Sminaire Lotharingien de Combinatoire **72** (2015), Article B72c

COMPARING ESTIMATES ON THE NUMBER OF ZEROS OF IRREDUCIBLE CHARACTERS IN SYMMETRIC GROUPS

F. PETÉNYI

To the memory of Professor David Chillag

Abstract. We compare lower estimates of Gallagher, Chillag and Wilde on the number of zeros of irreducible characters of finite groups. We formulate each estimate both for group elements and for conjugacy classes. We compare for each pair of estimates, if there are infinitely many symmetric groups where one of them is better than the other.

1. Introduction

In the literature zeros of characters are widely investigated, see e.g. [2], [3], [4], [5], [6], [7], [8], [9], [12], [14], [17], [18], [19] [21], [25], [26], [27], and [28].

In this paper we investigate lower estimates on the number of zeros of irreducible complex characters of finite groups.

The paper consists of two parts. In the first part we reformulate the lower estimate of Gallagher for the case of conjugacy classes, and the lower estimate of Chillag for the case of group elements. We also formulate estimates of Wilde.

We introduce for these estimates the following notation: the upper index (G or C) indicates if the estimate is for group elements or for conjugacy classes, respectively. The lower index Ga, Ch, W refers to the name of the person whose estimate is used.

We give the smallest finite groups for each ordered pair of the above mentioned estimates where the first one is better than the second for some irreducible character.

In the second part of the paper we consider symmetric groups. We examine, which estimate is better than the other for irreducible, non-linear characters of symmetric groups, namely which of them is bigger than the other for all but finitely many characters. In the paper we use standard notations; for example, $\mathrm{Cl}(G)$ denotes the set of conjugacy classes of G, and $C_G(g)$ denotes the centralizer of g in G.

2. Some lower estimates for the number of zeros of irreducible characters of finite groups

Let G be a finite group and χ one of its complex irreducible characters. We say that an element g is *vanishing* with respect to χ, if $\chi(g) = 0$. Our aim is to investigate and compare lower estimates for the number of vanishing conjugacy classes or elements.

2010 *Mathematics Subject Classification.* Primary 20C15; Secondary 20C30.

Key words and phrases. zeros of characters, estimates, partition, hook length formula, primes in interval.

The work reported in the paper has been developed in the framework of the project "Talent care and cultivation in the scientific workshops of BME" project. This project is supported by the grant TÁMOP -4.2.2.B-10/1-2010-0009.

The first well-known result is due to Burnside (see [11, Thm. 3.15]), which states that every nonlinear irreducible character of a finite group vanishes on some element of the group. We will use the results of Gallagher [8, Thm. 4], Chillag [4, Thm. 1.1], and Wilde [25, Thm. 2.1] to construct three different lower bounds.

We introduce the following notations. Denote by $n^G(\chi)$ and $n^C(\chi)$ the number of vanishing group elements and conjugacy classes of χ, respectively, i.e.,

$$n^G(\chi) = |\{g \in G \mid \ \chi(g) = 0\}| \text{ and } n^C(\chi) = |\{C \in \mathrm{Cl}(G) \mid \ \chi|_C = 0\}|.$$

Gallagher improved the result of Burnside as follows; see [11, Thm. 3.15].

Theorem 2.1 (Gallagher [8])**.** *For every irreducible character χ of a finite group, we have*

$$n^G(\chi) \geqslant (\chi(1)^2 - 1)|Z(\chi)|,$$

where $Z(\chi)$ is the center of χ. If $|\chi|$ takes only the values 0, 1, and $\chi(1)$, then equality holds.

Let us introduce the notation $\mathfrak{E}^G_{Ga}(\chi) = (\chi(1)^2 - 1)|Z(\chi)|$.

Remark 2.2. There are infinitely many groups where $n^G(\chi) = \mathfrak{E}^G_{Ga}(\chi)$ for some nonlinear irreducible character χ. An example is the case of groups of nilpotency class 2, see [11, Cor. 2.30, Thm. 2.31].

Let us recall the following result.

Theorem 2.3 (Chillag [4])**.** *Let G be a finite non-abelian group such that $G \neq G'$ and χ a non-linear irreducible character of G. Then one of the following holds:*

(i) *G has an element x such that $|C_G(x)| \leqslant 2n^C(\chi)$. In fact, one can choose such an x from any conjugacy class of maximal size among the classes on which χ vanishes.*

(ii) *$\phi = \chi_{G'} \in \mathrm{Irr}(G')$ and $\phi^G = \chi \sum\{\lambda \mid \lambda \in \mathrm{Irr}(G),\ \lambda(1) = 1\}$. Furthermore the set $\{\chi\lambda \mid \lambda \in \mathrm{Lin}(G)\}$ consists of exactly $|G : G'|$ extensions of ϕ, which are all the extensions of ϕ. In particular, ϕ is not linear and $G'' \neq 1$.*

We can use this theorem to give a lower bound for $n^C(\chi)$. This lower bound will be denoted by $\mathfrak{E}^C_{Ch}(\chi)$, which is given by

$$\mathfrak{E}^C_{Ch}(\chi) = \begin{cases} 0, & \text{if } \chi \text{ linear}, \\ \min_{\{x \in G \ \mid \ \chi(x)=0\}} \lceil |C_G(x)|/2 \rceil, & \text{if } G \neq G' \text{ and } \chi_{G'} \notin \mathrm{Irr}(G'), \\ 1, & \text{otherwise}. \end{cases}$$

Let us consider the case where $\mathfrak{E}^C_{Ch}(\chi) = 1$.

Proposition 2.4. *Let G be a finite group with $G \neq G'$, and let $\chi \in \mathrm{Irr}(G)$ be nonlinear and $\chi_{G'} \notin \mathrm{Irr}(G')$. Let $\mathfrak{E}^C_{Ch}(\chi) = 1$. Then $n^G(\chi) = \frac{|G|}{2}$ and $n^C(\chi) = 1$. Moreover, G is a Frobenius group with a complement of order 2.*

Proof. If $\mathfrak{E}^C_{Ch}(\chi) = 1$, then $\min_{\{g \mid \chi(g)=0\}} |\mathbf{C}_G(g)|$ is 1 or 2. Suppose the minimum is obtained at some element g_0.

If $|\mathbf{C}_G(g_0)| = 1$, then the group is the one-element group, which is a contradiction.

Otherwise, we have $|\mathbf{C}_G(g_0)| = 2$, so $|\operatorname{Cl}(g_0)| = \frac{|G|}{2} \leqslant n^G(\chi)$.

Consider the subgroup $H = \{1, g_0\}$. This is a trivial intersection set, i.e., $H \bigcap H^x = 1$ for all $x \in G \setminus H$. Therefore G is a Frobenius group with H as complement. In G every irreducible character is either linear or is induced from the Frobenius kernel N. Since χ has at least one vanishing conjugacy class, it cannot be linear. Furthermore, $\chi = \nu^G$, where $\nu \in \mathrm{Irr}(N)$. Since $|H|$ is even, the kernel N is abelian, see e.g. [11, Thm. 7.21]. Therefore ν is linear. Let $n \in N$ such that $\chi(n) = 0$. Then

$$\chi(n) = \frac{1}{|N|}\sum_{x\in G}\nu^{\circ}(xnx^{-1}) = \frac{1}{|N|}\sum_{x\in G}\nu(xnx^{-1}) = \nu(n) + \frac{1}{|N|}\sum_{x\in G\setminus N}\nu(xnx^{-1}).$$

However, every $g \in G\setminus N$ is inverting the elements of N. Hence $\chi(n) = \nu(n)+\overline{\nu(n)} = 0$. This holds exactly when $\nu(n) = i$ or $-i$. Thus 4 divides the order of n, and also $|N|$. This is a contradiction. This means that exactly the elements in $G \setminus N$ are the vanishing elements, and this is the only vanishing conjugacy class. Thus $n^C(\chi) = 1$ and $n^G(\chi) = \frac{|G|}{2}$. □

Suppose we know the sizes of the conjugacy classes of G. Then, using $\mathfrak{E}^G_{Ga}(\chi)$, we can construct another lower bound for $n^C(\chi)$. This lower bound will be denoted by

$$\mathfrak{E}^C_{Ga}(\chi) = \min\{|J| \;\mid\; J \subseteq I,\ \sum_{j\in J}|C_j| \geqslant \mathfrak{E}^G_{Ga}(\chi)\},$$

where $\mathrm{Cl}(G) = \{C_i\}_{i\in I}$.

Similarly, from $\mathfrak{E}^C_{Ch}(\chi)$ we can construct a lower estimate $\mathfrak{E}^G_{Ch}(\chi)$ of $n^G(\chi)$. If we use the result of Proposition 2.4, as well as the fact that χ cannot vanish on $Z(G)$, then we obtain

$$\mathfrak{E}^G_{Ch}(\chi) = \begin{cases} 0, & \text{if } \chi \text{ is linear,}\\ \frac{|G|}{2}, & \text{if } \mathfrak{E}^C_{Ch}(\chi) = 1,\ G \neq G',\\ & \qquad \text{and } \chi_{G'} \notin \mathrm{Irr}(G'),\\ \min_{J\subseteq I}\{\sum_{j\in J}|C_i| \;\mid & \\ \quad C_j \in \mathrm{Cl}(G),\ C_j \nsubseteq Z(G) \text{ and } |J| = \mathfrak{E}^C_{Ch}(\chi)\}, & \text{otherwise.}\end{cases}$$

We remark that, if $\chi_{G'} \in \mathrm{Irr}(G')$, then $\mathfrak{E}^G_{Ch}(\chi)$ is the size of the smallest non-central conjugacy class.

Before formulating the lower bound by Wilde, we mention the following fact.

Theorem 2.5 (Brauer, Nesbitt [11, Thm. 8.17])**.** *Let $\chi \in \mathrm{Irr}(G)$ and suppose $p \nmid \frac{|G|}{\chi(1)}$ for some prime p. Then $\chi(g) = 0$ whenever $p \mid o(g)$.*

This is equivalent to saying that, whenever $\chi(g) \neq 0$, then the squarefree part $o(g)_0$ of $o(g)$ divides $\frac{|G|}{\chi(1)}$.

In [25, Conj. 1.1], Wilde proposed the following conjecture.

Conjecture 2.6 (Wilde). Let G be a finite group. Furthermore, let $\chi \in \mathrm{Irr}(G)$ and $g \in G$, and suppose that $\chi(g) \neq 0$. Then $o(g)$ divides $\frac{|G|}{\chi(1)}$.

Wilde proved in [25] that Conjecture 2.6 holds for solvable groups, and he mentioned that it is true also for S_n, which we prove later for the sake of selfcontainedness, see Lemma 3.10. In [25, Thm. 2.1] the following partial result was proved.

Theorem 2.7 (WILDE). *Let χ be an irreducible character of G and $g \in G$ a group element with $\chi(g) \neq 0$. Then $o(g)o(g)_0 \mid (\frac{|G|}{\chi(1)})^2$ and $o(g)^3 \mid \frac{|G|^3}{\chi(1)^2}$.*

We use Theorem 2.7 and Burnside's theorem, see [11, Thm. 3.15], to give lower estimates for $n^G(\chi)$ and $n^C(\chi)$, which we will denote by $\mathfrak{E}^G_W(\chi)$, $\mathfrak{E}^C_W(\chi)$, respectively:

$$\mathfrak{E}^G_W(\chi) = \max\left\{\left|\left\{g \in G \mid o(g)o(g)_0 \nmid \frac{|G|^2}{\chi(1)^2} \text{ or } o(g)^3 \nmid \frac{|G|^3|}{\chi(1)^2}\right\}\right|, \min_{C\in \mathrm{Cl}(G), C\not\subseteq Z(G)} |C|\right\},$$

$$\mathfrak{E}^C_W(\chi) = \max\left\{\left|\left\{C \in \mathrm{Cl}(G) \mid o(g)o(g)_0 \nmid \frac{|G|^2}{\chi(1)^2} \text{ or } o(g)^3 \nmid \frac{|G|^3}{\chi(1)^2} \text{ for all } g \in C\right\}\right|, 1\right\}.$$

We know that Conjecture 2.6 is true for p-groups. In fact one can prove more.

Proposition 2.8. *If G is a p-group, then for every $g \in G$ and for every $\chi \in \mathrm{Irr}(G)$, we have $o(g) \mid \frac{|G|}{\chi(1)}$. In particular, for every $\chi \in \mathrm{Irr}(G)$, we have $\mathfrak{E}^C_W(\chi) = 1$ and $\mathfrak{E}^G_W(\chi) = \min_{C\in \mathrm{Cl}(G), C\not\subseteq Z(G)} |C|$.*

Proof. We use induction on the order of the group. We may suppose that $\chi(1) > 1$. Then $\langle g\rangle \neq G$.

Let M be a maximal subgroup in G containing g. Then χ_M is either irreducible or it is the sum of p irreducible characters. If $\chi_M = \psi_1 + \cdots + \psi_p$, where $\psi_i \in \mathrm{Irr}(M)$, then $\psi_i^G = \chi$, and so $\chi(1) = p\psi_i(1)$. Since M is a smaller p-group, by induction hypothesis we have $o(g) \mid \frac{|M|}{\psi_i(1)} = \frac{|G|}{\chi(1)}$.

If $\chi_M = \psi$, where $\psi \in \mathrm{Irr}(M)$, then by induction hypothesis we have

$$o(g) \mid \frac{|M|}{\psi(1)} = \frac{|M|}{\chi(1)} \mid \frac{|G|}{\chi(1)}.$$

□

For both $n^G(\chi)$ and $n^C(\chi)$ we have the following results on the above lower bounds. We checked the examples with the help of the GAP system [10], see the used programs in [22].

Theorem 2.9. *For each ordered pair of the above estimates on conjugacy classes $\{\mathfrak{E}^C_{Ch}(\ ),\ \mathfrak{E}^C_{Ga}(\ ), \mathfrak{E}^C_W(\ )\}$ and group elements $\{\mathfrak{E}^G_{Ch}(\ ),\ \mathfrak{E}^G_{Ga}(\ ), \mathfrak{E}^G_W(\ )\}$, there exists a group and an irreducible character of that group such that the first estimate is better than the second.*

We demonstrate the result using the following diagrams:

$$\begin{tikzcd}
\mathfrak{E}^C_{Ch}(\ ) \arrow[rr, bend left, "D_8\ (2)"] \arrow[dr, bend left, "C_5\rtimes C_4\ (20{,}3)\ (1)"] & & \mathfrak{E}^C_{Ga}(\ ) \arrow[ll, bend left, "C_3\rtimes Q_8\ (2)"] \arrow[dl, bend left, "C_7\rtimes C_4\ (2)"] \\
& \mathfrak{E}^C_W(\ ) \arrow[ul, bend left, "D_8\ (2)"] \arrow[ur, bend left, "D_8\ (2)"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{E}^G_{Ch}(\ ) \arrow[rr, bend left, "D_8\ (2)"] \arrow[dr, bend left, "C_3\rtimes C_4\ (2)"] & & \mathfrak{E}^G_{Ga}(\ ) \arrow[ll, bend left, "D_{10}\ (1)"] \arrow[dl, bend left, "D_{10}\ (1)"] \\
& \mathfrak{E}^G_W(\ ) \arrow[ul, bend left, "D_8\ (2)"] \arrow[ur, bend left, "D_8\ (2)"] &
\end{tikzcd}$$

Here, an arrow points to the estimate which is better on some irreducible character of the group corresponding to it. We labelled the arrows with the smallest possible groups having suitable character. When the semidirect product is not unique, we included the group ID of the GAP system. In brackets you can find the size of the centre of the character. With this data, the group and set of suitable characters are uniquely determined.

3. Estimates for symmetric groups

Let consider the estimates of the previous section for symmetric groups.

Lemma 3.1. *Let χ be an irreducible character of the symmetric group S_n for $n \geq 5$. Then $\mathfrak{E}_{Ga}^{G}(\chi) = \chi(1)^2 - 1$. Thus $n^G(\chi) \geq \chi(1)^2 - 1$.*

Proof. If χ is linear, then the statement is true. Otherwise $\ker(\chi) = 1$, and $Z(\chi) = Z(S_n) = 1$. □

Lemma 3.2. *Let $\chi \in \mathrm{Irr}(S_n)$ be the character corresponding to the partition λ. Then $\mathfrak{E}_{Ch}^{C}(\chi) = 1$ if and only if λ is not symmetric and χ is not linear, or $n = 3$ and χ is corresponding to the partition $(2, 1)$.*

Proof. According to [20, Sec. 6.7], the character χ_{A_n} is an irreducible character in A_n if and only if the diagram corresponding to χ is not symmetric. We then use Proposition 2.4 to conclude. □

We will need the following well-known result.

Lemma 3.3. *The largest conjugacy class in S_n corresponds to the partition $(n-1, 1)$, and the second largest is (n).*

Proof. The idea of the proof is that, if we replace one of the largest cycles and another cycle by a cycle of length equal to the sum of lengths of the two cycles, then the corresponding conjugacy class is bigger, unless we had just a 1-cycle and an $(n-1)$-cycle.

The size of the conjugacy class $(1^{j_1}, 2^{j_2}, \ldots, k^{j_k})$, which contains exactly j_l cycles of length l, is

$$\frac{n!}{\prod_{i=1, i\neq l}^{k-1} i^{j_i} j_i! \cdot l^{j_l} j_l! \cdot k^{j_k} j_k!},$$

and the size of the conjugacy class $(1^{j_1}, 2^{j_2}, \ldots, l^{j_l - 1}, \ldots, k^{j_k - 1}, (k+1)^0, \ldots, (k+l-1)^0, (k+l)^1)$ is

$$\frac{n!}{\prod_{i=1, i\neq l}^{k-1} i^{j_i} j_i! \cdot, l^{(j_l - 1)} (j_l - 1)! \cdot k^{(j_k - 1)} (j_k - 1)! \cdot (k+l)}.$$

The second one is always larger, unless $j_1 = j_{n-1} = 1$, and for the other indices k we have $j_k = 0$ or $j_n = 1$. Thus the largest conjugacy class is $(n-1, 1)$, and the second largest is (n). □

Now we can compute $\mathfrak{E}_{Ch}^{C}(\chi_\lambda)$ for symmetric λ as well.

Lemma 3.4. *If χ is an irreducible character of S_n corresponding to a symmetric partition, then $\mathfrak{E}_{Ch}^{C}(\chi) = \lceil \frac{n-1}{2} \rceil$.*

Proof. Since λ is symmetric, χ is induced from an irreducible character of A_n. Thus χ vanishes on the conjugacy classes of odd permutations. If n is odd, then the largest conjugacy class $\mathrm{Cl}(n-1,1)$ is vanishing. Thus

$$\mathfrak{E}_{Ch}^{C}(\chi) = \min_{\chi(g)=0} \left\lceil \frac{|C_G(g)|}{2} \right\rceil = \left\lceil \frac{n-1}{2} \right\rceil.$$

Let n be even. In Lemma 3.3 we have seen that the largest conjugacy class is the class $\mathrm{Cl}(n-1,1)$, and the second largest is $\mathrm{Cl}(n)$. Since n is even, the conjugacy class $\mathrm{Cl}(n)$ is vanishing. Therefore, we have

$$\mathfrak{E}_{Ch}^{C}(\chi) = \min_{\chi(g)=0} \left\lceil \frac{|C_G(g)|}{2} \right\rceil = \begin{cases} \left\lceil \frac{n-1}{2} \right\rceil, & n \text{ odd}, \\ \left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{n-1}{2} \right\rceil, & n \text{ even}. \end{cases}$$

□

Lemma 3.5. *If $\chi \in \mathrm{Irr}(S_n)$ and $n \geq 851$, then $\mathfrak{E}_{Ga}^{C}(\chi) \leq 1$.*

Proof. If χ is an irreducible character of S_n, then we know by [24] that

$$\chi(1) \leq e^{-0{,}11565\sqrt{n}}\sqrt{n!}.$$

This implies that

$$\chi^2(1) - 1 = \mathfrak{E}_{Ga}^{G}(\chi) < e^{-0{,}2313\sqrt{n}} n!.$$

Let $f(n) := e^{-0{,}2313\sqrt{n}} n!/(n(n-2)!)$. Since $f(n) \leq 1$ for $n = 851$ and $f(n)$ is a decreasing function for $n \geq 851$, for $n \geq 851$ the bound $\mathfrak{E}_{Ga}^{G}(\chi)$ is less than $n(n-2)!$, which is the size of the largest conjugacy class of S_n. Thus $\mathfrak{E}_{Ga}^{C}(\chi) \leq 1$ for $n \geq 851$. □

Moreover, we have the following conjecture.

Conjecture 3.6. Let χ be a non-linear irreducible character of a symmetric group. The above defined $\mathfrak{E}_{Ga}^{C}(\chi)$ is two exactly if χ corresponds to one of the following partitions:

$$(3,1^2),\ (3,2,1),\ (3,2,1^2),\ (4,2,1),\ (4,2,1^2),\ (4,2^2,1),\ (4,3,1^2),$$
$$(4,3,2,1),\ (4,3,2,1^2),\ (5,3,2,1),\ (5,3,2,1,1),\ (6,4,3,2,1,1).$$

In all other cases, $\mathfrak{E}_{Ga}^{C}(\chi)$ is exactly 1.

Remark 3.7. Using results of McKay in [16], we can see that this conjecture is true if $n \leq 75$. It is easy to check (for example with the help of GAP [10]) that the conjecture holds for 2-part and 3-part partitions. One can also prove by induction that the conjecture is true for hook-partitions. The conjecture shows for symmetric groups that Gallagher's estimate for conjugacy classes ($\mathfrak{E}_{Ga}^{C}(\ )$) is not really better than Burnside's, see [11, Thm. 3.15].

Proposition 3.8. (i) *The minimal degrees of non-linear irreducible characters of S_n are $n-1$, if $n > 4$.*
(ii) *The minimal degrees of those non-linear irreducible characters of S_n which correspond to symmetric partitions are at least $2^{n/4}$.*

Proof. (i). The proof can be found in [13, Thm. 4.10].

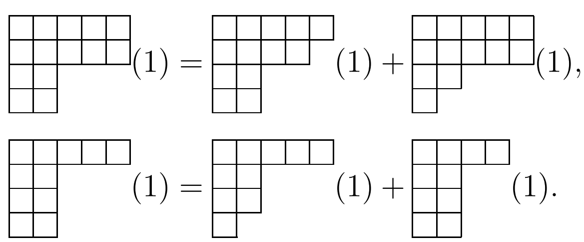
(ii). We can easily check the statement for $n = 3, \ldots, 7$ directly. Let n be at least 8. We will prove by induction that, if $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition not necessarily symmetric, and $\lambda_1 \geq k \geq 2$, then $\chi_\lambda(1) \geq \lambda_1 2^{\frac{n-\lambda_1-1}{2}}$. Since $\chi_\lambda(1) = \chi_{\lambda^T}(1)$, in this way we can estimate the degree of every non-linear irreducible character. Every partition not corresponding to a linear character contains a $(\lambda_1, 1)$ subdiagram. Let i be the number of the boxes outside of this subdiagram, and denote the n-partition λ by $\lambda(n,i)$. Rephrasing the statement for a function of i, we have

$$\chi_\lambda(1) \geq (n-i-1)2^{i/2}. \tag{3.1}$$

We will prove this by double induction on n and i. We checked the statement for $n = 8, 9$ by GAP [10]. For $i = 0$ and $i = 1$ it follows from this table that (3.1) holds.

i	diagram of λ filled by hook-lengths	$\chi_\lambda(1)$	$(n-i-1)2^{i/2}$
0	$\begin{array}{\|c\|c\|c\|c\|c\|}\hline n & n-2 & n-3 & \dots & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$	$n-1$	$n-1$
1	$\begin{array}{\|c\|c\|c\|c\|c\|}\hline n-1 & n-2 & n-4 & \dots & 1 \\ \hline 2 & 1 \\ \cline{1-2}\end{array}$	$n(n-3)/2$	$(n-2)\sqrt{2}$
1	$\begin{array}{\|c\|c\|c\|c\|c\|}\hline n & n-3 & n-4 & \dots & 1 \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$	$(n-1)(n-2)/2$	$(n-2)\sqrt{2}$

Let us suppose $n > 9$, $i > 1$, and that we know the statement (3.1) for n_1, i_1, where either $n_1 = n$ and $0 \leq i_1 < i$ or $8 \leq n_1 < n$. If $\lambda \neq (\lambda_1, \dots, \lambda_1)$, then there exist two rows each of them bigger than the one following it. Using the branching rule (see [23, Thm. 2.8.3]), $\chi_\lambda|_{S_{n-1}}$ is the sum of at least two irreducible characters corresponding to $(n-1)$-partitions. At most one deleted box could be in the first row. Therefore we have to check both options: $\chi_\lambda(1) \geq \chi_{\nu(n-1,i-1)}(1) + \chi_{\mu(n-1,i-1)}(1)$ or $\chi_\lambda(1) \geq \chi_{\nu(n-1,i-1)}(1) + \chi_{\mu(n-1,i)}(1)$. We illustrate this by two examples:



By the induction hypothesis, we get

$$\chi_\lambda(1) \geq \min\{(n-i-1)2^{(i-1)/2} \cdot 2, (n-i-1)2^{(i-1)/2} + (n-i-2)2^{i/2}\}. \tag{3.2}$$

In the other case, we have $\lambda = (\lambda_1, \dots, \lambda_1)$. For computing $\chi_\lambda(1)$ we use the branching rule (see [23, Thm. 2.8.3]) twice. In the first step we cannot delete a box from the first row. In the second step, the integer i decreases by 2 or 1:

$$\chi_\lambda(1) = \chi_{\theta(n-1,i-1)} \geq \chi_{\nu(n-2,i-2)}(1) + \chi_{\mu(n-2,i-2)}(1)$$

or

$$\chi_\lambda(1) = \chi_{\theta(n-1,i-1)}(1) \geq \chi_{\nu(n-2,i-2)}(1) + \chi_{\mu(n-2,i-1)}(1),$$

respectively. Here are again two examples for the sake of illustration:

$$\ydiagram{5,5,5,5}(1) = \ydiagram{5,5,5,4}(1) = \ydiagram{5,5,5,3}(1) + \ydiagram{5,5,4,4}(1),$$

$$\ydiagram{8,8}(1) = \ydiagram{8,7}(1) = \ydiagram{8,6}(1) + \ydiagram{7,7}(1).$$

Thus we know by induction that

$$\chi_\lambda(1) \geq \min\{(n-i-1)2^{(i-2)/2} \cdot 2, (n-i-1)2^{(i-2)/2} + (n-i-2)2^{(i-1)/2}\}. \tag{3.3}$$

Since $n > 9$, by the assumption on k we have $n-i-1 = \lambda_1 > 3$. Therefore $\sqrt{2}(n-i-2) > (n-i-1)$. Now the inequalities (3.2) and (3.3) imply that $\chi_\lambda(1) \geq (n-i-1)2^{i/2}$. The minimum of the function $f(i) = (n-i-1)2^{i/2}$ on the interval $[0, n-4]$ is at $i = 0$. (In this way, we also have an alternative proof of the first part of the proposition.) Since the function $f(i)$ is increasing on the interval $[0, n-4]$, if we want to compute a lower bound of the degrees of irreducible characters corresponding to symmetric partitions, then by inequality (3.1) it is enough to find the symmetric partition where i is minimal. This minimum is either $(k, 1^{k-1})$ if n is $2k-1$, or $(k, 2, 1^{k-2})$ if n is $2k$. Consequently, $\min\{\chi_\lambda(1) | \lambda \text{ symmetric}\} \geq \frac{n}{2}2^{(n-3)/4} \geq 2^{n/4}$ for $n \geq 4$. $\square$

Corollary 3.9. *If χ is a non-linear, irreducible character of S_n for $n > 4$, then $\mathfrak{E}^G_{Ga}(\chi) \geq (n-1)^2 - 1$ and, if χ corresponds to a symmetric partition, then $\mathfrak{E}^G_{Ga}(\chi_\lambda) \geq 2^{n/2} - 1$ for $n > 2$.*

Here we give a proof of Conjecture 2.6 for S_n.

Lemma 3.10. *If χ is an irreducible character of S_n and the order of g does not divide $\frac{|S_n|}{\chi(1)}$, then $\chi(g) = 0$.*

Proof. Let λ be a partition of n and $\chi \in \mathrm{Irr}(S_n)$ the character corresponding to λ. Consider a permutation $g \in S_n$. We know, by the hook-length formula (see [23, Thm. 3.10.2]), that $\frac{|S_n|}{\chi(1)} = \prod_{h \in H(\lambda)} h$, where $H(\lambda)$ is the multi-set of the hook-lengths of λ. Then there exists a prime p and an exponent α with $p^\alpha \mid o(g)$ and $p^\alpha \nmid \prod_{h \in H(\lambda)} h$. Let us choose α to be maximal. Then g has a cycle of length divisible by p^α, however no hook-length in λ is divisible by p^α. Thus there is no rim-hook of length divisible by p^α, either. Hence, by the Murnaghan–Nakayama rule (see [23, Thm. 4.10.2]), we have $\chi(g) = 0$. $\square$

Consequently we can use Burnside's Theorem (see [11, Thm. 3.15]) and Lemma 3.10 to give two other lower estimates for $n^G(\chi)$ and $n^C(\chi)$, for a nonlinear, irreducible character χ of S_n:

$$\mathfrak{E}^G_{Co}(\chi) = \max\left\{ \left| \left\{ g \in G \mid o(g) \nmid \frac{|S_n|}{\chi(1)} \right\} \right|, \min_{C \in \mathrm{Cl}(G), |C| \neq 1} |C| \right\},$$

$$\mathfrak{E}^C_{Co}(\chi) = \max\left\{ \left| \left\{ C \in \mathrm{Cl}(S_n) \mid o(g) \nmid \frac{|S_n|}{\chi(1)} \text{ for all } g \in C \right\} \right|, 1 \right\}.$$

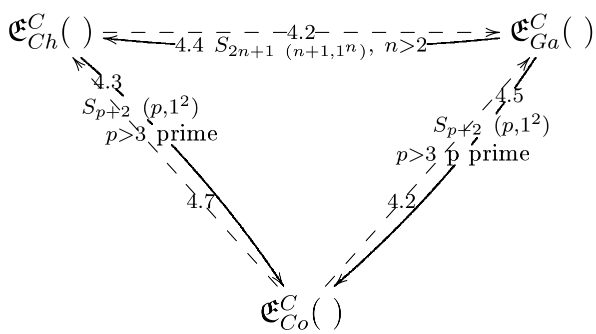
For linear characters, we define $\mathfrak{E}^G_{Co}(\chi) = \mathfrak{E}^C_{Co}(\chi) = 0$.

Of course, the inequalities $\mathfrak{E}^G_{Co}(\chi) \geq \mathfrak{E}^G_W(\chi)$ and $\mathfrak{E}^C_{Co}(\chi) \geq \mathfrak{E}^C_W(\chi)$ always hold. Thus we use $\mathfrak{E}^C_{Co}(\ ), \mathfrak{E}^G_{Co}(\ )$ instead of $\mathfrak{E}^C_W(\ ), \mathfrak{E}^G_W(\ )$ for symmetric groups.

4. Comparing estimates for symmetric groups

Theorem 4.1. *For each ordered pair of elements of the set $\{\mathfrak{E}^C_{Ch}, \mathfrak{E}^C_{Ga}, \mathfrak{E}^C_{Co}\}$, except for the pairs $(\mathfrak{E}^C_{Ga}, \mathfrak{E}^C_{Co})$, $(\mathfrak{E}^C_{Ga}, \mathfrak{E}^C_{Ch})$, and $(\mathfrak{E}^C_{Ch}, \mathfrak{E}^C_{Co})$, there are infinitely many symmetric groups having a non-linear, irreducible character, with the property that the first estimate is better then the second. In the remaining cases, there are just finitely many symmetric groups with this property.*

We illustrate the theorem in the following diagram. The broken arrow notation means that there are just finitely many symmetric groups where either $\mathfrak{E}^C_{Ch}(\chi) > \mathfrak{E}^C_{Co}(\chi)$ or $\mathfrak{E}^C_{Ga}(\chi) > \mathfrak{E}^C_{Ch}(\chi)$ or $\mathfrak{E}^C_{Ga}(\chi) > \mathfrak{E}^C_{Co}(\chi)$ for some irreducible character χ. Each ordered pair of estimates is handled in a separate proposition below.



We prove the theorem in Propositions 4.2–4.7.

Since for each nonlinear irreducible character $\chi \in \mathrm{Irr}(S_n)$ the estimates $\mathfrak{E}^C_{Co}(\chi)$ and $\mathfrak{E}^C_{Ch}(\chi)$ are at least 1, by Lemma 3.5, we obtain the following fact.

Proposition 4.2. *There are just finitely many symmetric groups having an irreducible character χ such that $\mathfrak{E}^C_{Ga}(\chi) > \mathfrak{E}^C_{Ch}(\chi)$ or $\mathfrak{E}^C_{Ga}(\chi) > \mathfrak{E}^C_{Co}(\chi)$.*

Despite of this, we show that there are infinitely many symmetric groups having an irreducible character χ such that $\mathfrak{E}^C_{Ch}(\chi) = \mathfrak{E}^C_{Ga}(\chi) = \mathfrak{E}^C_{Co}(\chi)$. For example, if $n > 5$ with $n-1$ being not prime, and, if in addition χ corresponds to the partition $(n-1, 1)$, then we get $\mathfrak{E}^C_{Ch}(\chi) = \mathfrak{E}^C_{Ga}(\chi) = \mathfrak{E}^C_{Co}(\chi) = 1$ (see Lemma 3.2, Proposition 4.6). However this value is smaller than $n^C(\chi)$.

Proposition 4.3. *Let p be a prime larger than 3. Then the symmetric group S_{p+2} possesses an irreducible character χ with $\mathfrak{E}^C_{Co}(\chi) > \mathfrak{E}^C_{Ch}(\chi)$ and $\mathfrak{E}^G_{Co}(\chi) > \mathfrak{E}^G_{Ch}(\chi)$.*

Proof. Let $\chi \in \mathrm{Irr}(S_{p+2})$ be the irreducible character corresponding to the partition $(p, 1^2)$. Since $(p, 1^2)$ does not contain a hook of length divisible by p, we have $p \nmid \frac{|S_{p+2}|}{\chi(1)}$. Thus

$$\mathfrak{E}^C_{Co}(\chi) \geq |\{C \in \mathrm{Cl}(S_{p+2}) \mid p \text{ divides } o(g) \text{ for all } g \in C\}|.$$

The set on the right-hand side contains the conjugacy classes corresponding to the partitions $(p, 2)$ and $(p, 1^2)$, respectively. Hence we have $\mathfrak{E}^C_{Co}(\chi) \geq 2$. Since $(p, 1^2)$ is not symmetric, by Lemma 3.2 we have $\mathfrak{E}^C_{Ch}(\chi) = 1$, and $\mathfrak{E}^G_{Ch}(\chi)$ is the size of a minimal noncentral conjugacy class. Thus the size of the conjugacy class $(p, 1^2)$ is at least $\mathfrak{E}^G_{Ch}(\chi)$. Therefore the second statement is trivially true. □

Proposition 4.4. *For $n > 2$, the symmetric group S_{2n+1} possesses an irreducible character χ with $\mathfrak{E}^C_{Ga}(\chi) < \mathfrak{E}^C_{Ch}(\chi)$.*

Proof. Let $\chi \in \operatorname{Irr}(S_{2n+1})$ be the character corresponding to the partition $(n+1, 1^n)$. Then we have $\mathfrak{E}^G_{Ga}(\chi) = \binom{2n}{n}^2 - 1$. By induction on n we can prove that, for $n > 2$, the latter quantity is at most $(2n+1)(2n-1)!$. Thus $\mathfrak{E}^C_{Ga}(\chi) = 1$. By Lemma 3.4, we have $\mathfrak{E}^C_{Ch}(\chi) = n$. So we are done. □

Proposition 4.5. *Let p be a prime larger than 3. Then the symmetric group S_{p+2} possesses an irreducible character χ with $\mathfrak{E}^C_{Co}(\chi) > \mathfrak{E}^C_{Ga}(\chi)$ and $\mathfrak{E}^G_{Co}(\chi) > \mathfrak{E}^G_{Ga}(\chi)$.*

Proof. Let $\chi \in \operatorname{Irr}(S_{p+2})$ correspond to the partition $(p, 1^2)$. Since $\mathfrak{E}^G_{Ga}(\chi) = \binom{p+1}{2}^2 - 1 \leq (p+2)p!$ for $p > 3$, we have $\mathfrak{E}^C_{Ga}(\chi) = 1$.

On the other hand, we have seen in Proposition 4.3 that

$$\left\{ C \in \operatorname{Cl}(S_{p+2}) \mid o(g) \nmid \frac{|G|}{\chi(1)} \text{ for all } g \in C \right\} \supseteq \left\{ (p, 1^2), (p, 2) \right\}.$$

Thus we have $\mathfrak{E}^C_{Co}(\chi) \geq 2$. So we are done with the first part.

We can easily check that $\chi^2(1) - 1 = \left\lfloor \frac{(p+1)p}{2} \right\rfloor^2 - 1 < |\operatorname{Cl}(p,2)| + |\operatorname{Cl}(p,1^2)| = \frac{(p+2)!}{p}$. This implies that $\mathfrak{E}^G_{Ga}(\chi) < \mathfrak{E}^G_{Co}(\chi)$. □

Proposition 4.6. *Let n be an integer such that $n-1$ is not a prime and $n > 5$. Then the symmetric group S_n possesses an irreducible character χ, with $\mathfrak{E}^C_{Ga}(\chi) = \mathfrak{E}^C_{Co}(\chi) = 1$ and $\mathfrak{E}^G_{Ga}(\chi) > \mathfrak{E}^G_{Co}(\chi)$.*

Proof. Let χ be the irreducible character of degree $n-1$ corresponding to the partition $(n-1, 1)$. We will prove that, for every group element g, $o(g)$ divides $\frac{|G|}{\chi(1)} = n \cdot (n-2)!$. Let us fix a prime p, and compute the exponents of p in the prime factorizations of $o(g)$ and $n \cdot (n-2)!$. Let l denote the exponent of p in $o(g)$. We have $l \leq k = \max\{j \mid p^j \leqslant n\}$. We will prove that k is at most the exponent of p in $n \cdot (n-2)!$. In order to that, we distinguish between three cases depending on the relative sizes of n and p^k.

If $n = p^k$, then the exponent of p in $n \cdot (n-2)!$ is $k + \left\lfloor \frac{n-2}{p} \right\rfloor + \cdots + \left\lfloor \frac{n-2}{p^{k-1}} \right\rfloor$, which is at least k.

If $n - 1 = p^k$, then $k > 1$ by our assumption, thus $\left\lfloor \frac{n-2}{p} \right\rfloor + \cdots + \left\lfloor \frac{n-2}{p^{k-1}} \right\rfloor \geqslant (p^{k-1} - 1) + \cdots + (p-1)$, and we can easily see by induction that $k \leqslant p^{k-1} - 1$, since $k = 2$ and $p = 2$ cannot hold at the same time.

If $n - 2 \geq p^k$, then the exponent of p in $n \cdot (n-2)!$ is larger than k.

Therefore $\mathfrak{E}^C_{Co}(\chi) = 1$ and $\mathfrak{E}^G_{Co}(\chi) = \min_{g \neq 1} |\operatorname{Cl}(g)| = \frac{n(n-1)}{2}$.

Furthermore, we have $\mathfrak{E}^G_{Ga}(\chi) = \chi^2(1) - 1 = (n-1)^2 - 1 = n(n-2)$. Since the size of the largest conjugacy class is $n(n-2)!$ (see Lemma 3.3), we know that $\mathfrak{E}^C_{Ga}(\chi) = 1$. This concludes the proof. □

Theorem 4.7. *There exist only finitely many symmetric groups having a non-linear, irreducible character χ with $\mathfrak{E}^C_{Ch}(\chi) > \mathfrak{E}^C_{Co}(\chi)$ or $\mathfrak{E}^G_{Ch}(\chi) > \mathfrak{E}^G_{Co}(\chi)$.*

Proof. Let us consider a symmetric group S_n with $n > 3$, and let χ be an irreducible character of S_n. If χ corresponds to a nonsymmetric partition, then $\mathfrak{E}^C_{Ch}(\chi)$ is 1 by Lemma 3.2 and $\mathfrak{E}^C_{Co}(\chi) \geq 1$. Therefore

$$\mathfrak{E}^C_{Ch}(\chi) \leq \mathfrak{E}^C_{Co}(\chi) \text{ and } \mathfrak{E}^G_{Ch}(\chi) \leq \mathfrak{E}^G_{Co}(\chi).$$

So at least one of $\mathfrak{E}^C_{Ch}(\chi) > \mathfrak{E}^C_{Co}(\chi)$ and $\mathfrak{E}^G_{Ch}(\chi) > \mathfrak{E}^G_{Co}(\chi)$ cannot hold.

Thus we may assume that the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ corresponding to χ is symmetric, so, by Lemma 3.4, we have $\mathfrak{E}^C_{Ch}(\chi) = \left\lceil \frac{n-1}{2} \right\rceil$. We prove the result in several steps.

Step 1. If there is a prime p with $p \nmid \frac{|G|}{\chi(1)}$, then

$$\mathfrak{E}^C_{Co}(\chi) \geq |\{\text{conjugacy classes containing a } p\text{-cycle}\}| = p(n-p).$$

(Here, $p(n)$ is the number of n-partitions.)

Step 2. If $n \geq 16$, then $\pi(3n) - \pi(2n) \geq 4$. (Here, $\pi(n)$ is the number of prime numbers, which are at most n.)

Proof of Step 2. We used a GAP program to check $\pi(3n)-\pi(2n) > 3$ for $16 \leq n \leq 1216$, see [22].

Using [1] we will now prove that, if $n > 1216$, then $\pi(3n) - \pi(2n) \geq 4$.

Let T_3 be the product of primes in the interval $(2n, 3n]$. By the proof of [1, Thm. 1.3], we have

$$T_3 > \left(\frac{6,5}{\sqrt{27}}\right)^n \frac{1}{(3n)^{\sqrt{3n}/2}} =: t(n).$$

It is enough to prove that $t(n) \geq (3n)^3$, since then we know that T_3 has at least 4 prime divisors, which means that $\pi(3n)-\pi(2n) \geq 4$. By taking the logarithm, we get what we have to prove, namely that $\log \frac{6,5}{\sqrt{27}} \geq \frac{\frac{\sqrt{3n}}{2}+3}{n} \log(3n)$. Since $\frac{\frac{\sqrt{3n}}{2}+3}{n} \log(3n)$ is a decreasing function for $n \geq 16$ and, if $n = 1217$, it is smaller than $\log \frac{6,5}{\sqrt{27}}$, we are done. □

Step 3. If $\left\lceil \frac{n}{4} \right\rceil \geq 16$ (equivalently, $n \geq 61$), we have

$$4 \leq \pi\left(3\left\lceil \tfrac{n}{4} \right\rceil\right) - \pi\left(2\left\lceil \tfrac{n}{4} \right\rceil\right) \leq \pi\left(3\left\lceil \tfrac{n}{4} \right\rceil\right) - \pi\left(\left\lceil \tfrac{n}{2} \right\rceil\right).$$

Step 4. We have $3 \geq \left|\left\{p \text{ prime} \mid p \geq \left\lceil \frac{n}{2} \right\rceil \text{ and } p \mid \frac{|G|}{\chi(1)} = \prod_{h \in H(\lambda)} h\right\}\right|$.

Proof of Step 4. We count the number of hook-lengths that are at least $\left\lceil \frac{n}{2} \right\rceil$.

Denote the hook-length of the i-th row and j-th column by $h_{i,j}(\lambda)$. Thus $h_{i,j}(\lambda) = h_{j,i}(\lambda) = \lambda_i + \lambda_j - i - j + 1$. Let $i_0 := \max\{k \mid \lambda_k \geq k\}$. Given this notation, a hook-length which is at least $\left\lceil \frac{n}{2} \right\rceil$ occurs only in the first i_0 rows and first i_0 columns.

The number of cells in the diagram is $n = 2 \cdot \sum_{j=1}^{i_0} \lambda_j - {i_0}^2$. Consider this sum as a function in i, and write $f(i) := 2 \cdot \sum_{j=1}^{i} \lambda_j - i^2$. Then it easy see that,

$$\text{if } i < i_0, \text{ then } f(i) < f(i+1).$$

If $h_{1,2}(\lambda) = \lambda_1 + \lambda_2 - 2 < \left\lceil \frac{n}{2} \right\rceil$, then only the hook-length $h_{1,1}(\lambda)$ could be at least $\left\lceil \frac{n}{2} \right\rceil$.

Assume that $h_{1,2}(\lambda) = \lambda_1 + \lambda_2 - 2 \geq \left\lceil \frac{n}{2} \right\rceil$. This means that i_0 is at least 2 and, if $i_0 > 2$, the above monotonicity of f implies

$$n = 2 \cdot \sum_{j=1}^{i_0} \lambda_j - {i_0}^2 > 2(\lambda_1 + \lambda_2) - 4 \geq 2\left\lceil \frac{n}{2} \right\rceil.$$

This is a contradiction. Thus $i_0 \leq 2$, and consequently there are at most 3 different hook-lengths that are at least $\left\lceil \frac{n}{2} \right\rceil$. Clearly, there cannot be more prime divisors with the desired properties. □

Step 5. We know by Steps 3 and 4 that, if $n \geq 61$, there exists a prime $p \in \left(\left\lceil \frac{n}{2} \right\rceil, 3\left\lceil \frac{n}{4} \right\rceil\right]$ such that $p \nmid \frac{|S_n|}{\chi(1)}$, and so for this prime p we have

$$\mathfrak{E}^C_{Co}(\chi) \geq p(n-p) \geq p\left(n - 3\left\lceil \frac{n}{4} \right\rceil\right) \geq p\left(\left\lfloor \frac{n}{4} \right\rfloor - 2\right).$$

According to [15, Cor. 3.1], we have $p(k) > \frac{e^{2\sqrt{k}}}{14}$. Therefore

$$\mathfrak{E}^C_{Co}(\chi) > \frac{e^{2\sqrt{(\left\lfloor \frac{n}{4} \right\rfloor - 2)}}}{14} \geq \frac{e^{\sqrt{n-11}}}{14}, \quad \text{for } n \geq 61.$$

Step 6. Since $\mathfrak{E}^C_{Co}(\chi) \geq \frac{e^{\sqrt{n-11}}}{14}$ and $\mathfrak{E}^C_{Ch}(\chi) = \left\lceil \frac{n-1}{2} \right\rceil \leq \frac{n}{2}$, it is enough to prove that $\frac{n}{2} \leq \frac{e^{\sqrt{n-11}}}{14}$. This inequality holds for $n \geq 44$.

Thus, for $n \geq 61$, we have

$$\mathfrak{E}^C_{Co}(\chi) \geq \mathfrak{E}^C_{Ch}(\chi).$$

Since in $\mathfrak{E}^G_{Ch}(\chi)$ we summed up the sizes of the $\mathfrak{E}^C_{Ch}(\chi)$ smallest different non-central conjugacy classes and in $\mathfrak{E}^G_{Co}$ we summed up the sizes of $\mathfrak{E}^C_{Co}(\chi)$ different non-central conjugacy classes, the inequality $\mathfrak{E}^G_{Ch}(\chi) \leq \mathfrak{E}^G_{Co}(\chi)$ holds as well for $n \geq 61$. □

Theorem 4.8. *There exist only finitely many symmetric groups having a non-linear, irreducible character χ, with $\mathfrak{E}^G_{Ch}(\chi) > \mathfrak{E}^G_{Ga}(\chi)$.*

Proof. If $n \leq 4$, then $\mathfrak{E}^G_{Ch}(\chi) \leq \mathfrak{E}^G_{Ga}(\chi)$. Let χ be an irreducible character of S_n corresponding to a symmetric partition. Then, by Lemma 3.4, we have $\mathfrak{E}^C_{Ch}(\chi) = \left\lceil \frac{n-1}{2} \right\rceil$. Hence $\mathfrak{E}^G_{Ch}(\chi)$ is less than the sum of the sizes of $\left\lceil \frac{n-1}{2} \right\rceil + 1 \leq \frac{n+2}{2}$ different conjugacy classes. Let k be the smallest number having at least $\frac{n+2}{2}$ partitions.

According to [15, Cor. 3.1], we have $p(k) > \frac{e^{2\sqrt{k}}}{14}$. Thus,

$$k \leq \frac{\log^2(7n+14)}{4} + 1.$$

Let $P(k)$ denote the set of partitions of k. For every $\lambda \in P(k)$, consider the conjugacy class $(\lambda, 1^{n-k})$. Let $\mathrm{Cl}(S_n) = \{C_i \mid i \in I\}$. Then we have

$$\mathfrak{E}^G_{Ch}(\chi) = \min_{\{J \subseteq I \mid \ |J| = \lceil \frac{n-1}{2} \rceil\}} \left\{\sum_{j \in J} |C_j|\right\} - 1 < \min_{\{J \subseteq I \mid \ |J| = \frac{n+2}{2}\}} \sum_{j \in J} |C_j|,$$

$$\mathfrak{E}^G_{Ch}(\chi) \leq \min_{\{J \subseteq I \mid \ |J| = p(k)\}} \sum_{j \in J} |C_j| \leq \sum_{\lambda \in P(k)} |\operatorname{Cl}_{S_n}(\lambda, 1^{n-k})|.$$

Using the inequality $|\operatorname{Cl}_{S_n}(\lambda, 1^{n-k})| \leq \frac{n!}{k!(n-k)!} |\operatorname{Cl}_{S_k}(\lambda)|$, we get

$$\mathfrak{E}^G_{Ch}(\chi) < \frac{n!}{k!(n-k)!} \sum_{\lambda \in P(k)} |\operatorname{Cl}_{S_k}(\lambda)| = \frac{n!}{(n-k)!} < n^k.$$

Thus

$$\mathfrak{E}^G_{Ch}(\chi) \leq \frac{n!}{(n-k)!} - 1 < n^{\frac{\log^2(7n+14)}{4} + 1} - 1.$$

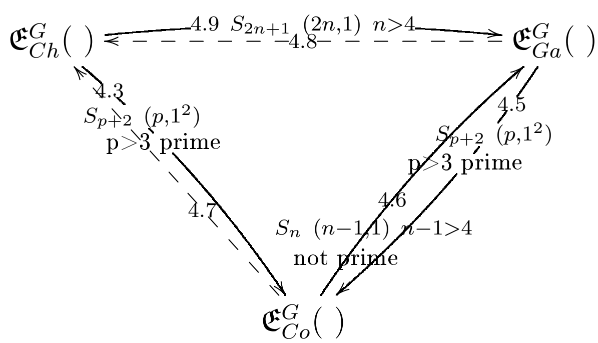
Furthermore, in Corollary 3.9 we have seen that $2^{n/2} - 1 \leq \mathfrak{E}^G_{Ga}(\chi)$. Since $n^{\frac{\log^2(7n+14)}{4}+1} - 1 \leq 2^{n/2} - 1$ holds for $n > 230$, the inequality $\mathfrak{E}^G_{Ga}(\chi) \geq \mathfrak{E}^G_{Ch}(\chi)$ holds for $\chi \in \operatorname{Irr}(S_n)$

and $n > 230$. If χ does not correspond to a symmetric partition, then by Lemma 3.2 we have $\mathfrak{C}^C_{Ch}(\chi) = 1$. Thus $\mathfrak{C}^G_{Ch}(\chi) \leq |\operatorname{Cl}(2, 1^{n-2})| = \frac{n(n-1)}{2}$. By Proposition 3.9, we conclude $\mathfrak{C}^G_{Ga}(\chi) \geq n(n-2)$. Therefore we are done. □

By the above results concerning the vanishing group elements, we have the following fact.

Remark 4.9. There are infinitely many symmetric groups which have a non-linear, irreducible character such that $\mathfrak{C}^G_{Ch}(\chi) < \mathfrak{C}^G_{Ga}(\chi)$. For example, let $n > 4$ and χ be the irreducible character of S_{2n+1} corresponding to $(2n, 1)$. Then $\mathfrak{C}^G_{Ga}(\chi) = (2n)^2 - 1$ and $\mathfrak{C}^G_{Ch}(\chi) \leq |\operatorname{Cl}(2, 1^{2n-1})| = (2n+1)n$.

Theorem 4.10. *For each ordered pair from $\{\mathfrak{C}^G_{Ch}, \mathfrak{C}^G_{Ga}, \mathfrak{C}^G_{Co}\}$ there are infinitely many groups, having an irreducible character, for which the second estimate is better then the first, except for the pairs $(\mathfrak{C}^G_{Ga}, \mathfrak{C}^G_{Ch})$, $(\mathfrak{C}^G_{Co}, \mathfrak{C}^G_{Ch})$.*



5. Acknowledgements

I thank my advisor Erzsébet Horváth for proposing this problem to work on, and for all the guidance and support throughout. I also thank Sándor Kiss for the help given in connection with number theory issues. I express my gratitude to Christine Bessenrodt, Mihály Szalay and Attila Maróti for their advice.

References

[1] Bachraoui, M. E. Primes in the interval $[2n, 3n]$, *Int. J. Contemp. Math. Sci.* **1** (2006), 617–621.
[2] Berkovich, Y., Kazarin, L. Finite groups in which the zeros of every nonlinear irreducible character are conjugate modulo its kernel, *Houston Journal of Mathematics* **24** (1998), 619–630.
[3] Bianchi, M., Chillag, D., Gillio, A. Finite groups in which every irreducible character vanishes on at most two conjugacy classes, *Houston Journal of Mathematics* **26** (2000), 451–461.
[4] Chillag, D. On zeros of characters of finite groups, *Proceedings of the American Mathematical Society* **127** (1999), 977–983.
[5] Dolfi, S., Navarro, G., Pacifici, E. Sanus, P., Tiep, P. H. Non-vanishing elements of finite groups, *Journal of Algebra* **323** (2010), 540–545.
[6] Dolfi, S., Pacifici, E., Sanus, L., Spiga, P. On the orders of zeros of irreducible characters, *Journal of Algebra* **321** (2009), 345–352.
[7] Gagola, Jr., S. M. Characters vanishing on all but two conjugacy classes, *Pacific Journal of Mathematics* **109** (1983), 363–385.
[8] Gallagher, P. X. Group characters and commutators, *Mathematische Zeitschrift* **79** (1962), 122–126.
[9] Gallagher, P. X. Zeros of characters of finite groups, *Journal of Algebra* **4** (1966), 42–45.

[10] The GAP Group, GAP — Groups, Algorithms and Programs, Version 4.4.12, Aachen, St. Andrews 120, `http://www.gap-system.org`, 2008.

[11] Isaacs, I. M. (1994) *Character Theory of Finite Groups*, Dover, New York.

[12] Isaacs, I. M., Navarro, G., Wolf, T. R. Finite group elements where no irreducible character vanishes, *Journal of Algebra* **222** (1999), 413–423.

[13] James, G., Kerber, A. (1981) *The Representation Theory of the Symmetric Group*, London, Amsterdam.

[14] Malle, G., Navarro, G., Olsson, J. B. Zeros of characters of finite groups, *Journal of Group Theory* **3** (2000), 353–368.

[15] Maróti, A. On elementary lower bounds for the partition function, *Integers (Electronic Journal of Combinatorial Number Theory)* **3** (2003), 1–9.

[16] McKay, J. The largest degrees of irreducible characters of symmetric groups, *Mathematics of Computation* **30** (1976), 624–631.

[17] Moretó, A., Sangroniz, J. On the number of conjugacy classes of zeros of characters, *Israel Journal of Mathematics* **142** (2004), 163–187.

[18] Moretó, A., Sangroniz, J. On the number of zeros in the columns of the character table of a group, *Journal of Algebra* **279** (2004), 726–736.

[19] Navarro, G. Zeros of primitive characters in solvable groups, *Journal of Algebra* **221** (1999) 644–650.

[20] Olsson, J. B. (1993) *Combinatorics and Representations of Finite Groups*, Vorlesungen aus dem Fachbereich Mathematik der Universität Essen 20.

[21] Qian, G. Bounding the Fitting height of a solvable group by the number of zeros in a character table, *Proceedings of the American Mathematical Society* **130** (2002), 3171–3176.

[22] Petényi, F. Programs, `www.math.bme.hu/~pfranci/programs.html`.

[23] Sagan, B. E. (2001) *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, Springer, New York.

[24] Vershik, A. M., Kerov, S. V. Asymptotic of the largest and typical dimensions of irreducible representations of a symmetric group, *Functional analysis and its applications* **19** (1985), 21–31.

[25] Wilde, T. Orders of elements and zeros and heights of characters in a finite group, arχiv.org/abs/math/0604337 (2006).

[26] Zhmud, É. M. On finite groups all irreducible characters of which take at most two nonzero values, *Ukrainian Mathematical Journal* **47** (1995), 1308–1313.

[27] Zhmud, É. M. On estimates of the number of zeros of irreducible characters of a finite group, *Publicationes Mathematicae Debrecen* **33** (1986), 125–146.

[28] Zhmud, É. M. The zeros of group characters, *Uspekhi Matematicheskikh Nauk* **32** (1977), 223–224.

Department of Algebra, Budapest University of Technology and Economics, H-1111 Budapest, Műegyetem rkp. 3–9., Hungary

E-mail address: `petenyi.franciska@gmail.com`