w & \mapsto & w' \\
\mathbf{F}\,w & = & (n+1-\mathbf{L}\,)w'
\end{array}
\tag{1.4}
$$

and

$$
\begin{array}{ccc}
\mathrm{And}_n^I & \xrightarrow{\phi\circ \theta} & \mathrm{And}_n^{II} \\
w & \mapsto & w'' \\
\mathbf{F}\,w & = & (n-\mathbf{grn})w''
\end{array}
\tag{1.5}
$$

thereby completing the proof of Theorem 1.1.

1.4. *The fundamental bijection ϕ.* For proving (1.4) and (1.5) and also the next Theorem 1.2, two new statistics are to be introduced, the *spike* "**spi**" and the *pit* "**pit**", related to the *left* minimum records for the former one, and the *right* minimum records for the latter one. In Section 3 the bijection ϕ between And_n^I and And_n^{II} will be shown to have the further property:

$$
(\mathbf{F}\,,\mathbf{spi},\mathbf{NL})w=(\mathbf{pit},\mathbf{L}\,,\mathbf{grn})\phi(w).
\tag{1.6}
$$

This implies that

(1.7) *for each pair (m,k) the two sets $\{w\in \mathrm{And}_n^I : (\mathbf{spi},\mathbf{NL})w=(m,k)\}$ and $\{w\in \mathrm{And}_n^{II} : (\mathbf{L}\,,\mathbf{grn})w=(m,k)\}$ are equipotent.*

It also follows from Theorem 1.1 that "$(n+1)-\mathbf{spi}$" on And_n^I and "**pit**" on And_n^{II} are two further Entringerian statistics.

1.5. *The twin Seidel matrix sequence.* The next step is to say something about the joint distributions of the pairs $(\mathbf{F}\,,\mathbf{NL})$ on And_n^I and $(\mathbf{L}\,,\mathbf{grn})$ on And_n^{II}, whose marginal distributions are Entringerian, as announced in Theorem 1.1. We shall proceed in the following way: first, the notion of *twin Seidel matrix sequence* (A_n,B_n) $(n\ge 2)$ will be introduced (see Definition below), then the entry in cell (m,k) of A_n (respectively B_n) will be shown to be the number of André I (respectively II) permutations w, whose values $(\mathbf{F}\,,\mathbf{NL})w$ (respectively $(\mathbf{L}\,,\mathbf{grn})w$) are equal to (m,k). The definition involves the *partial difference operator* $\underset{(1)}{\Delta}$ acting on sequences $(a_n(m,k))$ $(n\ge 2)$ by

$$\underset{(1)}{\Delta}\, a_n(m,k) := a_n(m+1,k) - a_n(m,k).$$

The subscript (1) indicates that the difference operator is to be applied to the variable occurring at the *first* position, which is 'm' in the previous equation.

Definition. The *twin Seidel matrix sequence* (A_n,B_n) $(n\ge 2)$ is a sequence of finite square matrices that obey the following five rules (TS1)–(TS5) (see Diagram 1.3 for the values of the first matrices, where zero entries are replaced by dots):

(TS1) each matrix $A_n = (a_n(m,k))$ (respectively $B_n = b_n(m,k))$ $(1\le m,k\le n)$ is a square matrix of dimension n $(n\ge 2)$ with nonnegative entries, and zero entries along its diagonal, except for $a_2(1,1)=1$; let $a_n(m,\bullet)=\sum_k a_n(m,k)$ (respectively $a_n(\bullet,k)=\sum_m a_n(m,k)$) be the m-th row sum (respectively k-th column sum) of the matrix A_n with an analogous notation for B_n;

(TS2) for $n\ge 3$ the entries along the rightmost column in both A_n and B_n are zero, as well as the entries in the bottom row of A_n and the top row of B_n, i.e., $a_n(\bullet,n)=b_n(\bullet,n)=a_n(n,\bullet)=b_n(1,\bullet)=0$, as all the entries are supposed to be nonnegative; furthermore, $b_n(n,1)=0$;

(TS3) the first two matrices of the sequence are supposed to be: $A_2 = \begin{matrix} 1 & \cdot \\ \cdot & \cdot \end{matrix}$, $B_2 = \begin{matrix} \cdot & \cdot \\ 1 & \cdot \end{matrix}$;

(TS4) for each $n\ge 3$ the matrix B_n is derived from the matrix A_{n-1} by means of a transformation $\Psi : (a_{n-1}(m,k)) \to (b_n(m,k))$ defined as follows

$$\text{(TS4.1)}\qquad b_n(n,k) := a_{n-1}(\bullet,k-1)\quad (2\le k\le n-1);$$
$$\text{(TS4.2)}\qquad b_n(n-1,k) := a_{n-1}(\bullet,k)\quad (2\le k\le n-2);$$

and, by induction,

$$\text{(TS4.3)}\quad \underset{(1)}{\Delta}\, b_n(m,k) - a_{n-1}(m,k) = 0\quad (2\le k+1\le m\le n-2);$$
$$\text{(TS4.4)}\quad \underset{(1)}{\Delta}\, b_n(m,k) - a_{n-1}(m,k-1) = 0\quad (3\le m+2\le k\le n-1);$$

(TS5) for each $n \geq 3$ the matrix A_n is derived from the matrix B_{n-1} by means of a transformation $\Phi : (b_{n-1}(m,k)) \to (a_n(m,k))$ defined as follows

$$\text{(TS5.1)} \qquad a_n(1,k) := b_{n-1}(\bullet, k-1) \quad (2 \leq k \leq n-1);$$

and, by induction,

$$\text{(TS5.2)} \quad \mathop{\Delta}_{(1)} a_n(m,k) + b_{n-1}(m,k-1) = 0 \quad (3 \leq m+2 \leq k \leq n-1);$$

$$\text{(TS5.3)} \quad \mathop{\Delta}_{(1)} a_n(m,k) + b_{n-1}(m,k) = 0 \quad (2 \leq k+1 \leq m \leq n-1).$$

$$A_2 = \begin{matrix} 1 & \cdot \\ \cdot & \cdot \end{matrix} \xrightarrow{\Psi} B_3 = \begin{matrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{matrix} \xrightarrow{\Phi} A_4 = \begin{matrix} \cdot & 1 & 1 & \cdot \\ 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix} \xrightarrow{\Psi} B_5 = \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot \\ 1 & 1 & \cdot & 2 & \cdot \\ 1 & 2 & 2 & \cdot & \cdot \\ \cdot & 1 & 2 & 2 & \cdot \end{matrix} \xrightarrow{\Phi}$$

$$\xrightarrow{\Phi} A_6 = \begin{matrix} \cdot & 2 & 4 & 5 & 5 & \cdot \\ 2 & \cdot & 4 & 5 & 5 & \cdot \\ 2 & 4 & \cdot & 4 & 4 & \cdot \\ 1 & 3 & 4 & \cdot & 2 & \cdot \\ \cdot & 1 & 2 & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \xrightarrow{\Psi} B_7 = \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & 4 & 5 & 5 & \cdot \\ 2 & 2 & \cdot & 8 & 10 & 10 & \cdot \\ 4 & 6 & 8 & \cdot & 14 & 14 & \cdot \\ 5 & 9 & 12 & 14 & \cdot & 16 & \cdot \\ 5 & 10 & 14 & 16 & 16 & \cdot & \cdot \\ \cdot & 5 & 10 & 14 & 16 & 16 & \cdot \end{matrix} \xrightarrow{\Phi} A_8 = \begin{matrix} \cdot & 16 & 32 & 46 & 56 & 61 & 61 & \cdot \\ 16 & \cdot & 32 & 46 & 56 & 61 & 61 & \cdot \\ 16 & 32 & \cdot & 44 & 52 & 56 & 56 & \cdot \\ 14 & 30 & 44 & \cdot & 44 & 46 & 46 & \cdot \\ 10 & 24 & 36 & 44 & \cdot & 32 & 32 & \cdot \\ 5 & 15 & 24 & 30 & 32 & \cdot & 16 & \cdot \\ \cdot & 5 & 10 & 14 & 16 & 16 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

$$B_2 = \begin{matrix} \cdot & \cdot \\ 1 & \cdot \end{matrix} \xrightarrow{\Phi} A_3 = \begin{matrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix} \xrightarrow{\Psi} B_4 = \begin{matrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ 1 & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot \end{matrix} \xrightarrow{\Phi} A_5 = \begin{matrix} \cdot & 1 & 2 & 2 & \cdot \\ 1 & \cdot & 2 & 2 & \cdot \\ 1 & 2 & \cdot & 1 & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \xrightarrow{\Psi}$$

$$\xrightarrow{\Psi} B_6 = \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 2 & 2 & \cdot \\ 1 & 1 & \cdot & 4 & 4 & \cdot \\ 2 & 3 & 4 & \cdot & 5 & \cdot \\ 2 & 4 & 5 & 5 & \cdot & \cdot \\ \cdot & 2 & 4 & 5 & 5 & \cdot \end{matrix} \xrightarrow{\Phi} A_7 = \begin{matrix} \cdot & 5 & 10 & 14 & 16 & 16 & \cdot \\ 5 & \cdot & 10 & 14 & 16 & 16 & \cdot \\ 5 & 10 & \cdot & 13 & 14 & 14 & \cdot \\ 4 & 9 & 13 & \cdot & 10 & 10 & \cdot \\ 2 & 6 & 9 & 10 & \cdot & 5 & \cdot \\ \cdot & 2 & 4 & 5 & 5 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \xrightarrow{\Psi} B_8 = \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 5 & 10 & 14 & 16 & 16 & \cdot \\ 5 & 5 & \cdot & 20 & 28 & 32 & 32 & \cdot \\ 10 & 15 & 20 & \cdot & 41 & 46 & 46 & \cdot \\ 14 & 24 & 33 & 41 & \cdot & 56 & 56 & \cdot \\ 16 & 30 & 42 & 51 & 56 & \cdot & 61 & \cdot \\ 16 & 32 & 46 & 56 & 61 & 61 & \cdot & \cdot \\ \cdot & 16 & 32 & 46 & 56 & 61 & 61 & \cdot \end{matrix}$$

Diagram 1.3: First values of the twin Seidel matrices

It is worth noting that the twin Seidel matrix sequence involves two infinite subsequences: $\text{Twin}^{(1)} = (A_2, B_3, A_4, B_5, A_6, \ldots)$ and $\text{Twin}^{(2)} = (B_2, A_3, B_4, A_5, B_6, \ldots)$. They are independent in the sense that the matrices A_{2n} (respectively B_{2n}) depend only on the matrices B_{2m+1} and A_{2m} (respectively A_{2m+1} and B_{2m}) with $m < n$, with an analogous statement for the matrices A_{2n+1} (respectively B_{2n+1}).

As is easily verified, Rules (TS1)–(TS5) define the twin Seidel matrix sequence by induction in a unique manner. At each step Rules (TS1) and (TS2) furnish all the zero entries indicated by dots, and Rules (TS4.1), (TS4.2), (TS5.1) the initial values. It remains to use the finite difference equations (TS4.3), (TS4.4), (TS5.2), (TS5.3) to calculate the other entries.

Theorem 1.2. *The twin Seidel matrix sequence* $(A_n = (a_n(m,k)),\ B_n = (b_n(m,k)))$ $(n \geq 2,\ 1 \leq m, k \leq n)$ *defined by relations* (TS1)–(TS5) *provides the joint distributions of the pairs* $(\mathbf{F}\,,\mathbf{NL})$ *on* And_n^I *and* $(\mathbf{L}\,,\mathbf{grn})$ *on* And_n^{II} *in the sense that for* $n \geq 2$ *the following relations hold:*

$$a_n(m,k) = \#\{w \in \mathrm{And}_n^I : (\mathbf{F}\,,\mathbf{NL})w = (m,k)\}; \tag{1.8}$$

$$b_n(m,k) = \#\{w \in \mathrm{And}_n^{II} : (\mathbf{L}\,,\mathbf{grn})w = (m,k)\}. \tag{1.9}$$

By Theorems 1.1 and 1.2, the row and column sums of the matrices A_n and B_n have the following interpretations:

$$a_n(m,\bullet) = E_n(m), \quad b_n(m,\bullet) = E_n(n+1-m), \quad (1 \leq m \leq n); \tag{1.10}$$

$$a_n(\bullet,k) = b_n(\bullet,k) = E_n(n-k) \quad (1 \leq k \leq n); \tag{1.11}$$

and furthermore the matrix-analog of the refinement of E_n holds:

$$\sum_{m,k} a_n(m,k) = \sum_{m,k} b_n(m,k) = E_n. \tag{1.12}$$

1.6. *Tight and hooked permutations.* For proving Theorem 1.2, the crucial point is to show that the $a_n(m,k)$'s and $b_n(m,k)$'s satisfy the partial difference equations (TS4.3), (TS4.4), (TS5.2), (TS5.3), when these numbers are defined by the right-hand sides of (1.8) and (1.9). For each pair (m,k) let

$$A_n(m,k) := \{w \in \mathrm{And}_n^I : (\mathbf{F}\,,\mathbf{NL})w = (m,k)\};$$
$$B_n(m,k) := \{w \in \mathrm{And}_n^I : (\mathbf{spi},\mathbf{grn})w = (m,k)\}.$$

As the latter set is equinumerous with the set $\{w \in \mathrm{And}_n^{II} : (\mathbf{L}\,,\mathbf{grn})w = (m,k)\}$ by (1.7), we also have $a_n(m,k) = \#A_n(m,k)$ and $b_n(m,k) = \#B_n(m,k)$, by (1.8) and (1.9).

For the partial difference equation (TS5.2) (respectively (TS5.3)) the plan of action may be described by the diagram

$$\begin{tikzcd}
 & B_{n-1}(m,k-1) \quad \text{(respectively } B_{n-1}(m,k)\,\text{)} \arrow[d, "\phi"] & & \\
A_n(m,k) = & T_n(m,k) & \uplus & NT_n(m,k) \arrow[d, "f"] \\
 & & & A_n(m+1,k)
\end{tikzcd} \tag{1.13}$$

This means that the set $A_n(m,k)$ is to be split into two disjoint subsets, $A_n(m,k) = T_n(m,k) \uplus NT_n(m,k)$, in such a way that the first component is in bijection with $B_{n-1}(m,k-1)$ (respectively $B_{n-1}(m,k)$) by using the bijection ϕ defined in (6.6), and the second one with $A_n(m+1,k)$ by means of the bijection f defined in (6.5). If this plan is realized, the above partial difference equations are satisfied, as

$$\begin{aligned}
\mathop{\Delta}_{(1)} a_n(m,k) &= \#A_n(m+1,k) - \#A_n(m,k)\\
&= \#NT_n(m,k) - \#A_n(m,k)\\
&= -\#T_n(m,k)\\
&= -\#B_{n-1}(m,k-1) \text{ (respectively } -\#B_{n-1}(m,k)\\
&= -b_{n-1}(m,k-1) \text{ (respectively } -b_{n-1}(m,k)).
\end{aligned}$$

For the partial difference equation (TS4.3) (respectively (TS4.4)) the corresponding diagram is the following

$$\begin{array}{lccc}
 & & A_{n-1}(m,k) & \text{(respectively } A_{n-1}(m,k-1\,)\\
 & & \big\downarrow \Theta & \\
(1.14)\quad & B_n(m+1,k) = & H_n(m+1,k) & \uplus\ NH_n(m+1,k)\\
 & & & \big\uparrow \beta\\
 & & & B_n(m,k)
\end{array}$$

where Θ and β are two explicit bijections, defined in (6.8) and (6.13), (6.14), respectively. The elements in $T_n(m,k)$ from (1.13) (respectively in $H_n(m+1,k)$ from (1.14)) are the so-called *tight* (respectively *hooked*) *permutations.* All details will be given in Section 6 and constitute the bulk of the proof of Theorem 1.2.

1.7. *Trivariate generating functions.* The final step is to show that the partial difference equation systems (TS4.3), (TS4.4), (TS5.2), (TS5.3) satisfied by the twin Seidel matrix sequence (A_n, B_n) $(n \geq 2)$ make it possible to derive *closed expressions* for the trivariate generating functions for the sequences (A_{2n}), (A_{2n+1}), (B_{2n}), (B_{2n+1}). We list them all in the following theorems. See Section 8 for the detailed proofs.

The calculations are all based on the Seidel triangle sequence technique developed in our previous paper [FH14]. Note that the next generating functions for the matrices A_n do not involve the entries of the rightmost columns and bottom rows, which are all zero; they do not involve the entries of the rightmost columns of the matrices B_n either, since these are also equal to zero, as assumed in (TS2). Finally, the generating functions for the bottom rows of the matrices B_n are calculated separately: see (1.23) and (1.24).

Also, note that the right-hand sides of identities (1.15)–(1.18) are all symmetric with respect to x and z, in agreement with their combinatorial interpretation stated in Theorem 2.4. The property is less obvious for (1.16), but an easy exercise on trigonometry shows that the right-hand side is equal to the fraction $\dfrac{\cos x \cos z \sin(x+y+z) - \sin y}{\cos^2(x+y+z)}$. Finally, the summations below are taken over triples $\{(m,k,n)\}$ or pairs $\{(k,n)\}$ for the last two ones; only the ranges of the summations have been written.

Theorem 1.3 [The sequence (A_{2n}) $(n \geq 1)$]. *The generating function for the upper triangles is given by*

$$\sum_{2\leq m+1\leq k\leq 2n-1} a_{2n}(m,k)\frac{x^{m-1}}{(m-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^{2n-k-1}}{(2n-k-1)!} = \frac{\cos x\ \cos z \sin(x+y+z)}{\cos^2(x+y+z)}, \tag{1.15}$$

and for the lower triangles by

$$\sum_{2\leq k+1\leq m\leq 2n-1} a_{2n}(m,k)\frac{x^{2n-m-1}}{(2n-m-1)!}\frac{y^{m-k-1}}{(m-k-1)!}\frac{z^{k-1}}{(k-1)!} = \frac{\cos x\ \sin z}{\cos(x+y+z)} + \frac{\sin x\ \cos(x+y)}{\cos^2(x+y+z)}. \tag{1.16}$$

Theorem 1.4 [The sequence (A_{2n+1}) $(n \geq 1)$]. *The generating function for the upper triangles is given by*

$$\sum_{2\leq m+1\leq k\leq 2n} a_{2n+1}(m,k)\frac{x^{m-1}}{(m-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^{2n-k}}{(2n-k)!} = \frac{\cos x \cos z}{\cos^2(x+y+z)}, \tag{1.17}$$

and for the lower triangles by

$$\sum_{2\leq k+1\leq m\leq 2n} a_{2n+1}(m,k)\frac{x^{2n-m}}{(2n-m)!}\frac{y^{m-k-1}}{(m-k-1)!}\frac{z^{k-1}}{(k-1)!} = \frac{\cos(x+y)\ \cos(y+z)}{\cos^2(x+y+z)}. \tag{1.18}$$

Theorem 1.5 [The sequence (B_{2n}) $(n \geq 1)$]. *The generating function for the upper triangles is given by*

$$\sum_{2\leq m+1\leq k\leq 2n-1} b_{2n}(m,k)\frac{x^{m-1}}{(m-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^{2n-1-k}}{(2n-1-k)!} = \frac{\sin x \cos z}{\cos^2(x+y+z)}, \tag{1.19}$$

and for the lower triangles by

$$\text{(1.20)}\qquad \sum_{2\le k+1\le m\le 2n-1} b_{2n}(m,k)\frac{x^{2n-m-1}}{(2n-m-1)!}\,\frac{y^{m-k-1}}{(m-k1)!}\,\frac{z^{k-1}}{(k-1)!}$$
$$=\frac{\cos(x+y)\,\sin(y+z)}{\cos^2(x+y+z)}.$$

Theorem 1.6 [The sequence (B_{2n+1}) $(n\ge 1)$]. *The generating function for the upper triangles is given by*

$$\text{(1.21)}\qquad \sum_{2\le m+1\le k\le 2n} b_{2n+1}(m,k)\frac{x^{m-1}}{(m-1)!}\,\frac{y^{k-m-1}}{(k-m-1)!}\,\frac{z^{2n-k}}{(2n-k)!}$$
$$=\frac{\sin x\,\cos z\sin(x+y+z)}{\cos^2(x+y+z)},$$

and for the lower triangles by

$$\text{(1.22)}\qquad \sum_{2\le k+1\le m\le 2n} b_{2n+1}(m,k)\frac{x^{2n-m}}{(2n-m)!}\,\frac{y^{m-k-1}}{(m-k-1)!}\,\frac{z^{k-1}}{(k-1)!}$$
$$=-\frac{\sin x\,\sin z}{\cos(x+y+z)}+\frac{\cos x\,\cos(x+y)}{\cos^2(x+y+z)}.$$

The *bivariate* generating functions for the bottom rows $b_n(n,k)$ $(k=1,2,\dots)$ are computed as follows:

$$\text{(1.23)}\qquad \sum_{1\le k\le 2n-1} b_{2n}(2n,k)\frac{x^{2n-k-1}}{(2n-k-1)!}\,\frac{y^{k-1}}{(k-1)!}=\frac{\cos x}{\cos(x+y)};$$

$$\text{(1.24)}\qquad \sum_{1\le k\le 2n} b_{2n+1}(2n+1,k)\frac{x^{k-1}}{(k-1)!}\,\frac{y^{2n-k}}{(2n-k)!}=\frac{\sin x}{\cos(x+y)}.$$

The previous generating functions for the matrices A_n, B_n will be derived analytically in Section 8, from the sole definition of twin Seidel matrix sequence given in § 1.5, without reference to any combinatorial interpretation. It will be shown in Section 9 that, conversely, the closed expressions thereby obtained provide an analytical proof of identity (1.12) by means of the formal Laplace transform, that is, the fact that the entries of these matrices furnish a refinement of the tangent and secant numbers.

2. From alternating to André permutations of the first kind

Two further equivalent definitions of André permutations of the two kinds will be given (see Definitions 2.1 and 2.2). They were actually introduced in [Str74, FSt74, FSt76]. First, let x be a letter of a permutation $w=x_1x_2\cdots x_n$ of a set of positive integers $Y=\{y_1<y_2<\cdots<y_n\}$.

The *x-factorization* of w is defined to be the sequence (w_1, w_2, x, w_4, w_5), where

(1) the juxtaposition product $w_1 w_2 x w_4 w_5$ is equal to w;
(2) w_2 is the longest right factor of $x_1 x_2 \cdots x_{i-1}$, all letters of which are greater than x;
(3) w_4 is the longest left factor of $x_{i+1} x_{i+2} \cdots x_n$, all letters of which are greater than x.

Next, say that x is *of type I* (respectively *of type II*) in w, if, whenever the juxtaposition product $w_2 w_4$ is non-empty, its maximum (respectively minimum) letter belongs to w_4. Also, say that x is of type I *and* II, if w_2 and w_4 are both empty.

Definition 2.1. A permutation $w = x_1 x_2 \cdots x_n$ of $Y = \{y_1 < y_2 < \cdots < y_n\}$ is said to be an *André permutation of the first kind* (respectively *of the second kind*) [in short, "w is *André I*" (respectively "*André II*")], if x_i is of type I (respectively of type II) in w for every $i = 1, 2, \ldots, n$.

Definition 2.2. A permutation $w = x_1 x_2 \cdots x_n$ of $Y = \{y_1 < y_2 < \cdots < y_n\}$ is said to be an *André permutation of the first kind* (respectively *of the second kind*), if it has no double descent (a factor of the form $x_{i-1} > x_i > x_{i+1}$) and its troughs (factors $x_{i-1} x_i x_{i+1}$ satisfying $x_{i-1} > x_i$ and $x_i < x_{i+1}$) are all of type I (respectively of type II). By convention, $x_{n+1} := 0$.

The following notations are being used. If $Y = \{y_1 < y_2 < \cdots < y_n\}$ is a finite set of positive integers, let ρ_Y be the increasing bijection of Y onto $\{1, 2, \ldots, n\}$. The inverse bijection of ρ_Y is denoted by ρ_Y^{-1}. If $v = y_{i_1} y_{i_2} \cdots y_{i_n}$ is a permutation of Y, written as a word, let $\rho_Y(v) := \rho_Y(y_{i_1}) \rho_Y(y_{i_2}) \cdots \rho_Y(y_{i_n}) = i_1 i_2 \cdots i_n$ be the *reduction* of the word v, which is then a permutation of $1\,2 \cdots n$. When dealing with a given word v, the subscript Y in $\rho_Y(v)$ may be omitted, so that $\rho(v) = \rho_Y(v)$. In the same way, the subscript Y in each composition product $\rho_Y^{-1} \alpha \rho_Y(v)$ may be omitted, so that $\rho^{-1} \alpha \rho(v) = \rho_Y^{-1} \alpha \rho_Y(v)$.

Also, let $\mathbf{c}$ be the bijection $i \mapsto n + 1 - i$ from $\{1, 2, \ldots, n\}$ onto itself. Furthermore, if $v = y_{i_1} y_{i_2} \cdots y_{i_n}$ is a permutation of Y, written as a word, let $C(v) := Y$ and the *length* of v be $|v| = n$. Finally, a *left maximum record* (respectively *left minimum record*) of v is defined to be a letter of v greater (respectively smaller) than all the letters to its left.

Proposition 2.1. *Let $n \geq 2$ and $w = x_1 x_2 \cdots x_n$ be André I.*

(1) *In $w = v \min(w) v'$, both factors v and v' are André I.*
(2) *If w is from And_n^I, then both permutations $1\,(x_1{+}1)\,(x_2{+}1)\,\cdots\,(x_n{+}1)$ and $x_1 x_2 \cdots x_n\,(n{+}1)$ belong to And_{n+1}^I, and $(x_2{-}1)(x_3{-}1) \cdots (x_n{-}1)$ belongs to And_{n-1}^I whenever $x_1 = 1$.*

(3) *The last letter* x_n *is the maximum letter.*

(4) *Let* $w = w'\,y\,w''x_n$ *with* y *being the second greatest letter of* w. *If* $w'' \neq e$, *then* $\mathbf{F}\,w'' = \min(w'')$.

(5) *For each left maximum record* y *of* v, *less than* $\max(w)$, *the two factors* uy *and* u' *in the factorization* $w = u\,y\,u'$ *are themselves André I.*

(6) *Let* $w = v\,y\,v'$ *be André I. If* y *is a left minimum record, then* v *is André I.*

(7) *Let* $w = v\,y\,v'$ *be an arbitrary permutation with* y *a letter. If both factors* v *and* $y\,v'$ *are André I and* y *is a left minimum record, then* w *is André I.*

Proof. (1) By the very definition given in Subsection 1.2.

(2) Clear.

(3) Write $w = v\min(w)v'$. By definition, $\max(v') = \max(vv') = \max(w)$, and by induction the last letter of v', which is also the last letter of w, is equal to $\max(v') = \max(w)$.

(4) If $w'' \neq e$, let $x := \mathbf{F}\,w''$, and let (w_1, w_2, x, w_4, w_5) be the x-factorization of w. As y is the maximum letter of w_2, the maximum letter of w_4 must be equal to $\max w$ to make x of type I. This can be achieved only if x is the minimum of w''.

(5) Let $x := \min(w)$ and y be a left maximum record less than $\max(w)$, so that $w = v\,x\,v' = u\,y\,u'$ for some factors v, $v' \neq e$, u, $u' \neq e$. Two cases are to be considered: (i) x to the left of y so that $w = v\,x\,v''\,y\,u'$; (ii) y to the left of x so that $w = u\,y\,u''\,x\,v'$ for some factors v'', u''. In case (i) both factors v and $v''\,y\,u'$ are André I, following the definition in § 2.1. Now, the letter y is also a left maximum record of the word $v''\,y\,u'$. By induction on the length, both $v''\,y$ and u' are André I, so that the two factors v and $v''y$ of the word $u\,y = v\,x\,v''y$ are André I, making the latter word also André I. Thus, both uy and u' are André I. In case (ii), the same argument applies: both factors $u\,y\,u''$ and v' are André I, then also $u\,y$ and u'' by induction, as well as the juxtaposition product $u''\,x\,v'$.

(6) If $y = \min w$, then v is André I by definition. Otherwise, y is to the left of $\min(w)$ in $\{w : w = v\,y\,u\,\min(w)\,u'\}$. But y is also a left minimum record of $v\,y\,u$. By induction on the length, v is André I.

(7) If $y = \min(w)$, then v' is André I by (2). Now, v and v' being both André I, the product $w = v\,y\,v'$ is André I by definition. If $y > \min(w)$, then $w = v\,y\,u\,\min(w)\,u'$. There is nothing to prove if $v = e$. Otherwise, as $y\,v'$ is André I, both factors $y\,u$ and u' are André I. As y is also a left minimum record of $v\,y\,u$, the juxtaposition product $v\,y\,u$ is André I by induction on the length. Finally, w itself is André I by definition, as u' has been proved to be also André I. ∎

In [FSch71] a bijection between And_n^I and Alt_n was constructed, but did not preserve the first letter. For proving Theorem 1.1(i), we need construct a bijection

$$\eta : w \to \eta(w), \quad \text{such that} \quad \mathbf{F}\, w = \mathbf{F}\, \eta(w), \tag{2.1}$$

of And_n^I onto the set Alt_n of all alternating permutations of length n. For $n = 1, 2, 3$ it suffices to take: $1 \mapsto 1$, $12 \mapsto 12$, $123 \mapsto 132$, $213 \mapsto 231$. When $n \geq 4$, each w from And_n^I can be written $w = w'\, 1\, w''$, where both factors w', w'' (with w' possibly empty) are André I.

If $w' = e$, let $v' := 1$ and $v'' := \rho^{-1}\, \mathbf{c}\, \eta\, \rho(w'')$;

if $w' \neq e$, let

$$v' := \rho^{-1}\, \eta\, \rho(w');$$
$$v'' := \begin{cases} \rho^{-1}\, \eta\, \rho(1\, w''), & \text{if } |w'| \text{ is even;} \\ \rho^{-1}\, \mathbf{c}\, \eta\, \rho(1\, w''), & \text{if } |w'| \text{ is odd;} \end{cases}$$

and

$$\eta(w) := v'\, v''. \tag{2.2}$$

For instance, let $w = 1234 \in \mathrm{And}_4^I$. Then, $w' = e$, $w'' = 234$; hence, $v' = 1$, $\rho(w'') = 123$, $\eta(123) = 132$, $\mathbf{c}(132) = 312$, $\rho^{-1}(312) = 423 = v''$ and $\eta(1234) = 1423$.

With $w = 4361257$ we get $w' = 436$, $w'' = 257$; hence, $\rho(436) = 213$, $\eta(213) = 231$, $\rho^{-1}(231) = 463 = v'$. Also, $\rho(1257) = 1234$, $\eta(1324) = 1423$, $\mathbf{c}(1423) = 4132$, $\rho^{-1}(4132) = 7152 = v''$ and $\eta(4361257) = 4637152$.

Theorem 2.2. *The mapping η defined by* (2.2) *is a bijection between And_n^I and Alt_n such that $\mathbf{F}\, w = \mathbf{F}\, \eta(w)$.*

Proof. Again, factorize an André I permutation w in the form $w = x_1 x_2 \cdots x_n = w' 1 w''$. When $w' = e$, then $\rho(w'')$ is an André I permutation starting with $\rho(x_2)$. By induction, $\eta\, \rho(w'')$ is an *increasing* alternating permutation if $|w''| \geq 2$. Then $\mathbf{c}\, \eta\, \rho(w'')$ will be a *falling* alternating permutation, as well as the permutation $v'' = \rho^{-1}\, \mathbf{c}\, \eta\, \rho(w'')$, which is also a permutation of $23 \cdots n$. Hence, $\eta(v) = 1\, v''$ will be an alternating permutation starting with 1.

When $w' \neq e$, then v' is an alternating permutation of the set $C(w')$. By induction, it starts with the same letter as the first letter of w, that is, x_1. If $|w'|$ is even, $v'' = \rho^{-1}\, \eta\, \rho(1\, w'')$ is an alternating permutation starting with 1, by induction. The juxtaposition product $v'v''$ will then be an alternating permutation starting with x_1, as the last letter of v' is necessarily greater than the first letter of v''. If $|w'|$ is odd, we just have to verify that $\mathbf{L}\, v' < \mathbf{F}\, v''$. But w, being an André I permutation, ends with its maximum letter n and so does w''. By induction, $\eta\, \rho(1\, w'')$ starts with 1, so that $\mathbf{c}\, \eta\, \rho(1\, w'')$ starts with the maximum letter n. Therefore,

v'' is a falling alternating permutation starting with n and $v'v''$ is an alternating permutation starting with x_1. $\square$

For each permutation $w = x_1x_2\cdots x_{n-1}x_n$ $(n \geq 2)$ the *next-to-last letter* $\mathbf{NL}\, w$ of w has been defined as $\mathbf{NL}\, w := x_{n-1}$. The construction of a bijection θ from And_n^I onto itself having the property

$$\mathbf{NL}\, \theta(w) + \mathbf{F}\, w = |w| = n \tag{2.3}$$

is quite simple. It suffices to define:

$$\theta(x_1 \cdots x_{n-2}x_{n-1}\, n) := (n - x_{n-1})\, (n - x_{n-2}) \,\cdots\, (n - x_1)\, n. \tag{2.4}$$

Property (2.3) is readily seen. It remains to prove that, if w belongs to And_n^I, so does $\theta(w)$. This is the object of the next Proposition.

Proposition 2.3. *Let $Y = \{y_1 < y_2 < \cdots < y_n\}$ be a finite set of positive integers and $w = x_1x_2\cdots x_n$ be an André I permutation from the set And_Y^I. Then $\theta(w) := (x_n - x_{n-1})(x_n - x_{n-2})\cdots(x_n - x_1)x_n$ is also André I.*

Proof. Proposition 2.3 is true for $n = 2$, as $\theta(y_1y_2) = (y_2 - y_1)y_2$. For $n = 3$ we have $\theta(y_1y_2y_3) = (y_3 - y_2)(y_3 - y_1)y_3$, $\theta(y_2y_1y_3) = (y_3 - y_1)(y_3 - y_2)y_3$, which are two André I permutations.

For $n \geq 4$ let $w = x_1x_2 \ldots x_n \in \mathrm{And}_Y^I$ be written $w = w'y_1w''$. If $w' = e$, let $w'' - y_1 := (x_2 - y_1)\cdots(x_{n-2} - y_1)(x_{n-1} - y_1)(y_n - y_1)$. Then, w'' is André I by Lemma 2.1 (b), as well as $w'' - y_1$, since y_1 is the smallest element of Y. By induction,

$$\begin{aligned}\theta(w'' - y_1) &= (y_n - y_1 - (x_{n-1} - y_1))\,(y_n - y_1 - (x_{n-2} - y_1))\\ &\qquad\qquad \cdots\,(y_n - y_1 - (x_2 - y_1))(y_n - y_1)\\ &= (y_n - x_{n-1})\,(y_n - x_{n-2})\,\cdots\,(y_n - x_2)(y_n - y_1)\end{aligned}$$

is André I. Therefore,

$$\begin{aligned}&(y_n - x_{n-1})\,(y_n - x_{n-2})\,\cdots\,(y_n - x_2)(y_n - y_1)y_n\\ &\qquad = (x_n - x_{n-1})\,(x_n - x_{n-2})\,\cdots\,(x_n - x_2)(x_n - x_1)y_n\end{aligned}$$

is André I a fortiori and is precisely the expression of $\theta(w)$ that was wanted.

Let $|w'| = k \geq 1$. By induction, both

$$\theta(w'y_n) = (y_n - x_k)\cdots(y_n - x_2)(y_n - x_1)\, y_n$$

and

$$\theta(y_1w'') = (y_n - x_{n-1})\cdots(y_n - x_{k+2})\,(y_n - y_1)\, y_n$$

are André I, and also

$$\check{\theta}(y_1 w'') := (y_n - x_{n-1}) \cdots (y_n - x_{k+2})\,(y_n - y_1).$$

The juxtaposition product $\check{\theta}(y_1 w'')\,\theta_n(w' y_n)$ reads

$$(y_n - x_{n-1}) \cdots (y_n - x_{k+2})\,(y_n - y_1)(y_n - x_k) \cdots (y_n - x_2)(y_n - x_1)\,y_n,$$

that is, precisely $\theta(w)$, since $y_n - y_1 = y_n - x_{k+1}$.

Now, note that x_k is the greatest letter of w' by Lemma 2.1 (3), so that $(y_n - x_k)$ is the smallest letter of the right factor

$$(y_n - x_k) \cdots (y_n - x_2)(y_n - x_1)\,y_n$$

of $\theta(w)$. On the other hand, $(y_n - y_1) > (y_n - x_k)$. Thus, $(y_n - x_k)$ is a trough of $\theta(w)$; moreover, the $(y_n - x_k)$-factorization $(w_1, w_2, (y_n - x_k), w_4, w_5)$ of $\theta(w)$ is of type I, since w_2 contains the letter $(y_n - y_1)$ and w_4 the letter y_n, which is greater than $(y_n - y_1)$. Finally, the x-factorizations of the other letters x from $\check{\theta}(y_1 w'')$ (respectively from $\theta(w' y_n)$) in each of these two factors are identical with their x-factorizations in $\theta(w)$. They are then all of type I, and $\theta(w)$ is André I. ∎

By Proposition 2.3 and Identity (2.3) we obtain the following result.

Theorem 2.4. *The statistics "$\mathbf{F}$" and "$(n - \mathbf{NL})$" are both Entringerian on* And_n^I. *Moreover, the distribution of the bivariate statistic* $(\mathbf{F}, n - \mathbf{NL})$ *on* And_n^I *is symmetric.*

3. The bijection ϕ between André I and André II permutations

For each permutation $w = x_1 x_2 \cdots x_n$ of $12 \cdots n$ $(n \geq 2)$ make the convention $x_{n+1} := 0$ and introduce the statistic *spike of* w, denoted by "$\mathbf{spi}\, w$," to be equal to the letter x_i $(1 \leq i \leq n)$ having the properties

$$x_1 \leq x_1,\ x_1 \leq x_2,\ \ldots\ ,\ x_1 \leq x_i, \quad \text{and} \quad x_1 > x_{i+1}. \tag{3.1}$$

The spike statistic may be regarded as the permutation analog of the classical statistic that measures the time spent by a particle starting at the origin and wandering in the $y > 0$ part of the xy-plane, before crossing the x-axis for the first time. For instance, $\mathbf{spi}(253416) = 4$, as all the letters to the left of 4 are greater than or equal to 2, but the letter following 4 is less than 2. Also, $\mathbf{spi}(425136) = 4$ and $\mathbf{spi}(14235) = 5$.

When w is an André I permutation and $\mathbf{spi}\, w = x_i$, then x_i is a left maximum record, i.e., greater than all the letters to its left. Otherwise, the minimum trough between the maximum letter within $x_1 x_2 \cdots x_{i-1}$ would not be of type I. Accordingly, when w is an André I permutation, the spike x_i of w can also be defined as the *smallest left maximum record* (or the *leftmost one*), whose successive letter x_{i+1} is less than x_1.

For introducing the statistic "**pit**", we restrict the definition to all permutations $w = x_1 x_2 \cdots x_n$ of $12\cdots n$ such that $n \geq 2$ and $x_{n-1} < x_n$. Let $1 = a_1 < a_2 < \cdots < x_{n-1} = a_{k-1} < x_n = a_k$ be the increasing sequence of the right minimum records of w, that is to say, the letters which are smaller than all the letters to their right. With the assumption $x_{n-1} < x_n$, there are always two right minimum records to the right of each letter greater than x_n. If $x_n = n$ ($= \max w$), let $\mathbf{pit}\, w := 1$ ($= \min w$). Otherwise, let x_i be the rightmost letter greater than x_n and $a_j < a_{j+1}$ be the closest pair of right minimum records to the right of x_i. Define $\mathbf{pit}\, w := a_{j+1}$.

For instance, $\mathbf{pit}(451236) = 1$, as the word ends with the maximum letter 6. In the permutation 614235, the letter 6 is the rightmost letter greater than $x_n = 5$, and $1 < 2$ is the closest pair of right minimum records to the right of 6, so that $\mathbf{pit}(614235) = 2$.

An alternate definition for "**pit**" is the following: if w ends with $\max w$, let $\mathbf{pit}\, w = \min w$. Otherwise, write $w = w_1(\min w)w_2$ and define $\mathbf{pit}\, w := \mathbf{pit}\, w_2$. If w_2 does not end with the maximum letter, let $w_2 = w_3(\min w_2)w_4$ and define $\mathbf{pit}\, w_2 := \mathbf{pit}\, w_4$, continue the process until finding a right factor w_{2j} ending with its maximum letter to obtain $\mathbf{pit}\, w := \mathbf{pit}\, w_2 = \cdots = \mathbf{pit}\, w_{2j} = \min w_{2j}$. For instance, $\mathbf{pit}(614235) = \mathbf{pit}(4235) = 2$.

Remember that we have introduced three other statistics, namely "**NL**" ("next-to-last"), "**L**" ("last"), and "**grn**" ("greater neighbor of the maximum"), and that $\mathbf{grn}\, w = \mathbf{NL} w$ whenever w is an André I permutation. Our goal is to prove the next theorem.

Theorem 3.1. *The triplets* $(\mathbf{F}, \mathbf{spi}, \mathbf{NL})$ *on* And_n^{I} *and* $(\mathbf{pit}, \mathbf{L}, \mathbf{grn})$ *on* And_n^{II} *are equidistributed.*

Let $X = \{a_1 < a_2 < \cdots < a_n\}$ be a set of positive integers (or any finite totally ordered set), and let And_X^{I} (respectively And_X^{II}) denote the set of all André I (respectively André II) permutations of X. To prove the previous statement, a bijection

$$\phi : \mathrm{And}_X^{I} \to \mathrm{And}_X^{II} \tag{3.2}$$

will be constructed having the property

$$(\mathbf{F}, \mathbf{spi}, \mathbf{NL})\, w = (\mathbf{pit}, \mathbf{L}, \mathbf{grn})\, \phi(w). \tag{3.3}$$

When $n = 0$, let $\phi(e) := e$ with e the empty word. Let $\phi(a_1) := a_1$ for $n = 1$; $\phi(a_1 a_2) := a_1 a_2$ for $n = 2$. For $n \geq 3$ each permutation w from

And_X^I has one of the two forms:

$$\text{(i) } w = v_0\, a_1\, v_1\, a_2\, v_2; \qquad \text{(ii) } w = v_0\, a_2\, v_2\, a_1\, v_1.$$

Note that the three factors v_0, v_1, v_2 and the product $v_0a_2v_2$ are all André I permutations, and v_0 is possibly empty. In case (i), v_1 may be empty, but not v_2 (which ends with a_n greater than a_2); in case (ii), v_2 may be empty, but not v_1 (which ends with a_n). For both cases (i) and (ii), define

$$\phi(w) := \phi(v_1)\, a_1\, \phi(v_0\, a_2\, v_2). \tag{3.4}$$

By induction, both factors $\phi(v_1)$ and $\phi(v_0a_2v_2)$ are André II, as well as $\phi(w)$, since a_2 is to the right of a_1.

Example. Consider the André I permutation

$$w = \underset{v_0}{7\;8\;5\;6\;9}\,\Big|\,\underset{a_2}{2}\,\Big|\,\underset{v_2}{10}\,\Big|\,\underset{a_1}{1}\,\Big|\,\underset{v_1}{11\;3\;12\;4\;13};$$

we then have:

$$\begin{aligned}
\phi(w) &\overset{\text{(ii)}}{=} \phi(11\;3\;12\;4\;13)\,1\,\phi(7\;8\;5\;6\;9\;2\;10);\\
\phi(\underset{v_0}{11}\;\underset{a_1}{3}\;\;\underset{v_1}{12}\;\underset{a_2}{4}\;\;\underset{v_2}{13}) &\overset{\text{(i)}}{=} 12\;3\;\phi(11\;4\;13);\\
\phi(\underset{a_2}{11}\;\underset{a_1}{4}\;\;\underset{v_1}{13}) &\overset{\text{(ii)}}{=} 13\;4\;11;\\
\phi(\underset{v_0}{7\;8}\;\big|\underset{a_2}{5}\;\big|\underset{v_2}{6\;9}\;\big|\underset{a_1}{2}\big|\underset{v_1}{10}) &\overset{\text{(ii)}}{=} 10\;2\;\phi(7\;8\;5\;6\;9);\\
\phi(\underset{v_0}{7\;8}\;\big|\underset{a_1}{5}\;\big|\underset{a_2}{6}\big|\underset{v_2}{9}\;) &\overset{\text{(i)}}{=} 5\;\phi(7\;9\;6\;9);\\
\phi(\underset{a_2}{7}\;\big|\underset{v_2}{8}\;\big|\underset{a_1}{6}\big|\underset{v_1}{9}\;) &\overset{\text{(ii)}}{=} 9\;6\;\phi(7\;8) = 9\;6\;7\;8;
\end{aligned}$$

so that

$$\phi(w) = 12\;3\;13\;4\;11\;1\;10\;2\;5\;9\;6\;7\;8.$$

We can verify that $\qquad (\mathbf{F}\,,\mathbf{spi},\mathbf{NL})\,w = (\mathbf{pit},\mathbf{L}\,,\mathbf{grn})\,\phi(w) = (7,8,4).$

With the previous definition of ϕ, we see that the maximum letter a_n of X occurs in $\phi(v_0a_2v_2)$ (respectively in $\phi(v_1)$) when w is of form (i) (respectively of form (ii)). For constructing the inverse ϕ^{-1} of ϕ, this suggests that we start with the factorization

$$v = w_0\, a_1\, w_1$$

of each permutation v from And_X^{II} with $\#X \ge 3$, after defining $\phi^{-1}(e) := e$; $\phi^{-1}(a_1) := a_1$ for $n=1$; $\phi^{-1}(a_1a_2) := a_1a_2$ for $n=2$. As both w_0, w_1 are André II permutations with fewer letters, the images $\phi^{-1}(w_0)$, $\phi^{-1}(w_1)$ are defined by induction. Let $v_1 := \phi^{-1}(w_0)$. As the minimum letter of w_1 is a_2, define v_0 and v_2 to be the factors in $\phi^{-1}(w_1) := v_0\,a_2\,v_2$. Next, let

$$\phi^{-1}(v) := \begin{cases} v_0\,a_1\,v_1\,a_2\,v_2, & \text{if } a_n \text{ is a letter of } w_1; \\ v_0\,a_2\,v_2\,a_1\,v_1, & \text{if } a_n \text{ is a letter of } w_0. \end{cases} \tag{3.5}$$

Lemma 3.2. *The André II permutation $\phi(w)$ ends with its maximum letter if and only if w starts with its minimum letter* $\min w$, *and then* $\mathbf{pit}\,\phi(w) = \mathbf{F}\,w = \min w$.

Proof. This is obviously true for $n=2$. For $n\ge 3$, we have $w = a_1v_1a_2v_2$ and $\phi(w) = \phi(v_1)a_1\phi(a_2v_2)$. As a_2v_2 is André I starting with its minimum letter a_2, then, by induction, $\phi(a_2v_2)$ ends with its maximum letter, which is equal to $\max w$. Hence, $\mathbf{pit}\,\phi(w) = \mathbf{pit}(\phi(v_1)a_1\phi(a_2v_2)) = a_1 = \mathbf{F}\,w$. For the converse take the notation $v = w_0a_1w_1$ of (3.5). When the maximum letter a_n occurs in w_1, then $\psi(v) = v_0a_1v_1a_2v_2$ with $v_1 = \psi(w_0)$ and $\psi(w_1) = v_0a_2v_2$. By assumption, a_n occurs at the end of v, hence, at the end of w_1. By induction, $\psi(w_1) = v_0a_2v_2$ starts with its minimum letter. This can be true only if $v_0 = e$. Therefore, $\psi(v) = a_1v_1a_2v_2$ and starts with its minimum letter a_1. $\square$

Theorem 3.3. *The mapping ϕ is a bijection between And_n^{I} and And_n^{II}. Moreover, relation* (3.3) *holds.*

Proof. The bijectivity is proved by the construction of the inverse ϕ^{-1} (see (3.5)). To prove identity (3.3), let w be a André I permutation, either of the form $v_0a_1v_1a_2v_2$, or of the form $v_0a_2v_2a_1v_1$. In both cases,

$$\begin{aligned} \mathbf{spi}\,w &= \mathbf{spi}\,v_0 = \mathbf{spi}(v_0a_2v_2) \\ &= \mathbf{L}\,\phi(v_0a_2v_2) && \text{[by induction]} \\ &= \mathbf{L}\,(\phi(v_1)a_1\phi(v_0a_2v_2)) = \mathbf{L}\,\phi(w). \end{aligned}$$

Next, if $w = v_0a_1v_1a_2v_2$, then

$$\begin{aligned} \mathbf{NL}\,w &= \mathbf{NL}(v_0a_2v_2) \\ &= \mathbf{grn}\,\phi(v_0a_2v_2) && \text{[by induction]} \\ &= \mathbf{grn}(\phi(v_1)a_1\phi(v_0a_2v_2)) = \mathbf{grn}\,\phi(w), \end{aligned}$$

because the maximum letter a_n is a letter of v_2. If $w = v_0a_2v_2a_1v_1$, then a_1v_1 has at least two letters and ends with a_n, so that a_1v_1 is André I of form (i). Consequently,

$$\begin{aligned}\mathbf{NL}\, w &= \mathbf{NL}(a_1 v_1)\\ &= \mathbf{grn}\,\phi(a_1 v_1) && \text{[by induction]}\\ &= \mathbf{grn}(\phi(v_1) a_1 \phi(v_0 a_2 v_2)) = \mathbf{grn}\,\phi(w).\end{aligned}$$

When w does not start with its minimum letter, then $\phi(w) = \phi(v_1)a_1\phi(v_0a_2v_2)$, and $\phi(v_0a_2v_2)$ does not end with $\max w$. Therefore, $\mathbf{pit}\,\phi(w) = \mathbf{pit}\,\phi(v_0a_2v_2) = \mathbf{F}\, v_0a_2v_2$. As $v_0 = e$ only in case (ii), we then have: $\mathbf{pit}\,\phi(w) = \mathbf{F}\, v_0a_2 = \mathbf{F}\, w$. $\square$

4. The bijection g from the set of André I permutations onto itself

When making up the tables of the distribution of the bivariate statistic $(\mathbf{spi}, \mathbf{F})$ on And_n^I for $n = 1, 2, \ldots, 7$, as shown in Table 4.1, it can be noticed that the matrices are symmetric with respect to their skew-diagonals. The property will hold in general if a bijection g from And_n^I onto itself can be constructed satisfying

$$(\mathbf{F}, \mathbf{spi})\, w = (n + 1 - \mathbf{spi}, n + 1 - \mathbf{F})\, g(w) \tag{4.1}$$

for all w from And_n^I.

$\mathbf{F} =$	1
$\mathbf{spi} = 2$	1

$n = 2$

$\mathbf{F} =$	1	2
$\mathbf{spi} = 2$	.	1
3	1	.

$n = 3$

$\mathbf{F} =$	1	2	3
$\mathbf{spi} = 2$	.	1	.
3	.	1	1
4	2	.	.

$n = 4$

$\mathbf{F} =$	1	2	3	4
$\mathbf{spi} = 2$	.	2	.	.
3	.	1	3	.
4	.	2	1	2
5	5	.	.	.

$n = 5$

$\mathbf{F} =$	1	2	3	4	5
$\mathbf{spi} = 2$	.	5	.	.	.
3	.	2	8	.	.
4	.	3	3	8	.
5	.	6	3	2	5
6	16	.	.	.	.

$n = 6$

$\mathbf{F} =$	1	2	3	4	5	6
$\mathbf{spi} = 2$	.	16	.	.	.	.
3	.	5	27	.	.	.
4	.	7	8	31	.	.
5	.	11	10	8	27	.
6	.	22	11	7	5	16
7	61	.	.	.	.	.

$n = 7$

Table 4.1: distribution of $(\mathbf{spi}, \mathbf{F})$ on And_n^I

For the construction of g we proceed as follows. An André I permutation $v = y_1 y_2 \cdots y_l$ on a set X (of cardinality $l \geq 2$) is called *simple*, if the first letter y_1 of v is equal to $\min X$. Consider an André I permutation $w = x_1 x_2 \cdots x_n$ from And_n^I. Let $1 = a_1 < a_2 < \cdots < a_r$ (respectively $1 = b_1 < b_2 < \cdots < b_s = n$) be the increasing sequence of subscripts such that $x_{a_1} > x_{a_2} > \cdots > x_{a_r}$ (respectively $x_{b_1} < x_{b_2} < \cdots < x_{b_s}$) is the increasing (respectively decreasing) sequence of the left minimum (respectively maximum) records of $w = x_1 x_2 \cdots x_n$ from And_n^I.

For the following André I permutation the left minimum (respectively maximum) records are underlined (respectively overlined):

$$w = \underline{\overline{7}}\,\overline{8}\,\underline{5}\,6\,\overline{9}\,\underline{2}\,\overline{10}\,\underline{1}\,\overline{11}\,3\,\overline{12}\,4\,\overline{13}\,.$$

Going back to the general case, let

$$v_1 := x_1 \cdots x_{a_2 - 1},\ v_2 := x_{a_2} \cdots x_{a_3 - 1},\ \ldots\ ,\ v_r := x_{a_r} \cdots x_n,$$

so that w is the juxtaposition product $v_1 v_2 \cdots v_r$, and the factors v_i are obtained by cutting the word w just before each left minimum record. The factorization $(v_1, v_2, \ldots, v_r)$ is called the *canonical factorization* of the André I permutation w. Furthermore, the sequence

$$(\,(\mathbf{F}\,v_1, \mathbf{L}\,v_1), (\mathbf{F}\,v_2, \mathbf{L}\,v_2), \ldots, (\mathbf{F}\,v_r, \mathbf{L}\,v_r)\,),$$

which is also equal to $(\,(x_1, x_{a_2 - 1}), (x_{a_2}, x_{a_3 - 1}), \ldots, (x_{a_r}, x_n)\,)$, is called the *type* of the canonical factorization of w.

In our running example, the canonical factorization reads

$$w = \underbrace{\underline{\overline{7}}\,\overline{8}}_{v_1} \mid \underbrace{\underline{5}\,6\,\overline{9}}_{v_2} \mid \underbrace{\underline{2}\,\overline{10}}_{v_3} \mid \underbrace{\underline{1}\,\overline{11}\,3\,\overline{12}\,4\,\overline{13}}_{v_4}$$

and is of type $(\,(7,8), (5,9), (2,10), (1,13)\,)$.

Proposition 4.1. *Let $(v_1, v_2, \ldots, v_r)$ be the canonical factorization of the André I permutation $w = x_1 x_2 \cdots x_n$ from And_n^I. Let s be the number of left maximum records of w. Then:*

(i) $r \le s$;
(ii) each factor v_i $(i = 1, 2, \ldots, r)$ is a simple André I permutation;
(iii) $\mathbf{L}\,v_i$ is a left maximum record, so that $\mathbf{L}\,v_1 < \mathbf{L}\,v_2 < \cdots < \mathbf{L}\,v_{r-1} < \mathbf{L}\,v_r = n$ and, of course, $\mathbf{F}\,v_1 > \mathbf{F}\,v_2 > \cdots > \mathbf{F}\,v_{r-1} > \mathbf{F}\,v_r = 1$;
(iv) $(\mathbf{F}\,v_1, \mathbf{L}\,v_1) = (\mathbf{F}\,w, \mathbf{spi}\,w)$.

Let $w = x_1 x_2 \cdots x_n$ be an André I permutation from And_n^I and $\overline{w} := \overline{x}_n \overline{x}_{n-1} \overline{x}_{n-2} \cdots \overline{x}_1$ be the permutation defined by $\overline{x}_i := N - x_{n+1-i}$ $(i = 1, 2, \ldots, n)$, where N is some integer greater than n.

Proposition 4.2. *If w is a simple André I permutation, so is $\overline{w}$.*

For constructing the bijection g, let $n \geq 3$ and $N := n+1$. If $(v_1, v_2, \ldots, v_r)$ is the canonical factorization of a permutation w from And_n^I, define $g(w)$ to be the juxtaposition product

$$g(w) := \overline{v}_1\, \overline{v}_2\, \ldots\, \overline{v}_r. \tag{4.1}$$

Furthermore, if $\tau = ((p_1, q_1), (p_2, q_2), \ldots, (p_r, q_r))$ is the canonical factorization type of w, let

$$\overline{\tau} := ((\overline{q}_1, \overline{p}_1), (\overline{q}_2, \overline{p}_2), \ldots, (\overline{q}_r, \overline{p}_r)).$$

We then have the following fundamental property of g.

Theorem 4.3. *The transformation g is a bijection from And_n^I onto itself. Furthermore, if τ is the canonical factorization type of w, then $\overline{\tau}$ is the canonical factorization type of $g(w)$. In particular,*

$$(\mathbf{F}\,, \mathbf{spi})\, g(w) = (n+1-\mathbf{spi},\ n+1-\mathbf{F}\,)\, w. \tag{4.2}$$

In the running example, where $n+1 = 14$, we get

$$g(w) = 6\,7\ \mid\ 5\,8\,9\ \mid\ 4\,12\ \mid\ 1\,10\,2\,11\,3\,13,$$

which is of type $(\,(6,7), (5,9), (4,12), (1,13)\,)$.

The proofs of Propositions 4.1, 4.2 and Theorem 4.3 do not present any difficulties and are therefore omitted.

5. The proof of Theorem 1.1 (iii) and (iv)

We reproduce the sequence (1.5) by decomposing the product $\phi \circ g$:

$$\begin{array}{ccccc} \mathrm{And}_n^I & \xrightarrow{g} & \mathrm{And}_n^I & \xrightarrow{\phi} & \mathrm{And}_n^{II} \\ w & \mapsto & g(w) & \mapsto & \phi(g(w)) \\ \mathbf{F}\, w & = & n+1-\mathbf{spi}\, g(w) & = & n+1-\mathbf{L}\, \phi(g(w)) \end{array} \tag{5.1}$$

The first (respectively second) identity $\mathbf{F}\, w = n+1-\mathbf{spi}\, g(w)$ (respectively $\mathbf{spi}\, g(w) = \mathbf{L}\, \phi(g(w))$) is a specialization of (4.2) (respectively of (3.3)).

Take the example of the previous section: $w = 7\,8\,5\,6\,9\,2\,10\,1\,11\,3\,12\,4\,13$ and $g(w) = 6\,7\,5\,8\,9\,4\,12\,1\,10\,2\,11\,3\,13$. By using the definition of ϕ given in (3.4), we get $\phi(g(w)) = 10\,1\,11\,2\,13\,3\,12\,4\,8\,9\,5\,6\,7$, which belongs to And_{13}^{II}, and $n+1-\mathbf{L}\, \phi(g(w)) = 14 - 7 = 7 = \mathbf{F}\, w$.

Next, reproduce the sequence (1.4) by decomposing the product $\phi\circ\theta$:

$$
\begin{array}{ccccc}
\mathrm{And}_n^{I} & \xrightarrow{\theta} & \mathrm{And}_n^{I} & \xrightarrow{\phi} & \mathrm{And}_n^{II}\\
w & \mapsto & \theta(w) & \mapsto & \phi(\theta(w))\\
\mathbf{F}\,w & = & n-\mathbf{NL}\,\theta(w) & = & n-\mathbf{grn}\,\phi(\theta(w))
\end{array}
\tag{5.2}
$$

The first (respectively second) identity $\mathbf{F}\,w=n-\mathbf{NL}\,\theta(w)$ (respectively $\mathbf{NL}\,\theta(w)=\mathbf{grn}(\phi(g(w)))$ is a specialization of (2.3) (respectively of (3.3)).

For example, with $w'=10\,2\,11\,3\,12\,1\,9\,4\,5\,8\,6\,7\,13$, we obtain $\theta(w')=6\,7\,5\,8\,9\,4\,12\,1\,10\,2\,11\,3\,13$ and $\phi(\theta(w'))=10\,1\,11\,2\,13\,3\,12\,4\,8\,9\,5\,6\,7$. Thus, $\mathbf{F}\,w'=10=n-\mathrm{NL}\,\theta(w')=13-3=n-\mathbf{grn}\,\phi(\theta(w'))$.

The proofs of (iii) and (iv) of Theorem 1.1 are now completed. Another proof of Theorem 1.1 (iii) and (iv) makes use of the properties of a rearrangement group G_n, acting on the group $\mathfrak{S}_n$ of all permutations of $\{1,2,\ldots,n\}$, which were developed in [FSt74, FSt76], and another correspondence Γ on binary increasing trees, introduced in [FH13]. They constitute the main ingredients for the constructions of three bijections, $\overline{\Gamma}$, Φ^{I}, and Φ^{II}, appearing in the diagram

$$
\begin{array}{ccccc}
\mathrm{And}_n^{II} & \xrightarrow{\overline{\Gamma}} & \mathfrak{S}_n/G_n & \xrightarrow{\Phi^{I}} & \mathrm{And}_n^{I}\\
 & & \big\downarrow{\scriptstyle\Phi^{II}} & & \\
 & & \mathrm{And}_n^{II} & &
\end{array}
$$

which have the property

$$\mathbf{L}\,w=1+\mathbf{NL}\,\Phi^{I}\overline{\Gamma}(w)=1+\mathbf{grn}\,\Phi^{II}\overline{\Gamma}(w).$$

6. Combinatorics of the twin Seidel matrix sequence

This section is devoted to proving Theorem 1.2. As announced in Subsection 1.5, the question is to show that the integers $a_n(m,k)$ and $b_n(m,k)$, when taken as $a_n(m,k)=\#A_n(m,k)$, $b_n(m,k)=\#B_n(m,k)$ with

$$A_n(m,k):=\{w\in\mathrm{And}_n^{I}:(\mathbf{F}\,,\mathbf{NL})w=(m,k)\};\tag{6.1}$$

$$B_n(m,k):=\{w\in\mathrm{And}_n^{I}:(\mathbf{spi},\mathbf{grn})w=(m,k)\};\tag{6.2}$$

satisfy all the properties (TS1)–(TS5) stated in Subsection 1.5.

The verifications of properties (TS1), (TS2), (TS3), (TS4.1), (TS4.2), (TS5.1) are easy and given in the next subsection. The proofs of the other properties are much harder and will be developed thereafter.

6.1. *The first evaluations.* By (1.7), the set $B_n(m,k)$ is equinumerous with

$$B'_n(m,k) := \{w \in \mathrm{And}_n^{II} : (\mathbf{L}\,, \mathbf{grn})w = (m,k)\}. \tag{6.3}$$

The evaluations in this subsection are made by using $B'_n(m,k)$ instead of $B_n(m,k)$.

(TS1) Nothing to prove, except for the diagonals of the twin Seidel matrices A_n and B_n. They have zero entries when $n \geq 3$, because the first and next-to-last letter of each André I permutation cannot be the same! On the other hand, the identity $\mathbf{L}\,w = \mathbf{grn}\,w = m$ would mean that the permutation w from And_n^{II} ends with a double descent $n > m > 0$.

(TS2) We have $a_n(k,n) = b_n(k,n) = 0$, because $\mathbf{grn}\,w \leq n-1$ for each w from either And_n^{I}, or And_n^{II}. Also, $a_n(n,k) = 0$, as each permutation from And_n^{I} ends with n. Finally, $b_n(1,k) = 0$, because each permutation from And_n^{II} cannot end with the letter 1.

(TS3) We have: $A_2 = \begin{matrix} 1 & \cdot \\ \cdot & \cdot \end{matrix}$ and $B_1 = \begin{matrix} \cdot & \cdot \\ 1 & \cdot \end{matrix}$, because $\mathrm{And}_2^{I} = \mathrm{And}_2^{II} = \{12\}$ and $(\mathbf{F}\,, \mathbf{NL}, \mathbf{L}\,, \mathbf{grn})(12) = (1,1,2,1)$.

(TS4.1) The entry $b_n(n,k)$ counts the André II permutations w from And_n^{II} ending with the two-letter factor $k\,n$. The deletion of the ending letter n maps w onto an André II permutation w' from And_{n-1}^{II} ending with k in a bijective manner. Hence, $b_n(n,k) = b_{n-1}(k,\bullet)$, which is equal to $a_{n-1}(\bullet, k-1)$ by Theorem 1.1 for $1 \leq k \leq n-1$.

(TS4.2) The entry $b_n(n-1,k)$ counts the André II permutations w from And_n^{II} of the form $w = x_1 \cdots x_{i-2}\, n\, x_i \cdots x_{n-1}\,(n-1)$ with $i \leq n-1$ and k equal to x_{i-2} or x_i. Such a permutation can be mapped onto a permutation w' from And_{n-1}^{II} defined by

$$w' := x_1 \cdots x_{i-2}\,(n-1)\,x_i \cdots x_{n-1}. \tag{6.4}$$

This defines a bijection between the set of all w from And_n^{II} such that $(\mathbf{L}\,, \mathbf{grn})w = (n-1,k)$ and the set of all w' from And_{n-1}^{II} such that $\mathbf{grn}\,w' = k$ $(1 \leq k \leq n-2)$. Thus, $b_n(n-1,k) = b_{n-1}(\bullet, k)$, which is also equal to $a_n(\bullet, k)$ by Theorem 1.1.

(TS5.1) The entry $a_n(1,k)$ counts the permutations w from A_n^I such that $(\mathbf{F}\,, \mathbf{NL})w = (1,k)$. The bijection $1\,x_2 \cdots k\,n \mapsto (x_2-1)\cdots(k-1)\,(n-1)$ maps the set of these permutations onto the set of all w' from A_{n-1}^I such that $\mathbf{grn}\,w' = k-1$. Hence, $a_n(1,k) = a_{n-1}(\bullet, k-1)$.

6.2. *Tight André I permutations.* As sketched in Subsection 1.6 and its display (1.13), proving (TS5.2) and (TS5.3) amounts to do the following points:

(a) split each set $A_n(m,k)$ into two disjoint subsets

$$A_n(m,k) = T_n(m,k) \uplus NT_n(m,k),$$

in such a way that

(b) when $2 \le k+1 \le m \le n-2$ or $3 \le m+2 \le k \le n-1$ a bijection

$$f : NT_n(m,k) \to A_n(m+1,k);$$

(c) and another bijection

$$\begin{aligned} &\phi : B_{n-1}(m,k) \to T_n(m,k), && \text{when } m > k;\\ &\phi : B_{n-1}(m,k-1) \to T_n(m,k), && \text{when } m < k; \end{aligned}$$

can be duly constructed.

Points (a) and (b). Let f be the transposition of the first letter $\mathbf{F}\, w = m$ within a permutation w and the letter equal to $(m+1)$ $(1 \le m \le n-2)$:

$$f : w = m\, v\, (m+1)\, v' \mapsto w' = (m+1)\, v\, m\, v'. \tag{6.5}$$

If w is an André I permutation, the image $w' = f(w)$ is not always an André I permutation. For example, 423516 belongs to And_6^I, but not $f(w) = 523416$, for the trough 2 is not of type I. However, the reverse transposition

$$f^{-1} : w' = (m+1)\, v\, m\, v' \mapsto f^{-1}(w') = w = m\, v\, (m+1)\, v',$$

whenever defined, maps each André I permutation to an André I permutation. The André I permutations w, whose images $f(w)$ are *not* André I permutations, are called *tight*. They are characterized as follows.

Definition 6.1. An André I permutation $w = m\, v\, (m+1)\, v'$ is said to be *tight*, if the following two conditions hold:

(i) either $v = e$, or $v \ne e$ and all its letters are less than m;

(ii) either $v' \ne e$ and $\mathbf{F}\, v'$ is less than all the letters of w to its left, or $v' = e$ and necessarily $m = n-1$.

Let T_n (respectively NT_n) be the subset of all André I permutations from And_n^I, which are tight (respectively not tight), and let $T_n(m,k) := T_n \cap A_n(m,k)$, $NT_n(m,k) := NT_n \cap A_n(m,k)$.

Note that the André I permutations from $A_n(1,k)$ are all of the form $1\, v\, 2\, v'$ and, either the letters of v are all greater than 2, or $v' \ne e$ but $2 < \mathbf{F}\, v'$, so that at least one of conditions (i), (ii) does not hold. Accordingly, $NT_n(1,k) = A_n(1,k)$ for all k, that is, all André I permutations starting with 1 are not tight. Also, note that each André I permutation from $A_n(n-1,k)$ is of the form $w = (n-1)\, v\, n$ and is necessarily tight, so that $T_n(n-1,k) = A_n(n-1,k)$ for all k.

Proposition 6.1. *Let $n \geq 3$ and let w be a tight André I permutation from* And_n^I*. Then, $f(w)$ (defined in (6.5)) cannot be an André I permutation.*

Proof. Take the notation of (6.5) for w and $w' = f(w)$. If $m = n-1$, then $w' = n\,v\,(n-1)$ is not André I. If $m \leq n-2$, $v = e$, and (ii) of Definition 6.1 holds, then w' contains the double descent $(m+1) > m > \mathbf{F}\,v'$, therefore is not André I. If $m \leq n-2$, $v \neq e$ and (ii) of Definition 6.1 holds, let x be the minimum trough in w' between $(m+1)$ and m; then the x-factorization (w_1, w_2, x, w_4, w_5) of w' is such that $\max w_2 w_4 = m+1$ with $(m+1)$ a letter of w_2. Again, w' cannot be an André I permutation. ∎

Proposition 6.2. *If $2 \leq k+1 \leq m \leq n-2$ or $3 \leq m+2 \leq k \leq n-1$, then f maps $\mathit{NT}_n(m,k)$ onto $A_n(m+1,k)$ in a bijective manner.*

Proof. To prove that w' is André I when w is not tight, we prove that (i) w' has no double descent; (ii) all the troughs of w' are of type I.

(i) The only double descent that could be created when going from w to w' is $(m+1) > m > \mathbf{F}\,v'$. This could occur only if $v = e$, $v' \neq e$ and $m > \mathbf{F}\,v'$, and this would mean that w is tight; a contradiction.

(ii) Let x_i (respectively x'_i) be the i-th letter counted from left to right of w (respectively w'). Also, let $(w_1, w_2, x_i, w_4, w_5)$ (respectively $(w'_1, w'_2, x'_i, w'_4, w'_5)$) be the x_i- (respectively x'_i-)factorization of w (respectively of w'). Several cases are to be considered.

(1) Suppose that x_i is to the right of $(m+1)$ in w. Then $x'_i = x_i$. If x_i is a trough of w, then either $(m+1)$ is a letter of w_2, or not. If it is, then w'_2 is derived from w_2 by replacing the letter $(m+1)$ by m. Therefore, $\max w'_2 \leq \max w_2 < \max w_4 = \max w'_4$, and the x'_i-factorization remains of type I in w'. If it is not, then $w'_2 = w_2$, $w'_4 = w_4$, and the same conclusion holds.

(2) Now, suppose that $x_i = (m+1)$, so that $x'_i = m$. If x_i is a trough of w — this is possible, as w is supposed to be not tight — then $v \neq e$, and m is not a letter of w_2. Furthermore, $(m+1)$ is a letter of w'_2 only when all the letters between m and $(m+1)$ are greater than $(m+1)$. Whatever the particular case may be, we have $\max w'_2 = \max w_2 < \max w_4 = \max w'_4$, so that x'_i is a trough of type I in w'.

(3) Next, let x_i lie between m and $(m+1)$ in w, so that $x'_i = x_i$, and suppose that x_i is trough of w. If x_i is greater than $(m+1)$, then $w'_2 = w_2$ and $w'_4 = w_4$. Moreover, $x'_i = x_i$ will be a trough of type I in w'. If x_i is less than m, the only problem arises when m and $(m+1)$ are the maximum letters of w_2 and w_4, respectively. In such a case, all the letters between m and $(m+1)$ are smaller than m and $m > \mathbf{F}\,v'$. Hence, w would be tight, a contradiction.

Thus, the image $f(w)$ of w, supposed to be not tight, is André I. If

$2 \le k+1 \le m \le n-2$, a fortiori, $k < m+1$, so that the next-to-last letter of a permutation w from $NT_n(m,k)$, which is equal to k, cannot be equal to $(m+1)$. Thus, $f(NT_n(m,k)) \subset A_n(m+1,k)$. In the same manner, if $3 \le m+1 < k \le n-1$, the inequality $m+1 < k$ implies the same inclusion. As f and f^{-1} are inverses of each other when applied to the sets $A_n(m,k)$ and $A_n(m+1,k)$, respectively, the restriction of f^{-1} to $A_n(m+1,k)$ is necessarily $NT_n(m,k)$ by Proposition 6.1. Thus, Proposition 6.2 is proved for $2 \le k+1 \le m \le n-2$ and $3 \le m+1 < k \le n-1$. ∎

In Table 6.1 the bijection $f : NT_5(m,k) \to A_5(m+1,k)$ is materialized by the vertical arrows. The five tight permutations in And_5^I are reproduced in boldface. They can only be targets of these arrows. This completes the program of points (a) and (b).

F \ NL	1	2	3	4
1		13425	12435 14235	12345 13245
2	23415		21435 24135	21345 **23145**
3	**32415**	31425 **34125**		31245
4		**41325**	**41235**	

Table 6.1: the bijection $f : NT_5(m,k) \to A_5(m+1,k)$ $(k \ne m+1)$.
$f : A_5(m,m+1) \to A_5(m+1,m)$

Remark. When $3 \le m+1 = k \le n-2$, in the permutation $w = m\,v\,(m+1)\,v'$ from $A_n(m,m+1)$ the right factor v' is equal to the one-letter word n. This implies that f maps $A_n(m,m+1)$ onto $A_n(m+1,m)$ in a bijective manner. In particular, $a_n(m,m+1) = a_n(m+1,m)$. This fact is illustrated in Table 6.1 by oblique arrows.

Point (c). Let $n \ge 3$, and consider a permutation $w = x_1x_2\cdots x_{n-1}$ from And_{n-1}^I. Let $x_j = \mathbf{spi}\,w$. Define $\phi(w) := x_jx'_1x'_2\cdots x'_{n-1}$, where

$$x'_i := \begin{cases} x_i, & \text{if } x_i \le x_j - 1; \\ x_i + 1, & \text{if } x_i \ge x_j. \end{cases} \tag{6.6}$$

The inverse bijection ϕ^{-1} is defined as follows: let $w' = x'_1x'_2\cdots x'_n$ belong to T_n; then $\phi^{-1}(w') := \rho(x'_2\cdots x'_n)$, where ρ is the reduction defined in Section 2.

Theorem 6.3. *The mapping ϕ is a bijection between And^I_{n-1} and the set T_n of all tight André I permutations having the following properties:*

(i) $\mathbf{spi}\, w = \mathbf{F}\, \phi(w)$;

(ii) $\mathbf{grn}\, \phi(w) = \begin{cases} \mathbf{grn}\ w, & \text{if } \mathbf{spi}\, w > \mathbf{grn}\, w; \\ \mathbf{grn}\, w + 1, & \text{if } \mathbf{spi}\, w < \mathbf{grn}\, w. \end{cases}$

$k=$	1	2	3	4	5
$m=1$	·	·	·	·	·
2				$\widehat{\mathbf{2}}\mathbf{1435}$ 231546	$\widehat{\mathbf{2}}\mathbf{1345}$ 231456
3	$\widehat{\mathbf{3}}\mathbf{2415}$ 342516	$\widehat{\mathbf{3}}\mathbf{1425}$ 341526			$\mathbf{2}\widehat{\mathbf{3}}\mathbf{145}$ 324156 $\widehat{\mathbf{3}}\mathbf{1245}$ 341256
4	$\mathbf{23}\widehat{\mathbf{4}}\mathbf{15}$ 423516	$\mathbf{3}\widehat{\mathbf{4}}\mathbf{125}$ 435126 $\widehat{\mathbf{4}}\mathbf{1325}$ 451326	$\mathbf{2}\widehat{\mathbf{4}}\mathbf{135}$ 425136 $\widehat{\mathbf{4}}\mathbf{1235}$ 451236		
5		$\mathbf{1342}\widehat{\mathbf{5}}$ 513426	$\mathbf{1243}\widehat{\mathbf{5}}$ 512436 $\mathbf{1423}\widehat{\mathbf{5}}$ 514236	$\mathbf{1234}\widehat{\mathbf{5}}$ 512346 $\mathbf{1324}\widehat{\mathbf{5}}$ 513246	

Table 6.2: The bijection $\phi : B_5(m, k-1)$ (respectively $B_5(m, k)$) $\to T_6(m, k)$.

In Table 6.2 the permutations in boldface are the elements of And^I_5. Their images under ϕ are the sixteen tight permutations from T_6, written in plain below them. The box (m, k) contains the permutations w from And^I_5 such that $\mathbf{spi}\, w = m$ and $\mathbf{grn}\, w = k$ (respectively $\mathbf{grn}\, w = k-1$) when $m > k$ (respectively when $m < k$). It also contains the elements w' from T_6 such that $\mathbf{F}\, w' = m$ and $\mathbf{grn}\, w' = k$. A hat sign $\widehat{\ }$ has been put on the spike of w.

Proof of Theorem 6.3. Let $w = x_1 x_2 \cdots x_{n-2} x_{n-1}$ be from And^I_{n-1}. Let $x_j = \mathbf{spi}\, w$. If $j = 1$, then $x_1 > x_2$ and $\phi(w) = x_1\,(x_1+1)\, x_2 \cdots x'_{n-2} x'_{n-1}$. Accordingly, $\phi(w)$ is tight. Moreover, $\mathbf{spi}\, w = x_1 = \mathbf{F}\, \phi(w)$, still since $x_1 > x_2$. Also, either $\mathbf{grn}\, w = x_{n-2} < x_1 = \mathbf{spi}\, w$, and then $\mathbf{grn}\, \phi(w) = x'_{n-2} = x_{n-2} = \mathbf{grn}\, w$, or $\mathbf{grn}\, w = x_{n-2} > x_1 = \mathbf{spi}\, w$, and then $\mathbf{grn}\, \phi(w) = x'_{n-2} = x_{n-2} + 1 = \mathbf{grn}\, w + 1$.

If $j \ge 2$, we have

$$\phi(w) = x_j\, x_1\, \cdots\, x_{j-1}\, (x_j + 1)\, x_{j+1}\, x'_{j+2}\, \cdots x'_{n-1}. \tag{6.7}$$

On the other hand, $\phi(w)$ is André I, because no double descent has been created; furthermore, the new trough x_1 is of type I, as the letter $(x_j + 1)$ is to its right. Also, $\phi(w)$ is tight, because x_{j+1} (respectively $(x_j + 1)$) is less (respectively greater) than all the letters to its left. Finally, $\mathbf{spi}\, w = x_j = \mathbf{F}\, \phi(w)$. Moreover, $\mathbf{grn}\, \phi(w) = x'_{n-2}$ is equal to $x_{n-2} = \mathbf{grn}\, w$ or $x_{n-2} + 1 = \mathbf{grn}\, w + 1$, depending on whether $x_{n-2} = \mathbf{grn}\, w$ is less than or at least equal to $x_n = \mathbf{spi}\, w$. ∎

This achieves the program of point (c), by definition of $B_n(m, k)$ given in (6.2).

6.3. *Hooked and unhooked permutations.* Let $n \ge 3$, and consider the mapping Θ, defined on And^I_{n-1} as follows. Let $w = x_1 x_2 \cdots x_{n-1}$ belong to And^I_{n-1}. Define

$$\Theta(w) := \begin{cases} (x_1 + 1) x_1 x'_2 \cdots x'_{n-1}, & \text{if } x_1 < x_2; \\ x_1 (x_1 + 1) x'_2 \cdots x'_{n-1}, & \text{if } x_1 > x_2; \end{cases} \tag{6.8}$$

where $x'_i := x_i$ (respectively $x_i + 1$) if $x_i < x_1$ (respectively if $x_i > x_1$). Clearly, Θ is an *injection* from And^I_{n-1} into And^I_n. The permutations belonging to the subset $\Theta(\mathrm{And}^I_{n-1})$ are said to be *hooked*. Their formal definition is stated next.

Definition 6.2. An André I permutation $w = x_1 x_2 \cdots x_n$ ($n \ge 3$) from And^I_n is called *hooked*, if $x_1 - 1 = x_2 < x_3$ or $x_1 + 1 = x_2 > x_3$.

Let H_n denote the subset of all the hooked permutations from And^I_n. The elements of the set-theoretic difference $NH_n := \mathrm{And}^I_n \setminus H_n$ are said to be *unhooked*. Let $H_n(m, k)$ (respectively $NH_n(m, k)$) denote the subset of H_n (respectively of NH_n) consisting of all w such that $(\mathbf{spi}, \mathbf{grn})w = (m, k)$.

Proposition 6.4. *The injection Θ defined in* (6.8) *from And^I_{n-1} into And^I_n maps And^I_{n-1} onto H_n. Moreover, for each w from And^I_n we have*

$$\begin{aligned} &\mathbf{spi}\, \Theta(w) = 1 + \mathbf{F}\, w; \\ &\mathbf{grn}\, \Theta(w) = \begin{cases} 1 + \mathbf{grn}\, w, & \text{if } \mathbf{F}\, w < \mathbf{grn}\, w; \\ \mathbf{grn}\, w, & \text{if } \mathbf{F}\, w > \mathbf{grn}\, w. \end{cases} \end{aligned} \tag{6.9}$$

Proof. With the notation of (6.8), we have $\mathbf{spi}\, \Theta(w) = x_1 + 1$ in both cases. The identity for "**grn**" follows from the very definition of Θ. ∎

Corollary 6.5. *The mapping Θ is a bijection between $A_{n-1}(m,k)$ and $H_n(m+1,k)$ when $1 \le k < m \le n-2$, and onto $H_n(m+1,k+1)$ when $3 \le m+2 \le k \le n-1$.*

In Table 6.3, the sixteen permutations from And_5^I are reproduced in boldface, and below them the hooked permutations from H_6 that are their images under Θ. A hat sign $\widehat{\ }$ has been put on the spike of each permutation from H_6.

$k=$	1	2	3	4	5
$m=1$	·	·	·	·	·
2			**13425** $\widehat{2}14536$	**12435** $\widehat{2}13546$ **14235** $\widehat{2}15346$	**12345** $\widehat{2}13456$ **13245** $\widehat{2}14356$
3	**23415** $\widehat{3}24516$			**21435** $2\widehat{3}1546$ **24135** $\widehat{3}25146$	**21345** $2\widehat{3}1456$ **23145** $\widehat{3}24156$
4	**32415** $3\widehat{4}2516$	**31425** $3\widehat{4}1526$ **34125** $\widehat{4}35126$			**31245** $3\widehat{4}1256$
5		**41325** $4\widehat{5}1326$	**41235** $4\widehat{5}1236$		

Table 6.3:
the bijection $\Theta : A_5(m,k) \to H_6(m+1,k)$ (respectively $H_6(m+1,k+1)$)

Referring to the program displayed in (1.14), the first bijection Θ has been constructed. The next step is devoted to the construction of the bijection β.

6.4. *A bijection between $B_n(m,k)$ and $NH_n(m+1,k)$.* We go back to the proofs of (TS4.1) and (TS4.2) made in Subsection 6.1. It was shown that $b_n(n-1,k) = b_n(n,k+1) = b_{n-1}(\bullet,k)$ for $1 \le k \le n-2$. By means of the bijections described in Subsection 6.1 and Section 4, and also the bijection ϕ constructed in (5.2), we can set up a bijection between $B_n(n-1,k)$ and $B_n(n,k+1)$. We can also proceed directly as follows. Let $n \ge 3$ and $w = x_1 \cdots x_{i-1}\,\widehat{(n-1)}\,x_{i+1} \cdots k\,n$ be an André I permutation such that $(\mathbf{spi},\mathbf{grn})w = (n-1,k)$. Then the mapping α defined by

$$\alpha(w) := 1\,(x_1+1)\cdots(x_{i-1}+1)\,(x_{i+1}+1)\cdots(k+1)\,\widehat{n} \tag{6.10}$$

fulfills our requirements.

The inverse α^{-1} is easy to find: let $w' = x'_1\ x'_2 \cdots x'_{n-1}\, n$ be a permutation from $B_n(n, k+1)$, so that $x'_1 = 1$. Then $\alpha^{-1}(w')$ is obtained by first determining the leftmost letter x'_{i+1} less than or equal to x'_2 and subsequently letting

$$\alpha^{-1}(w') := (x'_2 - 1) \cdots (x'_{i-1} - 1)\,(n-1)\,(x'_{i+1} - 1) \cdots k\, n. \tag{6.11}$$

The bijection α will be an ingredient for the next bijection, β, between $B_n(m,k)$ and $NH_n(m+1,k)$.

First, let

$$2 \le k+1 \le m \le n-2 \quad \text{or} \quad 3 \le m+2 \le k \le n-1, \tag{6.12}$$

and partition $B_n(m,k)$ into two subsets $B_n^{(1)}(m,k)$, $B_n^{(2)}(m,k)$ as follows. Note that each permutation w from $B_n(m,k)$ is of the form $w = w_1 m w_2 (m+1) w_3$ and the factor w_2 is never empty, as m is the spike of w. Also, $w_3 \neq e$ because of condition (6.12). Say that an element of $B_n(m,k)$ belongs to $B_n^{(1)}(m,k)$ (respectively to $B_n^{(2)}(m,k)$) if $\mathbf{F}\, w_3$ *is not* (respectively if $\mathbf{F}\, w_3$ *is*) a left minimum record, or equivalently, if $\min w_2 < \mathbf{F}\, w_3$ (respectively if $\min w_2 > \mathbf{F}\, w_3$).

Let $w = x_1 x_2 \cdots x_n = w_1\, m\, w_2\, (m{+}1)\, w_3$ be from $B_n(m,k)$ with (m,k) satisfying (6.12).

(1) If w belongs to $B_n^{(1)}(m,k)$, define $w' := \beta(w)$ to be the permutation derived from w *by transposing* the letters m and $(m+1)$:

$$\beta : w = w_1\, m\, w_2\, (m+1)\, w_3 \ \mapsto\ w' = w_1\, (m+1)\, w_2\, m\, w_3. \tag{6.13}$$

(2) If w belongs to $B_n^{(2)}(m,k)$, consider the factorization $w = v_1 w_3$, where $v_1 = w_1\, m\, w_2\, (m+1)$. Then v_1 is André I by Proposition 2.1 (6). Let n' be the length of v_1 and $\rho(v_1)$ be the *reduction* of v_1 (by using the increasing bijection from the set $\{x_1, \ldots, m, \ldots, m{+}1\}$ onto $\{1, 2, \ldots, n'\}$). Thus, $\rho(v_1)$ is an André I permutation from $\mathrm{And}^I_{n'}$ such that $\mathbf{spi}\, \rho(v_1) = n' - 1$. The bijection α, introduced in (6.10), can be applied to $\rho(v_1)$, and the permutation $w' := \beta(w)$ is defined by replacing the left factor v_1 of w by $\rho^{-1}\alpha\rho(v_1)$:

$$\beta : w = v_1 w_3 \mapsto w' := \rho^{-1}\alpha\rho(v_1)\, w_3. \tag{6.14}$$

Example. The permutation $w = 4\,\mathbf{5}\,3\,8\,1\,\mathbf{6}\,7\,2\,9$ belongs to $B_9^{(1)}(5,2)$, as $\min w_2 = \min 381 = 1 < 7 = \mathbf{F}\, w_3$. It then suffices to transpose 5 and 6 to get the permutation $w' = 4\,\mathbf{6}\ 3\,8\,1\,\mathbf{5}\,7\,2\,9$.

Next, $w = 3\,5\,\mathbf{6}\,2\,\mathbf{7}\,1\,8\,4\,9$ belongs to $B_9^{(2)}(6,4)$, as $\min w_2 = 2 > 1 = \mathbf{F}\,w_3$. Hence, $v_1 = 3\,5\,6\,2\,7$, $\rho(v_1) = 2\,3\,4\,1\,5$, $\alpha\rho(v_1) = 1\,3\,4\,2\,5$, $\rho^{-1}\alpha\rho(v_1) = 2\,5\,6\,3\,7$ and $w' = 2\,5\,\mathbf{6}\,3\,\mathbf{7}\,1\,8\,4\,9$.

When w belongs to $B_n^{(1)}(m,k)$, the letter $(m+1)$ occurs to the left of m in w'. On the other hand, as w_2 is non-empty and $m+1 > \mathbf{F}\,w_2$, the permutation w' is unhooked if $w_1 = e$. The same conclusion also holds if $w_1 \neq e$, because $\mathbf{L}\,w_1 < m+1$ and $\mathbf{L}\,w_1 \neq m$. Obviously, $\mathbf{spi}\,w' = m+1$ and $\mathbf{grn}\,w' = k$ by (6.12).

Let us now prove that w' is André I. Note that the troughs remain the same in both w and w'. Let x be a trough within w_2 and (v_1, v_2, x, v_4, v_5) (respectively $(v_1', v_2', x, v_4', v_5')$) be the x-factorization of w (respectively of w'). When going from w to w', the type of x is not modified when at least one of the following conditions holds: $\max v_2 \neq m$, $\max v_4 \neq m+1$. If both were violated for a given x, it would be the case for $x = \min w_2$, and all the letters of w_2 would be less than m. But $\max v_4 = m+1$ implies $\max v_4 > \mathbf{F}\,w_3 > \min w_2$, and $\mathbf{F}\,w_3$ is a trough of w. If $(v_1'', v_2'', \mathbf{F}\,w_3, v_4'', v_5'')$ is the $\mathbf{F}\,w_3$-factorization of w, the word $\mathbf{F}\,w_3\,v_4''$ is necessarily a factor of v_4, as all its letters are greater than $\min w_2$. Hence, $\max v_4 > m+1$, a contradiction. Thus,

w' *is an unhooked permutation from* And_n^I *such that* $\mathbf{spi}\,w' = m+1$, $\mathbf{grn}\,w' = k$ *with* $(m+1)$ *to the left of* m. In short, $w' \in NH_n^{(1)}(m+1,k)$.

As the transposition $w_1\,(m+1)\,w_2\,m\,w_3 \;\mapsto\; w_1\,m\,w_2\,(m+1)\,w_3$, when applied to André I permutations with $(m+1)$ to the left of m, always maps an André I onto an André I permutations,

the direct transposition β *defined in* (6.13) *is a bijection between* $B_n^{(1)}(m,k)$ *and* $NH_n^{(1)}(m+1,k)$.

Next, let w belong to $B_n^{(2)}(m,k)$, and consider the permutation $w' = \beta(w)$ defined in (6.14). The left factor $\rho^{-1}\alpha\rho(v_1)$ of $\beta(w)$ is André I and ends with $(m+1)$. Therefore, $\beta(w)$ is of the form $w_1'\,m\,w_2'\,(m+1)\,w_3$. Again, with the hypothesis (6.12), the letter x_{n-1}, equal to k in the permutation $w = x_1x_2\cdots k\,n$, remains untouched when going from w to w'. Thus, $\mathbf{grn}\,w' = \mathbf{grn}\,w = k$. Next, we get $\mathbf{spi}\,\rho(v_1) = n'-1$ and $\mathbf{spi}\,\alpha\rho(v_1) = n'$; hence, $\mathbf{spi}\,\rho^{-1}\alpha\rho(v_1) = m+1$. As w_3 starts with a letter less than all the letters in v_1, we have $\mathbf{spi}\,w' = \mathbf{spi}\,\rho^{-1}\alpha\rho(v_1)\,w_3 = m+1$. Moreover, w_3 is André I by Proposition 2.1 (5), so that $\beta(w)$ is André I by Proposition 2.1 (7). This shows that

the mapping β *defined in* (6.14) *is a bijection between* $B_n^{(2)}(m,k)$ *and the set* $NH_n^{(2)}(m+1,k)$, *defined as the set of all unhooked permutations from* And_n^I *such that* $\mathbf{spi}\,w' = m+1$, $\mathbf{grn}\,w' = k$ *with* m *to the left of* $(m+1)$.

This proves the following theorem.

Theorem 6.6. *Under condition* (6.12), *the mapping* $\beta : w \mapsto w'$ *defined in* (6.13) *and* (6.14) *is a bijection between* $B_n(m,k) = B_n^{(1)}(m,k) \uplus B_n^{(2)}(m,k)$ *and* $NH_n(m+1,k) = NH_n^{(1)}(m+1,k) \uplus NH_n^{(2)}(m+1,k)$.

$m \backslash k$	1	2	3	4	5
1					
2			**214536**	**213546** **215346**	**213456** **214356**
3	**324516**	314526		315246 312546 **231546** **325146**	314256 **231456** **324156** 312456
4	423516 **342516**	413526 **341526** **435126**	241536 412536 415236 425136		412356 234156 241356 413256 **341256**
5	243516 234516	345126 351426 513426 **451326**	245136 514236 512436 251436 **451236**		

Table 6.4: the bijection $\beta : B_6(m,k) \to NH_6(m+1,k)$

Example. In Table 6.4, the image $\beta(w)$ of each André I permutation w from $B_6(m,k)$, with (m,k) satisfying inequalities (6.12) for $n=6$, is indicated by a downarrow. The hooked permutations are reproduced in boldface. Note that they are not bottoms of any downarrows, as β is a bijection between $B_n(m,k)$ and $NH_n(m+1,k)$.

With the construction of the bijection $\beta : B_n(m,k) \to NH_n(m+1,k)$, the program displayed in (1.14) is completed, as

$$\begin{aligned}
\underset{(1)}{\Delta}\, b_n(m,k) &= \#B_n(m+1,k) - \#B_n(m,k)\\
&= \#B_n(m+1,k) - \#NH_n(m+1,k)\\
&= \#H_n(m+1,k)\\
&= \#A_{n-1}(m,k) \quad (\text{respectively } = \#A_{n-1}(m,k-1))
\end{aligned}$$

if $1 \le k < m \le n-2$ (respectively if $3 \le m+2 \le k \le n-1$).

7. The making of Seidel triangle sequences

7.1. *The Seidel tangent-secant matrix.* In the sequel, three exponential generating functions will be attached to each infinite matrix $A = (a(m,k))_{m,k\ge 0}$,

$$A(x,y) := \sum_{m,k\ge 0} a(m,k)\frac{x^m}{m!}\frac{y^k}{k!};$$
$$A_{m,\bullet}(y) := \sum_{k\ge 0} a(m,k)\frac{y^k}{k!}; \qquad A_{\bullet,k}(x) := \sum_{m\ge 0} a(m,k)\frac{x^m}{m!};$$

for A itself, its m-th row, its k-th column. Let $\overline{H} = (\overline{h}_{i,j})$ $(i,j\ge 0)$ be the infinite matrix whose entries are the Entringer numbers $E_n(m)$ displayed along the skew-diagonals with the following sign:

$$\overline{h}_{i,j} = \begin{cases} (-1)^n\, E_{i+j+1}(j+1), & \text{if } i+j=2n; \\ (-1)^n\, E_{i+j+1}(i+1), & \text{if } i+j=2n-1; \end{cases} \tag{7.1}$$

or still

$$E_{2n+1}(j+1) = (-1)^n\,\overline{h}_{2n-j,j} \quad (0\le j\le 2n); \tag{7.2}$$
$$E_{2n}(i+1) = (-1)^n\,\overline{h}_{i,2n-1-i} \quad (0\le i\le 2n-1); \tag{7.3}$$

or still in displayed form:

$$\overline{H} = \begin{pmatrix} E_1(1) & -E_2(1) & 0 & E_4(1) & 0 & -E_6(1) & 0\cdots \\ 0 & -E_3(2) & E_4(2) & E_5(4) & -E_6(2) & -E_7(6) & \\ -E_3(1) & E_4(3) & E_5(3) & -E_6(3) & -E_7(5) & & \\ 0 & E_5(2) & -E_6(4) & -E_7(4) & & & \\ E_5(1) & -E_6(5) & -E_7(3) & & & & \\ 0 & -E_7(2) & & & & & \\ -E_7(1) & & & & & & \\ \vdots & & & & & & \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -1 & 0 & 2 & 0 & -16 & 0 & \cdots \\ 0 & -1 & 2 & 2 & -16 & -16 & & \\ -1 & 1 & 4 & -14 & -32 & & & \\ 0 & 5 & -10 & -46 & & & & \\ 5 & -5 & -56 & & & & & \\ 0 & -61 & & & & & & \\ -61 & & & & & & & \\ \vdots & & & & & & & \end{pmatrix}.$$

As noted by Dumont [Du82], the definition of such a matrix $\overline{H}$ goes back to Seidel himself [Se1877]. Entringer [En66] rediscovered the absolute values of the entries, when he classified the alternating permutations according to their first letters. The entries of the top row are the coefficients of the Taylor expansion of $1-\tanh y = 2/(1+e^{2y})$:

$$\begin{aligned}\overline{H}_{0,\bullet}(y) = 1-\tanh y &= 1+\sum_{n\ge 1}\frac{y^{2n-1}}{(2n-1)!}(-1)^n E_{2n-1}\\ &= 1-\frac{y}{1!}1+\frac{y^3}{3!}2-\frac{y^5}{5!}16+\frac{y^7}{7!}272-\frac{y^9}{9!}7936+\cdots.\end{aligned}$$

The entries of the leftmost column are the coefficients of the Taylor expansion of $1/\cosh x = 2\,e^x/(1+e^{2x})$, so that

$$\begin{aligned}\overline{H}_{\bullet,0}(x) = \frac{1}{\cosh x} &= \sum_{n\ge 0}\frac{x^{2n}}{(2n)!}(-1)^n\,E_{2n}\\ &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}5-\frac{x^6}{6!}61+\frac{x^8}{8!}1385-\cdots.\end{aligned}$$

By means of recurrence (1.1) satisfied by the Entringer numbers and (7.1), we can verify that the entries $\overline{h}_{i,j}$ obey the following rule: $\overline{h}_{i,j} = \overline{h}_{i-1,j}+\overline{h}_{i-1,j+1}$ for $j\ge 0$, $i\ge 1$, so that the entries $\overline{h}_{i,j}$ can be obtained by applying such a rule inductively, the entries of the top row being given. Such a matrix is called a *Seidel matrix* by Dumont [Du82], and its exponential generating function is directly obtained from the exponential generating function for its top row by the formula $\overline{H}(x,y) = \overline{H}_{0,\bullet}(x+y)\,e^x$ (see, e.g., [DV80]). Accordingly,

$$\overline{H}(x,y) = \frac{2\,e^x}{1+e^{2x+2y}}. \tag{7.4}$$

Two further matrices are derived from $\overline{H}$. The first one, $\overline{H}_1$, is obtained by replacing all the entries $\overline{h}_{i,j}$ such that $i+j$ is *odd* by zero, so that

$$\overline{H}_1 = \begin{pmatrix} 1 & \cdot & 0 & \cdot & 0 & \cdot & 0 & \cdots \\ \cdot & -1 & \cdot & 2 & \cdot & -16 & & \\ -1 & \cdot & 4 & \cdot & -32 & & & \\ \cdot & 5 & \cdot & -46 & & & & \\ 5 & \cdot & -56 & & & & & \\ \cdot & -61 & & & & & & \\ -61 & & & & & & & \\ \vdots & & & & & & & \end{pmatrix}.$$

As $\overline{H}(x,y)=\dfrac{2e^x}{1+e^{2x+2y}}$, we get

$$\overline{H}_1(x,y)=\frac{\overline{H}(x,y)+\overline{H}(-x,-y)}{2}=e^x\frac{1+e^{2y}}{1+e^{2x+2y}}=\frac{\cosh y}{\cosh(x+y)}. \tag{7.5}$$

The second one, $\overline{H}_2$, is derived from $\overline{H}$ by replacing the entries $\overline{h}_{i,j}$ such that $i+j$ is *even* by 0, so that

$$\overline{H}_2=\begin{pmatrix}
\cdot & -1 & \cdot & 2 & \cdot & -16 & \cdot & 272 & \cdots\\
0 & \cdot & 2 & \cdot & -16 & \cdot & 272 & & \\
\cdot & 1 & \cdot & -14 & \cdot & 256 & & & \\
0 & \cdot & -10 & \cdot & 224 & & & & \\
\cdot & -5 & \cdot & 178 & & & & & \\
0 & \cdot & 122 & & & & & & \\
\cdot & 61 & & & & & & & \\
0 & & & & & & & & \\
\vdots & & & & & & & &
\end{pmatrix}.$$

Therefore,

$$\overline{H}_2(x,y)=\frac{\overline{H}(x,y)-\overline{H}(-x,-y)}{2}=e^x\frac{1-e^{2y}}{1+e^{2x+2y}}=\frac{-\sinh y}{\cosh(x+y)}. \tag{7.6}$$

In the sequel, further matrices will be derived from $\overline{H}_1$ and $\overline{H}_2$, essentially by transposing them and/or removing either their top rows or leftmost columns. The corresponding actions on their respective exponential generating functions $\overline{H}_1(x,y)$ and $\overline{H}_2(x,y)$ are the exchange of the variables x and y: $TH_i(x,y):=H_i(y,x)$; then, the *partial derivatives* with respect to x and y: $D_xH_i(x,y)$ and $D_yH_i(x,y)$ $(i=1,2)$.

7.2. *The generating function for the Entringer numbers.* The generating function for the Entringer numbers, already derived in [FH14], can be obtained from relations (7.5) and (7.6). In fact, they are simply equal to $\overline{H}_1(xI,yI)$ and $I\overline{H}_2(xI,yI)$, where $I=\sqrt{-1}$. Thus,

$$\sum_{1\le k\le 2n+1} E_{2n+1}(k)\frac{x^{2n+1-k}}{(2n+1-k)!}\frac{y^{k-1}}{(k-1)!}=\frac{\cos y}{\cos(x+y)}; \tag{7.7}$$

$$\sum_{1\le k\le 2n} E_{2n}(k)\frac{x^{k-1}}{(k-1)!}\frac{y^{2n-k}}{(2n-k)!}=\frac{\sin y}{\cos(x+y)}. \tag{7.8}$$

7.3. *Seidel triangle sequences.* For calculating the generating functions for the twin Seidel matrices, we shall take recourse to the techniques developed in our previous paper [FH14] for the so-called Seidel triangle sequences. Only definitions will be stated, as well as the main result.

A sequence of square matrices (C_n) $(n \ge 1)$ is called a *Seidel triangle sequence* if the following three conditions are fulfilled:

(STS1) each matrix C_n is of dimension n;

(STS2) each matrix C_n has zero entries along and below its diagonal; let $(c_n(m,k))$ $(0 \le m < k \le n-1)$ denote its entries strictly above its diagonal, so that

$$C_1 = (\,\cdot\,)\,; \quad C_2 = \begin{pmatrix} \cdot & c_2(0,1) \\ \cdot & \cdot \end{pmatrix}; \quad C_3 = \begin{pmatrix} \cdot & c_3(0,1) & c_3(0,2) \\ \cdot & \cdot & c_3(1,2) \\ \cdot & \cdot & \cdot \end{pmatrix}; \ \ldots\ ;$$

$$C_n = \begin{pmatrix} \cdot & c_n(0,1) & c_n(0,2) & \cdots & \cdot & c_n(0,n-2) & c_n(0,n-1) \\ \cdot & \cdot & c_n(1,2) & \cdots & \cdot & c_n(1,n-2) & c_n(1,n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & c_n(n-2,n-1) \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \end{pmatrix};$$

the dots "$\cdot$" along and below the diagonal referring to zero entries.

(STS3) for each $n \ge 3$, the following relation holds:

$$c_n(m,k) - c_n(m,k+1) = c_{n-1}(m,k) \quad (m<k).$$

Record the last columns of the triangles C_2, C_3, C_4, C_5, …, read from top to bottom, namely, $c_2(0,1)$; $c_3(0,2), c_3(1,2)$; $c_4(0,3), c_4(1,3), c_4(2,3)$; $c_5(0,4), c_5(1,4), c_5(2,4), c_5(3,4)$; …, as skew-diagonals of an infinite matrix $H = (h_{i,j})_{i,j\ge 0}$, as shown next:

$$\text{(7.9)}\qquad H := \begin{array}{c} \\ 0\\1\\2\\3\\4\\ \vdots \end{array} \!\!\begin{array}{c} \begin{array}{cccccc} \ \ 0 \quad & \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \cdots \end{array} \\ \begin{pmatrix} c_2(0,1) & c_3(1,2) & c_4(2,3) & c_5(3,4) & c_6(4,5) & \cdots \\ c_3(0,2) & c_4(1,3) & c_5(2,4) & c_6(3,5) & & \\ c_4(0,3) & c_5(1,4) & c_6(2,5) & & & \\ c_5(0,4) & c_6(1,5) & & & & \\ c_6(0,5) & & & & & \\ \vdots & & & & & \end{pmatrix} \end{array},$$

Equivalently, the entries of H are defined by

$$\text{(7.10)}\qquad h_{i,j} = c_{i+j+2}(j, i+j+1).$$

The next theorem has been proved in [FH14] and will be of great use in the next sections.

Theorem 7.1. *The three-variable generating function for the Seidel triangle sequence* $(C_n = (c_n(m,k)))_{n\ge 1}$ *is equal to*

$$\sum_{1\le m+1\le k\le n-1} c_n(m,k)\frac{x^{n-k-1}}{(n-k-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^m}{m!} = e^x H(x+y,z), \tag{7.11}$$

where H is the infinite matrix defined in (7.10).

With $I := \sqrt{-1}$, we get

$$\begin{aligned}\sum_{1\le m+1\le k\le n-1} I^{n-2} c_n(m,k)\frac{x^{n-k-1}}{(n-k-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^m}{m!}\\ = e^{Ix} H(Ix+Iy, Iz).\end{aligned} \tag{7.12}$$

8. Trivariate generating functions

Each of the sequences $\mathrm{Twin}^{(1)} := (A_2, B_3, A_4, B_5, A_6, \dots)$ and $\mathrm{Twin}^{(2)} := (B_2, A_3, B_4, A_5, B_6, \dots)$ (see Diagram 1.3) gives rise to two Seidel triangle sequences, by considering the upper and lower triangles of the matrices.

8.1. *The upper triangles of* $\mathrm{Twin}^{(1)}$. The Seidel triangle sequence to be constructed is the following: first, $C_1 := (\cdot)$, then for $n\ge 2$ each C_n will be derived from the upper triangle of A_{n+1} (respectively B_{n+1}) by (i) dropping the rightmost column; (ii) transposing the remaining triangle with respect to its skew-diagonal; (iii) changing the signs of its entries according the following rule. More precisely,

$$C_n := (-1)^{(n+1)/2}\begin{pmatrix} \cdot & a_{n+1}(n-1,n) & \cdots & a_{n+1}(2,n) & a_{n+1}(1,n)\\ & & \ddots & \vdots & \vdots\\ & & \cdot & a_{n+1}(2,3) & a_{n+1}(1,3)\\ & & & \cdot & a_{n+1}(1,2)\\ & & & & \cdot \end{pmatrix} \text{ if } n \text{ is odd;}$$

$$C_n := (-1)^{n/2}\begin{pmatrix} \cdot & b_{n+1}(n-1,n) & \cdots & b_{n+1}(2,n) & b_{n+1}(1,n)\\ & & \ddots & \vdots & \vdots\\ & & \cdot & b_{n+1}(2,3) & b_{n+1}(1,3)\\ & & & \cdot & b_{n+1}(1,2)\\ & & & & \cdot \end{pmatrix} \text{ if } n \text{ is even;}$$

By referring to Diagram 1.3, we get $C_1 = \cdot\,;\quad C_2 = \begin{matrix}\cdot & 0\\ & \cdot\end{matrix}\,;\quad C_3 = \begin{matrix}\cdot & 1 & 1\\ & \cdot & 1\\ & & \cdot\end{matrix}\,;$

$$C_4 = \begin{array}{cccc} \cdot & 2 & 1 & 0 \\ & \cdot & 1 & 0 \\ & & \cdot & 0 \\ & & & \cdot \end{array}\ ; \quad C_5 = \begin{array}{ccccc} \cdot & -2 & -4 & -5 & -5 \\ & \cdot & -4 & -5 & -5 \\ & & \cdot & -4 & -4 \\ & & & \cdot & -2 \\ & & & & \cdot \end{array}\ ; \quad C_6 = \begin{array}{cccccc} \cdot & -16 & -14 & -10 & -5 & 0 \\ & \cdot & -14 & -10 & -5 & 0 \\ & & \cdot & -8 & -4 & 0 \\ & & & \cdot & -2 & 0 \\ & & & & \cdot & 0 \\ & & & & & \cdot \end{array};$$

$$C_7 \ = \begin{array}{ccccccc} \cdot & 16 & 32 & 46 & 56 & 61 & 61 \\ & \cdot & 32 & 46 & 56 & 61 & 61 \\ & & \cdot & 44 & 52 & 56 & 56 \\ & & & \cdot & 44 & 46 & 46 \\ & & & & \cdot & 32 & 32 \\ & & & & & \cdot & 16 \\ & & & & & & \cdot \end{array}\ ; \quad C_8 \ = \begin{array}{cccccccc} \cdot & 272 & 256 & 224 & 178 & 122 & 61 & 0 \\ & \cdot & 256 & 224 & 178 & 122 & 61 & 0 \\ & & \cdot & 208 & 164 & 112 & 56 & 0 \\ & & & \cdot & 136 & 92 & 46 & 0 \\ & & & & \cdot & 64 & 32 & 0 \\ & & & & & \cdot & 16 & 0 \\ & & & & & & \cdot & 0 \\ & & & & & & & \cdot \end{array}.$$

Therefore,

$$c_n(m,k) = \begin{cases} (-1)^{(n+1)/2} a_{n+1}(n-k, n-m), & \text{if } n \text{ is odd;} \\ (-1)^{n/2} b_{n+1}(n-k, n-m), & \text{if } n \text{ is even.} \end{cases} \tag{8.1}$$

Proposition 8.1. *The sequence (C_n) $(n \ge 1)$ just defined is a Seidel triangle sequence.*

Proof. We only have to verify that rule (STS3) holds. If n is odd and $0 \le m < k \le n-2$, then $3 \le m' + 2 := (n-k-1) + 2 \le k' := n - m \le (n+1) - 1$ and

$$\begin{aligned} &c_n(m,k) - c_n(m,k+1) \\ &\qquad = (-1)^{(n+1)/2} \big(a_{n+1}(n-k, n-m) - a_{n+1}(n-k-1, n-m) \big) \\ &\qquad = (-1)^{(n+1)/2} \underset{(1)}{\Delta}\, a_{n+1}(n-k-1, n-m) \\ &\qquad = (-1)^{(n+1)/2} \underset{(1)}{\Delta}\, a_{n+1}(m', k') \\ &\qquad = (-1)^{(n-1)/2}\, b_n(m', k'-1) && \text{[by rule (TS5.2)]} \\ &\qquad = (-1)^{(n-1)/2}\, b_n(n-k-1, n-m-1) \\ &\qquad = (-1)^{(n-1)/2} (-1)^{(n-1)/2}\, c_{n-1}(m,k) = c_{n-1}(m,k). \end{aligned}$$

The case when n is even can be proved in a similar way. ∎

The next step is to determine the matrix H, as defined in (7.9), whose skew-diagonals are equal to the rightmost columns of the matrices C_n. For $n \ge 2$ the skew-diagonal $(c_n(0, n-1), c_n(1, n-1), \ldots, c_n(n-2, n-1))$ of H, being the rightmost column of C_n, is equal to

$$\begin{cases} (-1)^{(n+1)/2} (a_{n+1}(1,n), a_{n+1}(1,n-1), \ldots, a_{n+1}(1,2)), & \text{if } n \text{ is odd,} \\ (-1)^{n/2} (b_{n+1}(1,n), b_{n+1}(1,n-1), \ldots, b_{n+1}(1,2)), & \text{if } n \text{ is even;} \end{cases}$$

which is also equal to

$$\begin{cases} (-1)^{(n+1)/2}(b_n(\bullet, n-1), b_n(\bullet, n-2), \ldots, b_n(\bullet, 1)), & \text{if } n \text{ is odd;} \\ (0, 0, \ldots, 0\,), & \text{if } n \text{ is even;} \end{cases}$$

by Rules (TS5.1) and (TS2); finally, this is equal to

$$\begin{cases} (-1)^{(n+1)/2}(E_n(1), E_n(2), \ldots, E_n(n-1)), & \text{if } n \text{ is odd;} \\ (0, 0, \ldots, 0\,), & \text{if } n \text{ is even;} \end{cases}$$

by (1.11).

Thus,

$$H = \begin{pmatrix} 0 & E_3(2) & 0 & -E_5(4) & 0 & E_7(6) & 0 & \cdots \\ E_3(1) & 0 & -E_5(3) & 0 & E_7(5) & 0 & & \\ 0 & -E_5(2) & 0 & E_7(4) & 0 & & & \\ -E_5(1) & 0 & E_7(3) & 0 & & & & \\ 0 & E_7(2) & 0 & & & & & \\ E_7(1) & 0 & & & & & & \\ 0 & & & & & & & \\ \vdots & & & & & & & \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & -2 & 0 & 16 & 0 & \cdots \\ 1 & 0 & -4 & 0 & 32 & 0 & & \\ 0 & -5 & 0 & 46 & 0 & & & \\ -5 & 0 & 56 & 0 & & & & \\ 0 & 61 & 0 & & & & & \\ 61 & 0 & & & & & & \\ 0 & & & & & & & \\ \vdots & & & & & & & \end{pmatrix}. \tag{8.2}$$

This matrix is to be compared with the matrix $\overline{H}_1$ (see §7.1). For getting H, it suffices to delete the top row of $\overline{H}_1$ and change the signs of all the entries. As $\overline{H}_1(x,y) = \cosh y/\cosh(x+y)$ by (7.5), we have

$$H(x,y) = -D_x\overline{H}_1(x,y) = \frac{\cosh y\,\sinh(x+y)}{\cosh^2(x+y)}. \tag{8.3}$$

Hence, the right-hand side of (7.11) becomes

$$e^x H(x+y,z) = e^x \frac{\cosh z\,\sinh(x+y+z)}{\cosh^2(x+y+z)};$$

and the right-hand side of (7.12) is equal to

$$e^{Ix} H(Ix+Iy,Iz) = (\cos x + I\sin x)\frac{I\,\cos z\,\sin(x+y+z)}{\cos^2(x+y+z)}.$$

It remains to interpret the left-hand side of identity (7.12) by using (8.1). If $n = 2l+1$, then $I^{n-2} = (-1)^{l+1}I$ and $(-1)^{(n+1)/2} = (-1)^{l+1}$.

Thus, $I^{n-2}c_n(m,k)=I\,a_{n+1}(n-k,n-m)$. The imaginary part of identity (7.12) then reads

$$\sum_{\substack{1\le m+1\le k\le n-1\\ n\ \text{odd}}} a_{n+1}(n-k,n-m)\frac{x^{n-k-1}}{(n-k-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^m}{m!} = \frac{\cos x\,\cos z\sin(x+y+z)}{\cos^2(x+y+z)}.$$

With the change of variables $n\leftarrow 2n-1$, $n-k\leftarrow m$, $n-m\leftarrow k$, we get (1.15) from Theorem 1.3. Note that the above generating function involves all the matrices A_4, A_6, … of $\mathrm{Twin}^{(1)}$, but not the very first term $A_2=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$.

If $n=2l$, then $I^{n-2}=(-1)^{l-1}$ and $(-1)^{n/2}=(-1)^l$, so that $I^{n-2}c_n(m,k)=-b_{n+1}(n-k,n-m)$. As for the real part, we have

$$\sum_{\substack{1\le m+1\le k\le n-1\\ n\ \text{even}}} b_{n+1}(n-k,n-m)\frac{x^{n-k-1}}{(n-k-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^m}{m!} = \frac{\sin x\,\cos z\sin(x+y+z)}{\cos^2(x+y+z)}.$$

With the change of variables $n\leftarrow 2n$, $n-k\leftarrow m$, $n-m\leftarrow k$, we get (1.21) from Theorem 1.6.

8.2. *The upper triangles of* $\mathrm{Twin}^{(2)}$. The sequence of triangles to be considered is the following: $C_1=\cdot$ and for $n\ge 2$

$$C_n := (-1)^{(n-1)/2}\begin{pmatrix} \cdot & b_{n+1}(n-1,n) & \cdots & b_{n+1}(2,n) & b_{n+1}(1,n)\\ & & \ddots & \vdots & \vdots\\ & & & \cdot\ \ b_{n+1}(2,3) & b_{n+1}(1,3)\\ & & & \cdot & b_{n+1}(1,2)\\ & & & & \cdot \end{pmatrix}\ \text{if } n \text{ is odd;}$$

$$C_n := (-1)^{n/2}\begin{pmatrix} \cdot & a_{n+1}(n-1,n) & \cdots & a_{n+1}(2,n) & a_{n+1}(1,n)\\ & & \ddots & \vdots & \vdots\\ & & & \cdot\ \ a_{n+1}(2,3) & a_{n+1}(1,3)\\ & & & \cdot & a_{n+1}(1,2)\\ & & & & \cdot \end{pmatrix}\ \text{if } n \text{ is even;}$$

that is, $C_1=\cdot$, $\quad C_2=\begin{matrix}\cdot & -1\\ & \cdot\end{matrix}\,;\quad C_3=\begin{matrix}\cdot & -1 & 0\\ & \cdot & 0\\ & & \cdot\end{matrix}\,;\quad C_4=\begin{matrix}\cdot & 1 & 2 & 2\\ & \cdot & 2 & 2\\ & & \cdot & 1\\ & & & \cdot\end{matrix}\,;$

$$C_5 = \begin{array}{ccccc} \cdot & 5 & 4 & 2 & 0 \\ & \cdot & 4 & 2 & 0 \\ & & \cdot & 1 & 0 \\ & & & \cdot & 0 \\ & & & & \cdot \end{array}; \quad C_6 = \begin{array}{cccccc} \cdot & -5 & -10 & -14 & -16 & -16 \\ & \cdot & -10 & -14 & -16 & -16 \\ & & \cdot & -13 & -14 & -14 \\ & & & \cdot & -10 & -10 \\ & & & & \cdot & -5 \\ & & & & & \cdot \end{array};$$

$$C_7 = \begin{array}{ccccccc} \cdot & -61 & -56 & -46 & -32 & -16 & 0 \\ & \cdot & -56 & -46 & -32 & -16 & 0 \\ & & \cdot & -41 & -28 & -14 & 0 \\ & & & \cdot & -20 & -10 & 0 \\ & & & & \cdot & -5 & 0 \\ & & & & & \cdot & 0 \\ & & & & & & \cdot \end{array}; \quad C_8 = \begin{array}{cccccccc} \cdot & 61 & 122 & 178 & 224 & 256 & 272 & 272 \\ & \cdot & 122 & 178 & 224 & 256 & 272 & 272 \\ & & \cdot & 173 & 214 & 242 & 256 & 256 \\ & & & \cdot & 194 & 214 & 224 & 224 \\ & & & & \cdot & 173 & 178 & 178 \\ & & & & & \cdot & 122 & 122 \\ & & & & & & \cdot & 61 \\ & & & & & & & \cdot \end{array}.$$

Thus,

$$c_n(m,k) = \begin{cases} (-1)^{(n-1)/2} b_{n+1}(n-k, n-m), & \text{if } n \text{ is odd;} \\ (-1)^{n/2} a_{n+1}(n-k, n-m), & \text{if } n \text{ is even.} \end{cases} \tag{8.6}$$

The sequence of triangles (C_n) defined by (8.6) is a Seidel triangle sequence (same argument as in the proof of Proposition 8.1). Following the same pattern as in the preceding subsection, we form the matrix H, whose skew-diagonals carry the entries of the leftmost columns of the C_n's:

$$H = \begin{pmatrix} -1 & 0 & 1 & 0 & -5 & 0 & 61 & \cdots \\ 0 & 2 & 0 & -10 & 0 & 122 & & \\ 2 & 0 & -14 & 0 & 178 & & & \\ 0 & -16 & 0 & 224 & & & & \\ -16 & 0 & 256 & & & & & \\ 0 & 272 & & & & & & \\ 272 & & & & & & & \\ \vdots & & & & & & & \end{pmatrix}.$$

This matrix is to be compared with the matrix $\overline{H}_2$ (see Subsection 7.2). We see that H is obtained from $\overline{H}_2$ by transposition and deletion of the first row, so that

$$\begin{aligned} H(x,y) &= D_x \overline{H}_2(y,x) = D_x \Big(\frac{-\sinh x}{\cosh(x+y)} \Big) \\ &= \frac{-\cosh x\, \cosh(x+y) + \sinh x\, \sinh(x+y)}{\cosh^2(x+y)} \\ &= \frac{-\cosh y}{\cosh^2(x+y)}. \end{aligned}$$

Therefore,

$$e^x H(x+y,z) = e^x \frac{-\cosh z}{\cosh^2(x+y+z)};$$
$$e^{Ix} H(Ix+Iy,Iz) = (\cos x + I\sin x)\frac{-\cos z}{\cos^2(x+y+z)}.$$

By using (8.6), the left-hand side of identity (7.12) can be computed as follows. If $n=2l+1$, then $I^{n-2}=(-1)^{l+1}I$ and $(-1)^{(n-1)/2}=(-1)^l$. Thus, $I^{n-2}c_n(m,k) = -I\,b_{n+1}(n-k,n-m)$. The imaginary part of identity (7.12) reads

$$\sum_{\substack{1\le m+1\le k\le n-1\\ n \text{ odd}}} b_{n+1}(n-k,n-m)\frac{x^{n-k-1}}{(n-k-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^m}{m!} = \frac{\sin x\cos z}{\cos^2(x+y+z)}.$$

With the change of variables $n\leftarrow 2n-1$, $n-k\leftarrow m$, $n-m\leftarrow k$, we get (1.19) from Theorem 1.5.

If $n=2l$, then $I^{n-2}=(-1)^{l-1}$ and $(-1)^{n/2}=(-1)^l$, so that $I^{n-2}c_n(m,k) = -a_{n+1}(n-k,n-m)$. As for the real part, we have

$$\sum_{\substack{1\le m+1\le k\le n-1\\ n \text{ even}}} a_{n+1}(n-k;n-m)\frac{x^{n-k-1}}{(n-k-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^m}{m!} = \frac{\cos x\cos z}{\cos^2(x+y+z)}.$$

With the change of variables $n\leftarrow 2n$, $n-k\leftarrow m$, $n-m\leftarrow k$, we get (1.17) from Theorem 1.4.

8.3. *The bottom rows of the matrices B_n's.* By Rule (TS4.1) and (2.6), these bottom rows, after discarding the rightmost entry which is always zero, read $b_2(2,1)=1$; $(b_3(3,1),b_3(3,2))=(0,1)$, $(b_4(4,1),b_4(4,2),b_4(4,3))=(0,1,1)$, $(b_5(5,1),b_5(5,2),b_5(5,3),b_5(5,4))=(0,1,2,2)$, …, which are equal to the sequences of the Entringer numbers: $E_1(1)$, $(E_2(2),E_2(1))$, $(E_3(3),E_3(2),E_3(1))$, $(E_4(4),E_4(3),E_4(2),E_4(1))$, … By (7.7) and (7.8), we recover the two identities (1.23) and (1.24) written at the end of Section 1.

8.4. *The lower triangles of* $\mathrm{Twin}^{(1)}$. As for the upper triangles, a geometric transformation is to be made to configurate these lower triangles into Seidel triangles. The bottom rows of the A_n's and B_n's being discarded, we form the following sequence of triangles:

$$C_1 = \cdot;\ C_2 = \begin{matrix} \cdot & 1 \\ & \cdot \end{matrix};\ C_3 = \begin{matrix} \cdot & 1 & 0 \\ & \cdot & 1 \\ & & \cdot \end{matrix};\quad C_4 = \begin{matrix} \cdot & 0 & -1 & -1 \\ & \cdot & -1 & -2 \\ & & \cdot & -2 \\ & & & \cdot \end{matrix};\quad C_5 = \begin{matrix} \cdot & -2 & -2 & -1 & 0 \\ & \cdot & -4 & -3 & -1 \\ & & \cdot & -4 & -2 \\ & & & \cdot & -2 \\ & & & & \cdot \end{matrix};$$

$$C_6 = \begin{matrix} \cdot & 0 & 2 & 4 & 5 & 5 \\ & \cdot & 2 & 6 & 9 & 10 \\ & & \cdot & 8 & 12 & 14 \\ & & & \cdot & 14 & 16 \\ & & & & \cdot & 16 \\ & & & & & \cdot \end{matrix};\quad C_7 = \begin{matrix} \cdot & 16 & 16 & 14 & 10 & 5 & 0 \\ & \cdot & 32 & 30 & 24 & 15 & 5 \\ & & \cdot & 44 & 36 & 24 & 10 \\ & & & \cdot & 44 & 30 & 14 \\ & & & & \cdot & 32 & 16 \\ & & & & & \cdot & 16 \\ & & & & & & \cdot \end{matrix};$$

Thus, for $0 \le m < k \le n-1$, we have

$$c_n(m,k) = \begin{cases} (-1)^{(n+1)/2} a_{n+1}(k+1, m+1), & \text{if } n \text{ is odd;} \\ (-1)^{(n+2)/2} b_{n+1}(k+1, m+1), & \text{if } n \text{ is even.} \end{cases} \tag{8.9}$$

The sequence of triangles (C_n) defined by (8.9) is a Seidel triangle sequence. The corresponding matrix H reads

$$\begin{aligned} H &= \begin{pmatrix} 1 & 1 & -2 & -2 & 16 & 16 & \cdots \\ 0 & -2 & -2 & 16 & 16 & & \\ -1 & -1 & 14 & 14 & & & \\ 0 & 10 & 10 & & & & \\ 5 & 5 & & & & & \\ 0 & & & & & & \\ \vdots & & & & & & \end{pmatrix} \\ &= \begin{pmatrix} 1 & . & -2 & . & 16 & \cdots \\ . & -2 & . & 16 & . & \\ -1 & . & 14 & . & & \\ . & 10 & . & & & \\ 5 & . & & & & \\ \vdots & & & & & \end{pmatrix} + \begin{pmatrix} . & 1 & . & -2 & . & 16 & \cdots \\ 0 & . & -2 & . & 16 & & \\ . & -1 & . & 14 & & & \\ 0 & . & 10 & & & & \\ . & 5 & & & & & \\ 0 & & & & & & \\ \vdots & & & & & & \end{pmatrix} \\ &= -D_y \overline{H}_2 - \overline{H}_2. \end{aligned}$$

Thus,

$$\begin{aligned} H(x,y) &= D_y \frac{\sinh y}{\cosh(x+y)} + \frac{\sinh y}{\cosh(x+y)} \\ &= \frac{\cosh x}{\cosh^2(x+y)} + \frac{\sinh y}{\cosh(x+y)}; \end{aligned} \tag{8.10}$$

$$e^x H(x+y,z) = e^x\Bigl(\frac{\cosh(x+y)}{\cosh^2(x+y+z)} + \frac{\sinh z}{\cosh(x+y+z)}\Bigr);$$

$$e^{Ix} H(Ix+Iy,Iz) = (\cos x + I\sin x)\Bigl(\frac{\cos(x+y)}{\cos^2(x+y+z)} + \frac{I\,\sin z}{\cos(x+y+z)}\Bigr).$$

If $n=2l+1$, then $I^{n-2}=(-1)^{l+1}I$ and $(-1)^{(n+1)/2}=(-1)^{l+1}$. Thus, $I^{n-2}c_n(m,k)=I\,a_{n+1}(k+1,m+1)$. The imaginary part of identity (7.12) becomes

$$\sum_{\substack{1\le m+1\le k\le n-1\\ n\text{ odd}}} a_{n+1}(k+1,m+1)\frac{x^{n-k-1}}{(n-k-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^m}{m!}$$
$$= \frac{\cos x\,\sin z}{\cos(x+y+z)} + \frac{\sin x\,\cos(x+y)}{\cos^2(x+y+z)}.$$

With the change of variables $n\leftarrow 2n-1$, $k+1\leftarrow m$, $m+1\leftarrow k$, we get (1.16) from Theorem 1.3.

If $n=2l$, then $I^{n-2}=(-1)^{l-1}$ and $(-1)^{(n+2)/2}=(-1)^{l+1}$, so that $I^{n-2}c_n(m,k)=b_{n+1}(k+1,m+1)$. As for the real part, we have

$$\sum_{\substack{1\le m+1\le k\le n-1\\ n\text{ even}}} b_{n+1}(k+1,m+1)\frac{x^{n-k-1}}{(n-k-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^m}{m!}$$
$$= -\frac{\sin x\,\sin z}{\cos(x+y+z)} + \frac{\cos x\,\cos(x+y)}{\cos^2(x+y+z)}.$$

With the change of variables $n\leftarrow 2n$, $k+1\leftarrow m$, $m+1\leftarrow k$, we get (1.22) from Theorem 1.6.

8.5. *The lower triangles of* $\mathrm{Twin}^{(2)}$. Again, the bottom rows of the A_n's and B_n's having been discarded, the Seidel triangle sequence to be considered is

$$C_1=\cdot\,;\quad C_2=\begin{array}{cc}\cdot&1\\&\cdot\end{array};\quad C_3=\begin{array}{ccc}\cdot&0&-1\\&\cdot&-1\\&&\cdot\end{array};\quad C_4=\begin{array}{cccc}\cdot&-1&-1&0\\&\cdot&-2&-1\\&&\cdot&-1\\&&&\cdot\end{array}\quad C_5=\begin{array}{ccccc}\cdot&0&1&2&2\\&\cdot&1&3&4\\&&\cdot&4&5\\&&&\cdot&5\\&&&&\cdot\end{array};$$

$$C_6=\begin{array}{cccccc}\cdot&5&5&4&2&0\\&\cdot&10&9&6&2\\&&\cdot&13&9&4\\&&&\cdot&10&5\\&&&&\cdot&5\\&&&&&\cdot\end{array};\qquad C_7=\begin{array}{ccccccc}\cdot&0&-5&-10&-14&-16&-16\\&\cdot&-5&-15&-24&-30&-32\\&&\cdot&-20&-33&-42&-46\\&&&\cdot&-41&-51&-56\\&&&&\cdot&-56&-61\\&&&&&\cdot&-61\\&&&&&&\cdot\end{array};$$

the general formula being

$$c_n(m,k) = \begin{cases} (-1)^{(n-1)/2} b_{n+1}(k+1,m+1), & \text{if } n \text{ is odd;} \\ (-1)^{(n-2)/2} a_{n+1}(k+1,m+1), & \text{if } n \text{ is even.} \end{cases} \tag{8.13}$$

Next, form the matrix H, whose skew-diagonals carry the entries of the rightmost columns of the c_n's, and write it as the sum of the two matrices

$$H = \begin{pmatrix} 1 & -1 & -1 & 5 & 5 & -61 & -61 & \cdots \\ -1 & -1 & 5 & 5 & -61 & -61 & & \\ 0 & 4 & 4 & -56 & -56 & & & \\ 2 & 2 & -46 & -46 & & & & \\ 0 & -32 & -32 & & & & & \\ -16 & -16 & & & & & & \\ 0 & & & & & & & \\ \vdots & & & & & & & \end{pmatrix} := K_1 + K_2$$

$$= \begin{pmatrix} 1 & \cdot & -1 & \cdot & 5 & \cdot & -61 & \cdots \\ \cdot & -1 & \cdot & 5 & \cdot & -61 & & \\ 0 & \cdot & 4 & \cdot & -56 & & & \\ \cdot & 2 & \cdot & -46 & & & & \\ 0 & \cdot & -32 & & & & & \\ \cdot & -16 & & & & & & \\ 0 & & & & & & & \\ \vdots & & & & & & & \end{pmatrix} + \begin{pmatrix} \cdot & -1 & \cdot & 5 & \cdot & -61 & \cdots \\ -1 & \cdot & 5 & \cdot & -61 & \cdot & \\ \cdot & 4 & \cdot & -56 & \cdot & & \\ 2 & \cdot & -46 & \cdot & & & \\ \cdot & -32 & \cdot & & & & \\ -16 & \cdot & & & & & \\ \vdots & & & & & & \end{pmatrix}.$$

These matrices are to be compared with the matrix $\overline{H}_1$ (see Section 7). Clearly, K_2 can be obtained from $\overline{H}_1$ by deleting the top row and then transposing the matrix, so that $K_2(x,y) = TD_x\overline{H}_1(x,y)$. Also, $K_1 = T\overline{H}_1$, and then $K_1(x,y) = \overline{H}_1(y,x)$. As $\overline{H}_1(x,y) = \cosh y/\cosh(x+y)$, we get

$$\begin{aligned} H(x,y) &= TD_x\overline{H}_1(x,y) + \overline{H}_1(y,x) \\ &= -\frac{\cosh x\,\sinh(x+y)}{\cosh^2(x+y)} + \frac{\cosh x}{\cosh(x+y)}; \\ e^x H(x+y,z) &= e^x\Big(-\frac{\cosh(x+y)\sinh(x+y+z)}{\cosh^2(x+y+z)} + \frac{\cosh(x+y)}{\cosh(x+y+z)}\Big); \\ e^{Ix}H(Ix+Iy,Iz) &= (\cos x + I\sin x) \\ &\qquad \times\Big(-\frac{I\,\cos(x+y)\sin(x+y+z)}{\cos^2(x+y+z)} + \frac{\cos(x+y)}{\cos(x+y+z)}\Big). \end{aligned}$$

If $n = 2l+1$, then $I^{n-2} = (-1)^{l+1}I$ and $(-1)^{(n-1)/2} = (-1)^l$. Thus, $I^{n-2}c_n(m,k) = -I\,b_{n+1}(k+1,m+1)$. The imaginary part of identity

(7.12) becomes

$$\sum_{\substack{1\le m+1\le k\le n-1\\ n\ \text{odd}}} b_{n+1}(k+1,m+1)\frac{x^{n-k-1}}{(n-k-1)!}\,\frac{y^{k-m-1}}{(k-m-1)!}\,\frac{z^m}{m!}$$

$$= -\frac{\sin x\ \cos(x+y)}{\cos(x+y+z)}+\frac{\cos x\ \cos(x+y)\sin(x+y+z)}{\cos^2(x+y+z)}$$

$$= \frac{\cos(x+y)\ \sin(y+z)}{\cos^2(x+y+z)}.$$

With the change of variables $n\leftarrow 2n-1$, $m+1\leftarrow k$, $k+1\leftarrow m$, we get (1.20) from Theorem 1.5.

If $n=2l$, then $I^{n-2}=(-1)^{l-1}$ and $(-1)^{(n-2)/2}=(-1)^{l-1}$, so that $I^{n-2}c_n(m,k)=a_{n+1}(k+1,m+1)$. As for the real part, we have

$$\sum_{\substack{1\le m+1\le k\le n-1\\ n\ \text{even}}} a_{n+1}(k+1,m+1)\frac{x^{n-k-1}}{(n-k-1)!}\,\frac{y^{k-m-1}}{(k-m-1)!}\,\frac{z^m}{m!}$$

$$= \frac{\cos x\ \cos(x+y)}{\cos(x+y+z)}+\frac{\sin x\ \cos(x+y)\sin(x+y+z)}{\cos^2(x+y+z)}$$

$$= \frac{\cos(x+y)\ \cos(y+z)}{\cos^2(x+y+z)}.$$

With the change of variables $n\leftarrow 2n-1$, $k+1\leftarrow m$, $m+1\leftarrow k$, we get (1.18) from Theorem 1.4.

9. The formal Laplace transform

The purpose of this section is to show that, when the Entringer numbers $E_n(k)$ are defined by relations (1.1), without any reference to their combinatorial interpretations, they can be proved to be a refinement of the tangent/secant numbers: $\sum_k E_n(k)=E_n$ $(n\ge 1)$. In the same manner, when the twin Seidel matrix sequence (A_n), (B_n) is analytically defined, as it was stated in Subsection 1.5, also without reference to any combinatorial interpretation, their entries $(a_n(m,k))$, $(b_n(m,k))$ make up a refinement of the Entringer numbers, by row and by column, and

$$\sum_{m,k} a_n(m,k)=\sum_{m,k} b_n(m,k)=E_n.$$

The proofs of these results make use of the closed expressions found for the generating functions in the preceding section, and of a well-adapted formal Laplace transform technique.

Theorem 9.1. (1) *Let $(E_n(k))$ be the sequence of the Entringer numbers, defined by*

$$E_1(1) := 1; \quad E_n(n) := 0 \text{ for all } n \geq 2;$$
$$\Delta E_n(m) + E_{n-1}(n-m) = 0 \quad (n \geq 2; m = n-1, \ldots, 2, 1);$$

Then,

$$\sum_{1 \leq k \leq 2n-1} E_{2n-1}(k) = E_{2n-1}; \qquad \sum_{1 \leq k \leq 2n} E_{2n}(k) = E_{2n}; \quad (n \geq 1). \tag{9.1}$$

(2) *Let $(a_n(m,k))$, $(b_n(m,k))$ be the entries of the twin Seidel matrix sequence (A_n), (B_n), as they are defined in Subsection* 1.5. *Then,*

$$a_n(m,\bullet) = E_n(m), \quad b_n(m,\bullet) = E_n(n+1-m), \quad (1 \leq m \leq n); \tag{9.2}$$
$$a_n(\bullet,k) = b_n(\bullet,k) = E_n(n-k) \quad (1 \leq k \leq n). \tag{9.3}$$

The proof of (9.1) is fully given. Next, we reproduce the proof of $a_{2n}(m,\bullet) = E_{2n}(m)$, based on Theorem 1.3. The other identities in (9.2) and (9.3) can also be derived following the same method by using Theorems 1.4, 1.5, 1.6. Their proofs are omitted.

The formal Laplace transform, already used in our previous paper [FH14], maps a function $f(x)$ to a function $\mathcal{L}(f(x),x,s)$ defined by

$$\mathcal{L}(f(x),x,s) := \int_0^\infty f(x)e^{-xs}\,dx.$$

In particular, $\mathcal{L}(\bullet,x,s)$ maps $x^k/k!$ to $1/s^{k+1}$:

$$\mathcal{L}(\frac{x^k}{k!},x,s) = \frac{1}{s^{k+1}}.$$

For proving (9.1), start with identity (7.7) involving the generating function for the numbers $E_{2n+1}(k)$ and apply the Laplace transform twice with respect to (x,s), (y,t), respectively. We get

$$\sum_{1 \leq k \leq 2n+1} \frac{1}{s^{2n-k+2}} \frac{1}{t^k} E_{2n+1}(k) = \int_0^\infty \int_0^\infty \frac{\cos y}{\cos(x+y)} e^{-xs-yt} dx\,dy,$$

which becomes, with $t \leftarrow s$ and $r = x+y$,

$$\sum_{1 \leq k \leq 2n+1} \frac{1}{s^{2n+2}} E_{2n+1}(k) = \int_0^\infty \int_0^\infty \frac{\cos y}{\cos(x+y)} e^{-xs-ys} dx\,dy$$

$$\begin{aligned}
&= \int_0^\infty \int_0^r \frac{\cos y}{\cos r} e^{-rs} dy\, dr \\
&= \int_0^\infty \frac{\sin r}{\cos r} e^{-rs} dr \\
&= \int_0^\infty (\tan r) e^{-rs} dr \\
&= \sum_{n\ge 1} \frac{1}{s^{2n}} E_{2n-1}.
\end{aligned}$$

Hence,

$$\sum_{1\le k\le 2n-1} E_{2n-1}(k) = E_{2n-1}.$$

In the same manner, apply the Laplace transform to identity (7.8) twice with respect to (x,s), (y,t), respectively. We get

$$\sum_{1\le k\le 2n} \frac{1}{s^k} \frac{1}{t^{2n-k+1}} E_{2n}(k) = \int_0^\infty \int_0^\infty \frac{\sin y}{\cos(x+y)} e^{-xs-yt} dx\, dy,$$

which becomes, with $s \leftarrow t$ and $r = x+y$,

$$\begin{aligned}
\sum_{1\le k\le 2n} \frac{1}{t^{2n+1}} E_{2n}(k) &= \int_0^\infty \int_0^\infty \frac{\sin y}{\cos(x+y)} e^{-xt-yt} dx\, dy \\
&= \int_0^\infty \int_0^r \frac{\sin y}{\cos r} e^{-rt} dy\, dr \\
&= \int_0^\infty \frac{1-\cos r}{\cos r} e^{-rt} dr \\
&= \int_0^\infty (\sec r - 1) e^{-rt} dr \\
&= \sum_{n\ge 1} \frac{1}{t^{2n+1}} E_{2n}.
\end{aligned}$$

Hence,

$$\sum_{1\le k\le 2n} E_{2n}(k) = E_{2n}.$$

Next, to prove $a_{2n}(m,\bullet) = E_{2n}(m)$ start with identity (1.15) of Theorem 1.5 and apply the Laplace transform to its left-hand side three times with respect to (x,s), (y,t), (z,u), respectively. We get

$$\sum_{2\le m+1\le k\le 2n-1} \frac{1}{s^m} \frac{1}{t^{k-m}} \frac{1}{u^{2n-k}} a_{2n}(m,k),$$

which becomes

$$\sum_{2\le m+1\le k\le 2n-1} \frac{1}{s^m}\frac{1}{u^{2n}} a_{2n}(m,k), \tag{9.4}$$

when $t \leftarrow u$ and $s \leftarrow su$. Apply the Laplace transform to the right-hand side of (1.15) three times with respect to (x,s), (y,t), (z,u), respectively, and let $t \leftarrow u$, $s \leftarrow su$. With $r = y+z$, we get

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty \frac{\cos x\ \cos z \sin(x+y+z)}{\cos^2(x+y+z)} e^{-xsu-yu-zu} dx\, dy\, dz \\
&\qquad = \int_0^\infty\int_0^\infty\int_0^r \frac{\cos x \cos z \sin(x+r)}{\cos^2(x+r)} e^{-xsu-ru} dz\, dr\, dx \\
&\qquad = \int_0^\infty\int_0^\infty \frac{\cos x \sin r \sin(x+r)}{\cos^2(x+r)} e^{-xsu-ru} dr\, dx. \qquad (9.5)
\end{aligned}$$

With identity (1.16) apply the Laplace transform to its left-hand side three times with respect to (x,u), (y,s), (z,t), respectively. We get

$$\sum_{2\le k+1\le m\le 2n-1} \frac{1}{u^{2n-m}}\frac{1}{s^{m-k}}\frac{1}{t^k} a_{2n}(m,k),$$

which becomes

$$\sum_{2\le k+1\le m\le 2n-1} \frac{1}{s^m}\frac{1}{u^{2n}} a_{2n}(m,k), \tag{9.6}$$

when $s \leftarrow su$ and $t \leftarrow su$. Apply the Laplace transform to the right-hand side of (1.16) three times with respect to (x,u), (y,s), (z,t), respectively, and let $s \leftarrow su$, $t \leftarrow su$. With $r = y+z$ we get

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty \Bigl(\frac{\cos x\ \sin z}{\cos(x+y+z)} + \frac{\sin x\ \cos(x+y)}{\cos^2(x+y+z)}\Bigr) e^{-xu-ysu-zsu} dx\, dy\, dz \\
&\quad = \int_0^\infty\int_0^\infty\int_0^r \Bigl(\frac{\cos x\ \sin z}{\cos(x+r)} + \frac{\sin x\ \cos(x+r-z)}{\cos^2(x+r)}\Bigr) e^{-xu-rsu} dz\, dr\, dx \\
&\quad = \int_0^\infty\int_0^\infty \Bigl(\frac{\cos x\,(1-\cos r)}{\cos(x+r)} + \frac{\sin x\,(\sin(x+r)-\sin x)}{\cos^2(x+r)}\Bigr) e^{-xu-rsu} dr\, dx \\
&\quad = \int_0^\infty\int_0^\infty \Bigl(\frac{\cos r\,(1-\cos x)}{\cos(x+r)} + \frac{\sin r\,(\sin(x+r)-\sin r)}{\cos^2(x+r)}\Bigr) e^{-ru-xsu} dr\, dx. \qquad (9.7)
\end{aligned}$$

By (9.4)–(9.7), we have

$$\sum_{1\le k,m\le 2n-1;\, k\ne m} \frac{1}{s^m}\frac{1}{u^{2n}} a_{2n}(m,k) = \int_0^\infty\int_0^\infty F(x,r) e^{-xsu-ru} dr\, dx, \tag{9.8}$$

where

$$F(x,r)=\frac{\cos x\sin r\sin(x+r)}{\cos^2(x+r)}+\frac{\cos r\,(1-\cos x)}{\cos(x+r)}+\frac{\sin r\,(\sin(x+r)-\sin r)}{\cos^2(x+r)}$$
$$=\frac{\cos x}{\cos^2(x+r)}-1.$$

However, from (7.8), we conclude

$$\sum_{1\le m\le 2n} E_{2n}(m)\frac{x^{m-1}}{(m-1)!}\frac{r^{2n-m-1}}{(2n-m-1)!}=\frac{\partial}{\partial r}\frac{\sin r}{\cos(x+r)}$$
$$=\frac{\cos x}{\cos^2(x+r)}. \tag{9.9}$$

Apply the Laplace transform to (9.9) twice with respect to (x,s), (y,u), respectively. We get

$$\sum_{1\le m\le 2n}\frac{1}{s^m}\frac{1}{u^{2n-m}}E_{2n}(m)=\int_0^\infty\int_0^\infty\frac{\cos x}{\cos^2(x+r)}e^{-xs-ru}dx\,dr,$$

or still

$$\sum_{1\le m\le 2n}\frac{1}{s^m}\frac{1}{u^{2n}}E_{2n}(m)=\int_0^\infty\int_0^\infty\frac{\cos x}{\cos^2(x+r)}e^{-xsu-ru}dx\,dr. \tag{9.10}$$

By (9.8) and (9.10), we obtain

$$\sum_{1\le k,m\le 2n-1;\,k\ne m}\frac{1}{s^m}\frac{1}{u^{2n}}a_{2n}(m,k)=\sum_{1\le m\le 2n}\frac{1}{s^m}\frac{1}{u^{2n}}E_{2n}(m)-\frac{1}{su^2},$$

and then

$$\sum_{1\le k,m\le 2n-1}\frac{1}{s^m}\frac{1}{u^{2n}}a_{2n}(m,k)=\sum_{1\le m\le 2n}\frac{1}{s^m}\frac{1}{u^{2n}}E_{2n}(m).$$

Hence,

$$\sum_{1\le k\le 2n-1}a_{2n}(m,k)=E_{2n}(m).\qquad\square$$

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Dominique Foata
Institut Lothaire
1, rue Murner
F-67000 Strasbourg, France
`foata@unistra.fr`

Guo-Niu Han
I.R.M.A. UMR 7501
Université de Strasbourg et CNRS
7, rue René-Descartes
F-67084 Strasbourg, France
`guoniu.han@unistra.fr`