Séminaire Lotharingien de Combinatoire **68** (2012), Article B68a

QUADRANT MARKED MESH PATTERNS IN ALTERNATING PERMUTATIONS

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Abstract. This paper is a continuation of the systematic study of the distribution of quadrant marked mesh patterns initiated by the authors in [*J. Integer Sequences* **12** (2012), Article 12.4.7]. We study quadrant marked mesh patterns on up-down and down-up permutations, also known as alternating and reverse alternating permutations, respectively. In particular, we refine classical enumeration results of André [*C. R. Acad. Sci. Paris* **88** (1879), 965–967; *J. Math. Pur. Appl.* **7** (1881), 167–184] on alternating permutations by showing that the distribution with respect to the quadrant marked mesh pattern of interest is given by $(\sec(xt))^{1/x}$ on up-down permutations of even length and by $\int_0^t (\sec(xz))^{1+\frac{1}{x}}dz$ on down-up permutations of odd length.

1. Introduction

The notion of mesh patterns was introduced by Brändén and Claesson [4] to provide explicit expansions for certain permutation statistics as, possibly infinite, linear combinations of (classical) permutation patterns (see [6] for a comprehensive introduction to the theory of permutation patterns). This notion was further studied in [3, 5, 7, 9, 10, 13].

Let $\sigma = \sigma_1 \ldots \sigma_n$ be a permutation in the symmetric group S_n written in one-line notation. Then we will consider the graph of σ, $G(\sigma)$, to be the set of points (i, σ_i) for $i = 1, \ldots, n$. For example, the graph of the permutation $\sigma = 471569283$ is shown in Figure 1. Then, if we draw a coordinate system centered at a point (i, σ_i), we will be interested in the points that lie in the four quadrants I, II, III, and IV of that coordinate system as shown in Figure 1. For any $a, b, c, d \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of natural numbers and any $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we say that σ_i matches the quadrant marked mesh pattern $MMP(a, b, c, d)$ in σ if in $G(\sigma)$ relative to the coordinate system which has the point (i, σ_i) as its origin, there are $\geq a$ points in quadrant I, $\geq b$ points in quadrant II, $\geq c$ points in quadrant III, and $\geq d$ points in quadrant IV. For example, if $\sigma = 471569283$,

2010 *Mathematics Subject Classification.* 05A15, 05E05.

Key words and phrases. permutation statistics, marked mesh pattern, distribution.

the point $\sigma_4 = 5$ matches the quadrant marked mesh pattern $MMP(2,1,2,1)$ since relative to the coordinate system with origin $(4,5)$, there are 3 points of $G(\sigma)$ in quadrant I, there is 1 point of $G(\sigma)$ in quadrant II, there are 2 points in quadrant III, and 2 points in quadrant IV. Note that, if a coordinate in $MMP(a,b,c,d)$ is 0, then there is no condition imposed on the points in the corresponding quadrant. In addition, we shall consider patterns $MMP(a,b,c,d)$, where $a,b,c,d \in \mathbb{N} \cup \{\emptyset\}$. Here, when one of the parameters a, b, c, or d in $MMP(a,b,c,d)$ is the empty set, then, for σ_i to match $MMP(a,b,c,d)$ in $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, it must be the case that there are no points in $G(\sigma)$ relative to the coordinate system with origin (i, σ_i) in the corresponding quadrant. For example, if $\sigma = 471569283$, the point $\sigma_3 = 1$ matches the marked mesh pattern $MMP(4,2,\emptyset,\emptyset)$ since, relative to the coordinate system with origin $(3,1)$, there are 6 points of $G(\sigma)$ in quadrant I, 2 points in quadrant II, no points in quadrant III, and no points in quadrant IV. We let $\mathrm{mmp}^{(a,b,c,d)}(\sigma)$ denote the number of i such that σ_i matches the marked mesh pattern $MMP(a,b,c,d)$ in σ.

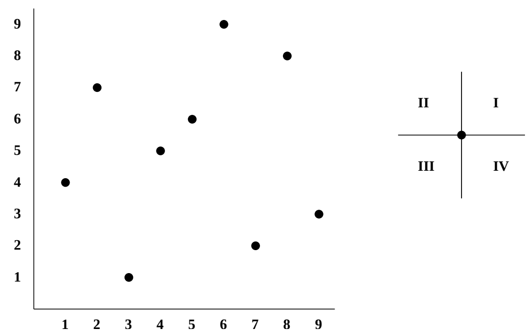
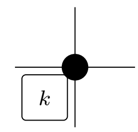
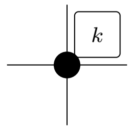


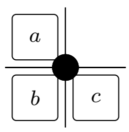
FIGURE 1. The graph of $\sigma = 471569283$.

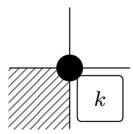
Note how the (two-dimensional) notation of Úlfarsson [13] for marked mesh patterns corresponds to our (one-line) notation for quadrant marked mesh patterns. For example,

$MMP(0,0,k,0) =$ 

, $MMP(k,0,0,0) =$ 

,

$MMP(0,a,b,c) =$ 

and $MMP(0,0,\emptyset,k) =$ 

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Kitaev and Remmel [7] studied the distribution of quadrant marked mesh patterns in the symmetric group S_n, and Kitaev, Remmel, and Tiefenbruck [9, 10] studied the distribution of quadrant marked mesh patterns in 132-avoiding permutations in S_n. The main goal of this paper is to study the distribution of the statistics $\mathrm{mmp}^{(1,0,0,0)}$, $\mathrm{mmp}^{(0,1,0,0)}$, $\mathrm{mmp}^{(0,0,1,0)}$, and $\mathrm{mmp}^{(0,0,0,1)}$ in the set of *up-down* and *down-up permutations*. We say that $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ is an up-down permutation if it is of the form

$$\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \sigma_5 < \cdots ,$$

and σ is a down-up permutation if it is of the form

$$\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \sigma_5 > \cdots .$$

Let UD_n denote the set of all up-down permutations in S_n and DU_n denote the set of all down-up permutations in S_n. Given a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we define the reverse of σ, σ^r, to be $\sigma_n\sigma_{n-1}\ldots\sigma_1$ and the complement of σ, σ^c, to be $(n+1-\sigma_1)(n+1-\sigma_2)\ldots(n+1-\sigma_n)$. For $n \geq 1$, we let

$$A_{2n}(x) = \sum_{\sigma \in UD_{2n}} x^{\mathrm{mmp}^{(1,0,0,0)}(\sigma)}, \qquad B_{2n-1}(x) = \sum_{\sigma \in UD_{2n-1}} x^{\mathrm{mmp}^{(1,0,0,0)}(\sigma)},$$
$$C_{2n}(x) = \sum_{\sigma \in DU_{2n}} x^{\mathrm{mmp}^{(1,0,0,0)}(\sigma)}, \text{ and } D_{2n-1}(x) = \sum_{\sigma \in DU_{2n-1}} x^{\mathrm{mmp}^{(1,0,0,0)}(\sigma)}.$$

We then have the following simple proposition.

Proposition 1. *For all $n \geq 1$,*

(1) $A_{2n}(x) = \sum_{\sigma \in DU_{2n}} x^{\mathrm{mmp}^{(0,1,0,0)}(\sigma)} = \sum_{\sigma \in DU_{2n}} x^{\mathrm{mmp}^{(0,0,0,1)}(\sigma)} = \sum_{\sigma \in UD_{2n}} x^{\mathrm{mmp}^{(0,0,1,0)}(\sigma)}$,

(2) $C_{2n}(x) = \sum_{\sigma \in UD_{2n}} x^{\mathrm{mmp}^{(0,1,0,0)}(\sigma)} = \sum_{\sigma \in UD_{2n}} x^{\mathrm{mmp}^{(0,0,0,1)}(\sigma)} = \sum_{\sigma \in DU_{2n}} x^{\mathrm{mmp}^{(0,0,1,0)}(\sigma)}$,

(3) $B_{2n-1}(x) = \sum_{\sigma \in UD_{2n-1}} x^{\mathrm{mmp}^{(0,1,0,0)}(\sigma)} = \sum_{\sigma \in DU_{2n-1}} x^{\mathrm{mmp}^{(0,0,0,1)}(\sigma)} = \sum_{\sigma \in DU_{2n-1}} x^{\mathrm{mmp}^{(0,0,1,0)}(\sigma)}$,

and

(4) $D_{2n-1}(x) = \sum_{\sigma \in DU_{2n-1}} x^{\mathrm{mmp}^{(0,1,0,0)}(\sigma)} = \sum_{\sigma \in UD_{2n-1}} x^{\mathrm{mmp}^{(0,0,0,1)}(\sigma)} = \sum_{\sigma \in UD_{2n-1}} x^{\mathrm{mmp}^{(0,0,1,0)}(\sigma)}$.

Proof. It is easy to see that for any $\sigma \in S_n$,

$$\mathrm{mmp}^{(1,0,0,0)}(\sigma) = \mathrm{mmp}^{(0,1,0,0)}(\sigma^r) = \mathrm{mmp}^{(0,0,0,1)}(\sigma^c) = \mathrm{mmp}^{(0,0,1,0)}((\sigma^r)^c).$$

Then part 1 easily follows since

$$\sigma \in UD_{2n} \iff \sigma^r \in DU_{2n} \iff \sigma^c \in DU_{2n} \iff (\sigma^r)^c \in UD_{2n}. \tag{1.1}$$

Parts 2, 3, and 4 are proved in a similar manner. □

It follows from Proposition 1 that the study of the distribution of the statistics $\mathrm{mmp}^{(1,0,0,0)}$, $\mathrm{mmp}^{(0,1,0,0)}$, $\mathrm{mmp}^{(0,0,1,0)}$, and $\mathrm{mmp}^{(0,0,0,1)}$ in the set of up-down and down-up permutations can be reduced to the study of the generating functions

$$A(t,x) = 1 + \sum_{n\geq 1} A_{2n}(x)\frac{t^{2n}}{(2n)!},$$
$$B(t,x) = \sum_{n\geq 1} B_{2n-1}(x)\frac{t^{2n-1}}{(2n-1)!},$$
$$C(t,x) = 1 + \sum_{n\geq 1} C_{2n}(x)\frac{t^{2n}}{(2n)!},$$

and

$$D(t,x) = \sum_{n\geq 1} D_{2n-1}(x)\frac{t^{2n-1}}{(2n-1)!}.$$

In the case when $x = 1$, these generating functions are well known. That is, the operation of complementation shows that $A_{2n}(1) = C_{2n}(1)$ and $B_{2n-1}(1) = D_{2n-1}(1)$ for all $n \geq 1$, and André [1, 2] proved that

$$\sum_{n\geq 0} A_{2n}(1)\frac{t^{2n}}{(2n)!} = \sec(t)$$

and

$$\sum_{n\geq 1} B_{2n-1}(1)\frac{t^{2n-1}}{(2n-1)!} = \tan(t).$$

Thus, the number of up-down permutations is given by the exponential generating function

$$\sec(t) + \tan(t) = \tan\left(\frac{t}{2} + \frac{\pi}{4}\right). \tag{1.2}$$

We shall prove the following theorem.

Theorem 1. *We have*

$$\begin{aligned} A(t,x) &= (\sec(xt))^{1/x}, \\ B(t,x) &= (\sec(xt))^{1/x}\int_0^t (\sec(xz))^{-1/x}dz, \\ C(t,x) &= 1 + \int_0^t (\sec(xy))^{1+\frac{1}{x}}\int_0^y (\sec(xz))^{1/x}dz\ dy, \end{aligned}$$

and

$$D(t,x) = \int_0^t (\sec(xz))^{1+\frac{1}{x}}dz.$$

As an immediate corollary of Theorem 1, we get, for example, that the bivariate exponential generating function for up-down permutations, where the variable x keeps track of occurrences of $MMP(1,0,0,0)$, is given by

$$A(t,x) + B(t,x) = (\sec(xt))^{1/x}\left(1 + \int_0^t (\sec(xz))^{-1/x}dz\right),$$

which refines (1.2).

One can use these generating functions to find some initial values of the polynomials $A_{2n}(x)$, $B_{2n-1}(x)$, $C_{2n}(x)$, and $D_{2n-1}(x)$. For example, we have used *Mathematica* to compute the following tables.

n	$A_{2n}(x)$
0	1
1	x
2	$x^2(3+2x)$
3	$x^3\left(15+30x+16x^2\right)$
4	$x^4\left(105+420x+588x^2+272x^3\right)$
5	$x^5\left(945+6300x+16380x^2+18960x^3+7936x^4\right)$
6	$x^6\left(10395+103950x+429660x^2+893640x^3+911328x^4+353792x^5\right)$

n	$B_{2n-1}(x)$
1	1
2	$2x$
3	$8x^2(1+x)$
4	$16x^3\left(3+8x+6x^2\right)$
5	$128x^4\left(3+15x+27x^2+17x^3\right)$
6	$256x^5\left(15+120x+381x^2+556x^3+310x^4\right)$
7	$1024x^6\left(45+525x+2562x^2+6420x^3+8146x^4+4146x^5\right)$

n	$C_{2n}(x)$
0	1
1	1
2	$x(2+3x)$
3	$x^2\left(8+28x+25x^2\right)$
4	$x^3\left(48+296x+614x^2+427x^3\right)$
5	$x^4\left(384+3648x+13104x^2+20920x^3+12465x^4\right)$
6	$x^5\left(3840+51840x+282336x^2+769072x^3+1039946x^4+555731x^5\right)$

n	$D_{2n-1}(x)$
1	1
2	$x(1+x)$
3	$x^2\left(3+8x+5x^2\right)$
4	$x^3\left(15+75x+121x^2+61x^3\right)$
5	$x^4\left(105+840x+2478x^2+3128x^3+1385x^4\right)$
6	$x^5\left(945+11025x+51030x^2+115350x^3+124921x^4+50521x^5\right)$
7	$x^6(10395+166320x+1105335x^2+3859680x^3$ $+7365633x^4+7158128x^5+2702765x^6)$

The outline of this paper is as follows. In Section 2, we shall prove Theorem 1. Then, in Section 3, we shall study the entries of the tables above explaining them either explicitly or through recursions.

2. Proof of Theorem 1

The proof of all parts of Theorem 1 proceeds in the same manner. Namely, there are simple recursions satisfied by the polynomials $A_{2n}(x)$, $B_{2n+1}(x)$, $C_{2n}(x)$, and $D_{2n+1}(x)$ based on the position of the largest value in the permutation.

2.1. **The generating function** $A(t,x)$**.** If $\sigma = \sigma_1 \ldots \sigma_{2n} \in UD_{2n}$, then $2n$ must occur in one of the positions $2, 4, \ldots, 2n$. Let $UD_{2n}^{(2k)}$ denote the set of permutations $\sigma \in UD_{2n}$ such that $\sigma_{2k} = 2n$. A schematic diagram of an element in $UD_{2n}^{(2k)}$ is shown in Figure 2.

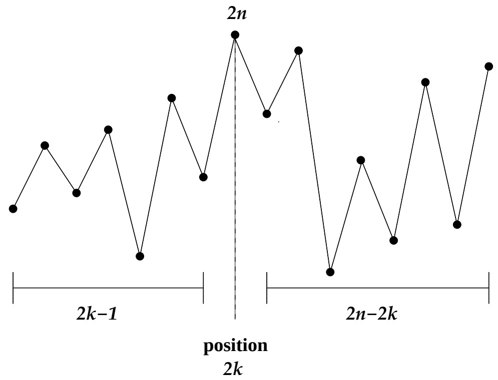

Figure 2. The graph of a permutation $\sigma \in UD_{2n}^{(2k)}$.

Note that there are $\binom{2n-1}{2k-1}$ ways to pick the elements which occur to the left of position $2k$ in such a permutation σ and there are $B_{2k-1}(1)$ ways to order them since the elements to the left of position $2k$ form an up-down permutation of length $2k-1$. Each of the elements to the left of position $2k$ contributes to $\mathrm{mmp}^{(1,0,0,0)}(\sigma)$. Thus the contribution of the elements to the left of position $2k$ to $\sum_{\sigma \in UD_{2n}^{(2k)}} x^{\mathrm{mmp}^{(1,0,0,0)}(\sigma)}$ is $B_{2k-1}(1)x^{2k-1}$. There are $A_{2n-2k}(1)$ ways to order the elements to the right of position $2k$ since they must form an up-down permutation of length $2n-2k$. Since the elements to the left of position $2k$ have no effect on whether an element to the right of position $2k$ contributes to $\mathrm{mmp}^{(1,0,0,0)}(\sigma)$, it follows that the contribution of the elements to the right of position $2k$ to $\sum_{\sigma \in UD_{2n}^{(2k)}} x^{\mathrm{mmp}^{(1,0,0,0)}(\sigma)}$ is $A_{2n-2k}(x)$. As a consequence, we obtain

$$A_{2n}(x) = \sum_{k=1}^{n} \binom{2n-1}{2k-1} B_{2k-1}(1) x^{2k-1} A_{2n-2k}(x),$$

or, equivalently,

$$\frac{A_{2n}(x)}{(2n-1)!} = \sum_{k=1}^{n} \frac{B_{2k-1}(1)x^{2k-1}}{(2k-1)!} \frac{A_{2n-2k}(x)}{(2n-2k)!}. \tag{2.1}$$

Multiplying both sides of (2.1) by t^{2n-1} and summing the result over $n \geq 1$, we see that

$$\sum_{n\geq 1} \frac{A_{2n}(x)t^{2n-1}}{(2n-1)!} = \left(\sum_{n\geq 1} \frac{B_{2n-1}(1)x^{2n-1}t^{2n-1}}{(2n-1)!}\right)\left(\sum_{n\geq 0} \frac{A_{2n}(x)t^{2n}}{(2n)!}\right).$$

By André's result, we have

$$\sum_{n\geq 1} \frac{B_{2n-1}(1)x^{2n-1}t^{2n-1}}{(2n-1)!} = \tan(xt),$$

so that

$$\frac{\partial}{\partial t}A(t,x) = \tan(xt)A(t,x).$$

Our initial condition is that $A(0,x) = 1$. It is easy to check that the solution to this differential equation is

$$A(t,x) = (\sec(xt))^{1/x}.$$

2.2. **The generating function** $B(t,x)$**.** If $\sigma = \sigma_1 \ldots \sigma_{2n+1} \in UD_{2n+1}$, then $2n+1$ must occur in one of the positions $2, 4, \ldots, 2n$. Let $UD_{2n+1}^{(2k)}$ denote the set of permutations $\sigma \in UD_{2n+1}$ such that $\sigma_{2k} = 2n+1$. A schematic diagram of an element in $UD_{2n}^{(2k)}$ is shown in Figure 3.

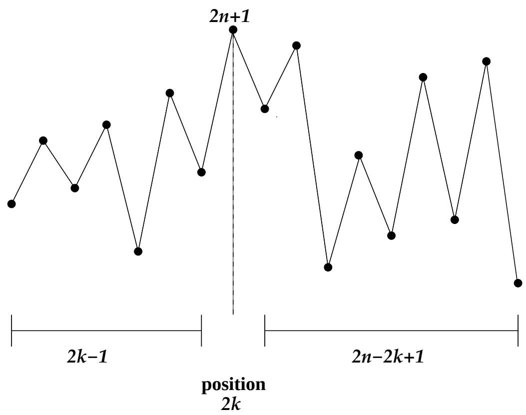

FIGURE 3. The graph of a permutation $\sigma \in UD_{2n+1}^{(2k)}$.

Again there are $\binom{2n}{2k-1}$ ways to pick the elements which occur to the left of position $2k$ in such a permutation σ, and the contribution of the elements to the left of position $2k$ to $\sum_{\sigma \in UD_{2n+1}^{(2k)}} x^{\mathrm{mmp}^{(1,0,0,0)}(\sigma)}$ is $B_{2k-1}(1)x^{2k-1}$. There are $B_{2n-2k+1}(1)$ ways to order the elements to the right of position $2k$ since they must form an up-down permutation of length $2n-2k+1$. Since the elements to the left of position $2k$ have no effect on whether an element to the right of position $2k$ contributes to $\mathrm{mmp}^{(1,0,0,0)}(\sigma)$, it follows that the contribution of the elements to the right of position $2k$ to $\sum_{\sigma \in UD_{2n+1}^{(2k)}} x^{\mathrm{mmp}^{(1,0,0,0)}(\sigma)}$ is $B_{2n-2k+1}(x)$. Consequently, if $n \geq 1$, then

$$B_{2n+1}(x) = \sum_{k=1}^{n} \binom{2n}{2k-1} B_{2k-1}(1)x^{2k-1}B_{2n-2k+1}(x).$$

Hence, for $n \geq 1$, we have

$$\frac{B_{2n+1}(x)}{(2n)!} = \sum_{k=1}^{n} \frac{B_{2k-1}(1)x^{2k-1}}{(2k-1)!} \frac{B_{2n-2k+1}(x)}{(2n-2k+1)!}. \tag{2.2}$$

Multiplying both sides of (2.2) by t^{2n}, summing the result over $n \geq 1$, and taking into account that $B_1(x) = 1$, we see that

$$\sum_{n\geq 0} \frac{B_{2n+1}(x)t^{2n}}{(2n)!} = 1 + \left(\sum_{n\geq 0} \frac{B_{2n+1}(1)x^{2n+1}t^{2n+1}}{(2n+1)!}\right)\left(\sum_{n\geq 0} \frac{B_{2n+1}(x)t^{2n+1}}{(2n+1)!}\right).$$

Since

$$\sum_{n\geq 1} \frac{B_{2n-1}(1)x^{2n-1}t^{2n-1}}{(2n-1)!} = \tan(xt),$$

we see that

$$\frac{\partial}{\partial t}B(t,x) = 1 + \tan(xt)B(t,x).$$

Our initial condition is that $B(0,x) = 0$. It is easy to check that the solution to this differential equation is

$$B(t,x) = (\sec(xt))^{1/x} \int_0^t (\sec(xz))^{-1/x} dz.$$

2.3. **The generating function $C(t,x)$.** If $\sigma = \sigma_1 \dots \sigma_{2n} \in DU_{2n}$, then $2n$ must occur in one of the positions $1, 3, \dots, 2n-1$. Let $DU_{2n}^{(2k+1)}$ denote the set of permutations $\sigma \in DU_{2n}$ such that $\sigma_{2k+1} = 2n$. A schematic diagram of an element in $DU_{2n}^{(2k+1)}$ is shown in Figure 4.

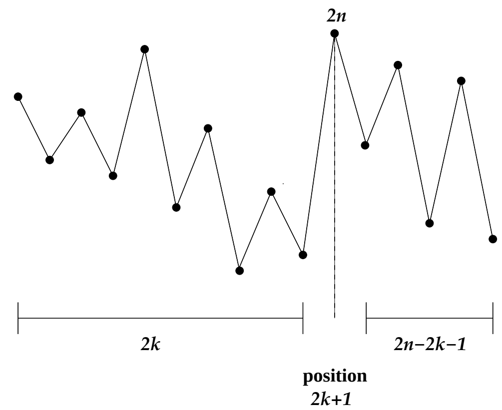

FIGURE 4. The graph of a permutation $\sigma \in DU_{2n}^{(2k+1)}$.

Note that there are $\binom{2n-1}{2k}$ ways to pick the elements which occur to the left of position $2k+1$ in such a permutation σ, and there are $C_{2k}(1) = A_{2k}(1)$ ways to order them since the elements to the left of position $2k+1$ form a down-up permutation of length $2k$. Each of the elements to the left of position $2k+1$ contributes to $\mathrm{mmp}^{(1,0,0,0)}(\sigma)$. Thus the contribution of the elements to the left of position $2k+1$ to $\sum_{\sigma \in UD_{2n}^{(2k)}} x^{\mathrm{mmp}^{(1,0,0,0)}(\sigma)}$ is $A_{2k}(1)x^{2k}$. There are $B_{2n-2k-1}(1)$ ways to order the elements to the right of position $2k+1$ since they must form an up-down permutation of length $2n-2k+1$. Since the elements to the left of position $2k+1$ have no effect on whether an element to the right of position $2k+1$ contributes to

$\mathrm{mmp}^{(1,0,0,0)}(\sigma)$, it follows that the contribution of the elements to the right of position $2k$ to $\sum_{\sigma \in UD_{2n}^{(2k)}} x^{\mathrm{mmp}^{(1,0,0,0)}(\sigma)}$ is $B_{2n-2k-1}(x)$. As a consequence, we obtain

$$C_{2n}(x) = \sum_{k=0}^{n-1} \binom{2n-1}{2k} A_{2k}(1) x^{2k} B_{2n-2k-1}(x),$$

or, equivalently,

$$\frac{C_{2n}(x)}{(2n-1)!} = \sum_{k=0}^{n-1} \frac{A_{2k}(1)x^{2k}}{(2k)!} \frac{B_{2n-2k-1}(x)}{(2n-2k-1)!}. \tag{2.3}$$

Multiplying both sides of (2.3) by t^{2n-1} and summing the result over $n \geq 1$, we see that

$$\sum_{n\geq 1} \frac{C_{2n}(x)t^{2n-1}}{(2n-1)!} = \left(\sum_{n\geq 1} \frac{B_{2n-1}(1)x^{2n-1}t^{2n-1}}{(2n-1)!}\right)\left(\sum_{n\geq 0} \frac{A_{2n}(x)t^{2n}}{(2n)!}\right).$$

By André's result, we have

$$\sum_{n\geq 0} \frac{A_{2n}(1)x^{2n}t^{2n}}{(2n)!} = \sec(xt),$$

so that

$$\frac{\partial}{\partial t} C(t,x) = \sec(xt)B(t,x) = (\sec(xt))^{1+\frac{1}{x}} \int_0^t (\sec(xz))^{\frac{-1}{x}} dz. \tag{2.4}$$

Our initial condition is that $C(0,x) = 1$. Both *Maple* and *Mathematica* will solve this differential equation, but the final expressions are complicated and not particularly useful for enumerative purposes. Thus we actually used the right-hand side of (2.4) to find the entries of the table for the initial values of $C_{2n}(x)$ given in the introduction. Nevertheless, we can record the solution of (2.4) as

$$C(t,x) = 1 + \int_0^t (\sec(xy))^{1+\frac{1}{x}} \int_0^y (\sec(xz))^{\frac{-1}{x}} dz\ dy.$$

2.4. **The generating function $D(t,x)$.** If $\sigma = \sigma_1 \dots \sigma_{2n+1} \in DU_{2n+1}$, then $2n+1$ must occur in one of the positions $1, 3, \dots, 2n+1$. Let $DU_{2n+1}^{(2k+1)}$ denote the set of permutations $\sigma \in DU_{2n+1}$ such that $\sigma_{2k+1} = 2n+1$. A schematic diagram of an element in $DU_{2n+1}^{(2k+1)}$ is shown in Figure 5.

Note that there are $\binom{2n}{2k}$ ways to pick the elements which occur to the left of position $2k+1$ in such a permutation σ, and there are $C_{2k}(1) = A_{2k}(1)$ ways to order them since the elements to the right of position $2k+1$ form a down-up permutation of length $2k$. Each of the elements to the left of position $2k+1$ contributes to $\mathrm{mmp}^{(1,0,0,0)}(\sigma)$. Thus the contribution of the elements to the left of position $2k+1$ to $\sum_{\sigma \in DU_{2n+1}^{(2k+1)}} x^{\mathrm{mmp}^{(1,0,0,0)}(\sigma)}$ is $A_{2k}(1)x^{2k}$. There are $A_{2n-2k}(1)$ ways to order the elements to the right of position $2k+1$ since they must form an up-down permutation of length $2n-2k$. Since the elements to the left of position $2k+1$ have no effect on whether an element to the right of position $2k+1$

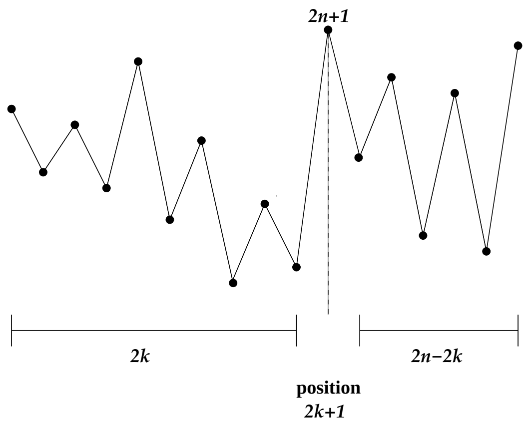

FIGURE 5. The graph of a permutation $\sigma \in DU_{2n+1}^{(2k+1)}$.

contributes to $\mathrm{mmp}^{(1,0,0,0)}(\sigma)$, it follows that the contribution of the elements to the right of position $2k+1$ to $\sum_{\sigma \in DU_{2n+1}^{(2k+1)}} x^{\mathrm{mmp}^{(1,0,0,0)}(\sigma)}$ is $A_{2n-2k}(x)$. Consequently, if $n \geq 1$, then

$$D_{2n+1}(x) = \sum_{k=0}^{n} \binom{2n}{2k} A_{2k}(1) x^{2k} A_{2n-2k}(x).$$

Hence, for $n \geq 1$, we have

$$\frac{D_{2n+1}(x)}{(2n)!} = \sum_{k=0}^{n} \frac{A_{2k}(1)x^{2k}}{(2k)!} \frac{A_{2n-2k}(x)}{(2n-2k)!}. \tag{2.5}$$

Multiplying both sides of (2.5) by t^{2n} and summing the result over $n \geq 0$, we see that

$$\sum_{n\geq 0} \frac{D_{2n+1}(x)t^{2n}}{(2n)!} = \left(\sum_{n\geq 0} \frac{A_{2n}(1)x^{2n}t^{2n}}{(2n)!}\right)\left(\sum_{n\geq 0} \frac{A_{2n}(x)t^{2n}}{(2n)!}\right),$$

so that

$$\frac{\partial}{\partial t} D(t,x) = \sec(x,t)A(t,x) = (\sec(xt))^{1+\frac{1}{x}}.$$

Our initial condition is that $D(0,x) = 0$ so that the solution to this differential equation is

$$D(t,x) = \int_0^t (\sec(xz))^{1+\frac{1}{x}} dz.$$

2.5. **A remark on $MMP(k,0,0,0)$ for $k \geq 2$.** We note that we cannot apply the same techniques to find the distribution of marked mesh patterns $MMP(k,0,0,0)$ in up-down and down-up permutations when $k \geq 2$. For example, suppose that we try to set up a recursion for $A_{2n}^{(2,0,0,0)}(x) = \sum_{\sigma \in UD_{2n}} x^{\mathrm{mmp}^{(2,0,0,0)}(\sigma)}$. Then, if we consider the permutations $\sigma = \sigma_1 \ldots \sigma_{2n} \in UD_{2n}$ such that $\sigma_{2k} = 2n$, we still have $\binom{2n-1}{2k-1}$ ways to pick the elements for $\sigma_1 \ldots \sigma_{2k-1}$. However, in this case the question of whether some σ_i with $i < 2k$ matches the marked mesh pattern $MMP(2,0,0,0)$ in σ is dependent on which values occur in $\sigma_{2k+1} \ldots \sigma_{2n}$. For example, if $2n-1 \in \{\sigma_{2k+1}, \ldots, \sigma_{2n}\}$, then every σ_i with $i \leq k$ will match the marked mesh pattern $MMP(2,0,0,0)$ in σ. However, if $2n-1 \in \{\sigma_1,\ldots,\sigma_{2k-1}\}$, this will not be the case. Thus we were not able to find a simple recursion for $A_{2n}^{(2,0,0,0)}(x)$.

3. The coefficients of the polynomials $A_{2n}(x)$, $B_{2n+1}(x)$, $C_{2n}(x)$, and $D_{2n+1}(x)$

The main goal of this section is to explain several of the coefficients of the polynomials $A_{2n}(x)$, $B_{2n+1}(x)$, $C_{2n}(x)$, and $D_{2n+1}(x)$. For any polynomial $P(x)$, we write $\langle x^k \rangle P(x)$ for the coefficient of x^k in $P(x)$. First, it is easy to understand the coefficients of the lowest power of x in each of these polynomials. More precisely, we have the following theorem, where $0!! = 1$ and, for $n \geq 1$, $(2n)!! = \prod_{i=1}^{n}(2i)$ and $(2n-1)!! = \prod_{i=1}^{n}(2i-1)$.

Theorem 2.

(1) *For all $n \geq 1$, we have*

$$\langle x^k \rangle A_{2n}(x) = \begin{cases} 0 & \text{if } 0 \leq k < n, \\ (2n-1)!! & \text{if } k = n. \end{cases}$$

(2) *For all $n \geq 1$, we have*

$$\langle x^k \rangle B_{2n+1}(x) = \begin{cases} 0 & \text{if } 0 \leq k < n, \\ (2n)!! & \text{if } k = n. \end{cases}$$

(3) *For all $n \geq 1$, we have*

$$\langle x^k \rangle C_{2n}(x) = \begin{cases} 0 & \text{if } 0 \leq k < n-1, \\ (2(n-1))!! & \text{if } k = n-1. \end{cases}$$

(4) *For all $n \geq 1$, we have*

$$\langle x^k \rangle D_{2n+1}(x) = \begin{cases} 0 & \text{if } 0 \leq k < n, \\ (2n-1)!! & \text{if } k = n. \end{cases}$$

Proof. For (1), note that, if $\sigma = \sigma_1 \ldots \sigma_{2n} \in UD_{2n}$, then σ_{2i+1} always matches the pattern $MMP(1,0,0,0)$ for $i = 0,\ldots,n-1$. Thus $\mathrm{mmp}^{(1,0,0,0)}(\sigma) \geq n$. We now proceed by induction on n to prove that $\langle x^n \rangle A_{2n}(x) = (2n-1)!!$ for all $n \geq 1$. This is obvious for $n=1$ since $A_2(x) = x$. Now suppose that $\sigma = \sigma_1 \ldots \sigma_{2n} \in UD_{2n}$ and $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = n$. It is then easy to see that $\sigma_2 = 2n$ since otherwise σ_2 is an unwanted occurrence of the pattern $MMP(1,0,0,0)$. Moreover, if $\tau = \mathrm{red}(\sigma_3 \ldots \sigma_{2n})$, then $\tau \in UD_{2n-2}$ and $\mathrm{mmp}^{(1,0,0,0)}(\tau) = n-1$. Thus, since we are assuming by induction that $\langle x^{n-1} \rangle A_{2n-2}(x) = (2n-3)!!$, we have $2n-1$ choices of σ_1 and $(2n-3)!!$ choices for τ. Hence $\langle x^n \rangle A_{2n}(x) = (2n-1)!!$.

For (2), note that, if $\sigma = \sigma_1 \ldots \sigma_{2n+1} \in UD_{2n+1}$, then σ_{2i+1} always matches the pattern $MMP(1,0,0,0)$ for $i = 0,\ldots,n-1$. Thus $\mathrm{mmp}^{(1,0,0,0)}(\sigma) \geq n$. We now proceed by induction on n to prove that $\langle x^n \rangle B_{2n+1}(x) = (2n)!!$ for all $n \geq 1$. This is obvious for $n=1$ since $B_3(x) = 2x$. Now suppose that $\sigma = \sigma_1 \ldots \sigma_{2n+1} \in UD_{2n+1}$ and $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = n$. It is then easy to see that $\sigma_2 = 2n+1$. Moreover, if $\tau = \mathrm{red}(\sigma_3 \ldots \sigma_{2n+1})$, then $\tau \in UD_{2n-1}$ and $\mathrm{mmp}^{(1,0,0,0)}(\tau) = n-1$. Thus, since we are assuming by induction that $\langle x^{n-1} \rangle B_{2n-1}(x) = (2n-2)!!$, we have $2n$ choices of σ_1 and $(2n-2)!!$ choices for τ. Hence $\langle x^n \rangle B_{2n+1}(x) = (2n)!!$ for $n \geq 1$.

For (3), note that, if $\sigma = \sigma_1 \ldots \sigma_{2n} \in DU_{2n}$, then σ_{2i} always matches the pattern $MMP(1,0,0,0)$ for $i = 1, \ldots, n-1$. Thus $\mathrm{mmp}^{(1,0,0,0)}(\sigma) \geq n-1$. Now suppose that $\sigma = \sigma_1 \ldots \sigma_{2n} \in DU_{2n}$ and $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = n-1$. It is then easy to see that $\sigma_1 = 2n$. Moreover, if $\tau = \sigma_2 \ldots \sigma_{2n}$, then $\tau \in UD_{2n-1}$ and $\mathrm{mmp}^{(1,0,0,0)}(\tau) = n-1$. Thus we have $(2(n-1))!!$ choices for τ by part (2). Hence $\langle x^{n-1} \rangle\, C_{2n}(x) = (2(n-1))!!$.

For (4), note that, if $\sigma = \sigma_1 \ldots \sigma_{2n+1} \in DU_{2n+1}$, then σ_{2i} always matches $MMP(1,0,0,0)$ for $i = 1, \ldots, n$. Thus $\mathrm{mmp}^{(1,0,0,0)}(\sigma) \geq n$. Now suppose that $\sigma = \sigma_1 \ldots \sigma_{2n+1} \in DU_{2n+1}$ and $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = n$. It is then easy to see that $\sigma_1 = 2n+1$. Moreover, if $\tau = \sigma_2 \ldots \sigma_{2n+1}$, then $\tau \in UD_{2n}$ and $\mathrm{mmp}^{(1,0,0,0)}(\tau) = n$. Thus we have $(2n-1)!!$ choices for τ by part (1). Hence $\langle x^{n} \rangle\, D_{2n+1}(x) = (2n-1)!!$ for $n \geq 1$. ☐

We can easily explain the coefficients of the highest power of x in each of the polynomials $A_{2n}(x)$, $B_{2n+1}(x)$, $C_{2n}(x)$, and $D_{2n+1}(x)$. More precisely, we have the following proposition.

Proposition 2.

(1) *For all $n \geq 1$, the highest power of x that appears in $A_{2n}(x)$ is x^{2n-1}, which appears with coefficient $B_{2n-1}(1)$.*
(2) *For all $n \geq 1$, the highest power of x that appears in $B_{2n+1}(x)$ is x^{2n-1}, which appears with coefficient $(2n)B_{2n-1}(1)$.*
(3) *For all $n \geq 1$, the highest power of x that appears in $C_{2n}(x)$ is x^{2n-2}, which appears with coefficient $(2n-1)A_{2n-2}(1)$.*
(4) *For all $n \geq 1$, the highest power of x that appears in $D_{2n+1}(x)$ is x^{2n}, which appears with coefficient $A_{2n}(1)$.*

Proof. For (1), it is easy to see that $\mathrm{mmp}^{(1,0,0,0)}(\sigma)$ is maximized for a permutation $\sigma = \sigma_1 \ldots \sigma_{2n} \in UD_{2n}$ with $\sigma_{2n} = 2n$. In such a case $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = 2n-1$ and $\sigma_1 \ldots \sigma_{2n-1}$ can be any element of UD_{2n-1}.

For (2), it is easy to see that $\mathrm{mmp}^{(1,0,0,0)}(\sigma)$ is maximized for a permutation $\sigma = \sigma_1 \ldots \sigma_{2n+1} \in UD_{2n+1}$ with $\sigma_{2n} = 2n+1$. In such a case $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = 2n-1$. We then have $2n$ choices for σ_{2n+1} and $\mathrm{red}(\sigma_1 \ldots \sigma_{2n-1})$ can be any element of UD_{2n-1}. Thus $\langle x^{2n-1} \rangle\, B_{2n+1}(x) = (2n)B_{2n-1}(1)$.

For (3), it is easy to see that $\mathrm{mmp}^{(1,0,0,0)}(\sigma)$ is maximized for a permutation $\sigma = \sigma_1 \ldots \sigma_{2n} \in DU_{2n}$ with $\sigma_{2n-1} = 2n$. In such a case $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = 2n-2$. We then have $2n-1$ choices for σ_{2n} and $\mathrm{red}(\sigma_1 \ldots \sigma_{2n-2})$ can be any element of DU_{2n-2}. Thus $\langle x^{2n-2} \rangle\, C_{2n}(x) = (2n-1)C_{2n-2}(1) = (2n-1)A_{2n-2}(1)$.

For (4), it is easy to see that $\mathrm{mmp}^{(1,0,0,0)}(\sigma)$ is maximized for a permutation $\sigma = \sigma_1 \ldots \sigma_{2n+1} \in DU_{2n+1}$ with $\sigma_{2n+1} = 2n+1$. In such a case $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = 2n$. Then $\sigma_1 \ldots \sigma_{2n}$ can be any element of DU_{2n}. Thus $\langle x^{2n} \rangle\, D_{2n+1}(x) = C_{2n}(1) = A_{2n}(1)$. ☐

3.1. **Recursions on up-down permutations of even length.** By Theorem 2, the lowest power of x that appears with a non-zero coefficient in $A_{2n}(x)$ is x^n. Next we consider $\left\langle x^{n+k} \right\rangle A_{2n}(x)$ for fixed k. That is, we let

$$A_{2n}^{=n+k} = |\{\sigma \in UD_{2n} : \mathrm{mmp}^{(1,0,0,0)}(\sigma) = n+k\}|$$

for fixed $k \geq 1$. Our goal is to show that $A_{2n}^{=n+k} = p_k(n)(2n-1)!!$ for some fixed polynomial $p_k(n)$ in n. More precisely, we shall prove the following theorem, where we let

$$(x)\downarrow_n = x(x-1)\cdots(x-n+1) \text{ if } n \geq 1 \text{ and } (x)\downarrow_0 = 1.$$

Theorem 3. *There is a sequence of polynomials $p_0(x), p_1(x), \ldots$ such that, for all $k \geq 0$,*

$$A_{2n}^{=n+k} = p_k(n)(2n-1)!! \text{ for all } n \geq k+1.$$

Moreover, for $k \geq 1$, the values $p_k(n)$ are defined by the recursion

$$p_k(n) = \frac{B_{2k+1}(1)}{(2k+1)!!} + \sum_{j=1}^{k} \sum_{t=k+2}^{n} \frac{B_{2j+1}(1)2^j(t-1)\downarrow_j}{(2j+1)!} p_{k-j}(t-j-1), \tag{3.1}$$

where $p_0(x) = 1$.

Proof. We proceed by induction on k. For $k = 0$, we know by Theorem 2 that $A_{2n}^{=n} = (2n-1)!!$ for all $n \geq 1$ so that we may let $p_0(x) = 1$.

Now assume that $k \geq 1$ and the theorem is true for $s < k$. That is, assume that for $0 \leq s < k$, there is a polynomial $p_s(x)$ such that for $n \geq s+1$, $A_{2n}^{=n+s} = p_s(n)(2n-1)!!$.

It is easy to see that, for $\sigma = \sigma_1 \ldots \sigma_{2n} \in UD_{2n}$, we have $\text{mmp}^{(1,0,0,0)}(\sigma) > n+k$ if $\sigma_{2j} = 2n$ with $j > k+2$, because then $\sigma_2, \sigma_4, \ldots, \sigma_{2k+2}$ as well as σ_{2i+1}, $i = 0, \ldots, n-1$, will match the pattern $MMP(1,0,0,0)$ in σ. Thus, if $\text{mmp}^{(1,0,0,0)}(\sigma) = n+k$, then $2n \in \{\sigma_2, \sigma_4, \ldots, \sigma_{2k+2}\}$. Now suppose that $j \leq k+1$ and $\sigma_{2j} = 2n$. Then we have $\binom{2n-1}{2j-1}$ ways to choose the elements $\sigma_1, \ldots, \sigma_{2j-1}$, and we have $B_{2j-1}(1)$ ways to order them. Then we know that σ_i matches the marked mesh pattern $MMP(1,0,0,0)$ in σ for i odd and for $i \in \{2, 4, \ldots, 2j-2\}$. Hence, we have $\text{mmp}^{(1,0,0,0)}(\text{red}(\sigma_{2j+1} \ldots \sigma_{2n})) = n-j+k-j+1$. Thus it follows that, for $n \geq k+1$,

$$A_{2n}^{=n+k} = \sum_{j=1}^{k+1} \binom{2n-1}{2j-1} B_{2j-1}(1) A_{2(n-j)}^{=(n-j)+k-j+1}. \tag{3.2}$$

Now define $p_k(n) = \frac{A_{2n}^{=n+k}}{(2n-1)!!}$ for $n \geq k+1$. Note that $A_{2k+2}^{(k+1)+k} = B_{2k+1}(1)$, since for a permutation $\tau = \tau_1 \ldots \tau_{2k+2} \in UD_{2k+2}$ to have $\text{mmp}^{(1,0,0,0)}(\tau) = 2k+1$, it must satisfy $\tau_{2k+2} = 2k+2$, and, hence, we have $B_{2k+1}(1)$ choices for $\tau_1 \ldots \tau_{2k+1}$. Hence, $p_k(k+1) = \frac{B_{2k+1}(1)}{(2k+1)!!}$.

We may rewrite (3.2) as

$$\begin{aligned} p_k(n)(2n-1)!! &= (2n-1)p_k(n-1)(2n-3)!! \\ &\quad + \sum_{j=2}^{k+1} \frac{\prod_{i=0}^{2j-2}(2n-1-j)}{(2j-1)!} B_{2j-1}(1) p_{k-j+1}(n-j)(2n-2j-1)!!. \end{aligned} \tag{3.3}$$

Dividing both sides of (3.3) by $(2n-1)!!$, we obtain

$$\begin{aligned} p_k(n) - p_k(n-1) &= \sum_{j=2}^{k+1} \frac{B_{2j-1}(1)\prod_{s=1}^{j-1}(2n-2s)}{(2j-1)!} p_{k-j+1}(n-j) \\ &= \sum_{j=1}^{k} \frac{B_{2j+1}(1)2^j(n-1)\downarrow_j}{(2j+1)!} p_{k-j}(n-j-1). \end{aligned}$$

Hence, for $n \geq k+1$, we have

$$\begin{aligned} p_k(n) - p_k(k+1) &= \sum_{t=k+2}^{n} p_k(t) - p_k(t-1) \\ &= \sum_{t=k+2}^{n} \sum_{j=1}^{k} \frac{B_{2j+1}(1)2^j(t-1)\downarrow_j}{(2j+1)!} p_{k-j}(t-j-1). \end{aligned}$$

It follows that, for $n \geq k+1$,

$$p_k(n) = \frac{B_{2k+1}(1)}{(2k+1)!!} + \sum_{j=1}^{k} \sum_{t=k+2}^{n} \frac{B_{2j+1}(1)2^j(t-1)\downarrow_j}{(2j+1)!} p_{k-j}(t-j-1).$$

This proves (3.1).

Since $p_s(x)$ is a polynomial for $s < k$, it is easy to see that

$$\sum_{t=k+2}^{n} \frac{B_{2j+1}(1)2^j(t-1)\downarrow_j}{(2j+1)!} p_{k-j}(t-j-1)$$

is a polynomial in n for $j = 1, \ldots, k$. Thus $p_k(n)$ is a polynomial in n. □

One may use *Mathematica* and (3.1) to compute the first few expressions for $p_k(n)$. For example, we have computed that

$$\begin{aligned} p_0(n) &= 1, \\ p_1(n) &= \frac{2}{3}\binom{n}{2}, \\ p_2(n) &= \frac{n(2+7n-14n^2+5n^3)}{90}, \end{aligned}$$

and

$$p_3(n) = \frac{n(192-478n+213n^2+227n^3-198n^4+35n^5)}{5670}.$$

3.2. **Recursions on up-down permutations of odd length.**

Theorem 4. *There is a sequence of polynomials $q_0(x), q_1(x), \ldots$ such that, for all $k \geq 0$, we have*

$$B_{2n+1}^{=n+k} = q_k(n)(2n)!! \text{ for all } n \geq k+1.$$

Moreover, for $k \geq 1$, the values $q_k(n)$ are defined by the recursion

$$q_k(n) = \frac{B_{2k+1}(1)}{(2k)!!} + \sum_{j=1}^{k} \sum_{t=k+2}^{n} \frac{B_{2j+1}(1) \prod_{s=0}^{j-1}(2t-1-2s)}{(2j+1)!} q_{k-j}(t-j-1), \tag{3.4}$$

where $q_0(x) = 1$.

Proof. We proceed by induction on k. For $k = 0$, we know by Theorem 2 that $B_{2n+1}^{=n} = (2n)!!$ for all $n \geq 1$ so that we may let $q_0(x) = 1$.

Now assume that $k \geq 1$ and the theorem is true for $s < k$. That is, assume that, for $0 \leq s < k$, there is a polynomial $q_s(x)$ such that, for $n \geq s+1$, $B_{2n+1}^{=n+s} = q_s(n)(2n)!!$.

We can argue as in Theorem 3 that, if $\sigma = \sigma_1 \ldots \sigma_{2n+1} \in UD_{2n+1}$ and $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = n+k$, then $2n \in \{\sigma_2, \sigma_4, \ldots, \sigma_{2k+2}\}$. Now suppose that $j \leq k+1$ and $\sigma_{2j} = 2n$. Then we have $\binom{2n}{2j-1}$ ways to choose the elements $\sigma_1, \ldots, \sigma_{2j-1}$, and we have $B_{2j-1}(1)$ ways to order them. Then we know that σ_i matches the marked mesh pattern $MMP(1,0,0,0)$ in σ for $i \in \{2,4,\ldots,2j-2\} \cup \{1,3,\ldots,2n-1\}$. Hence, we have $\mathrm{mmp}^{(1,0,0,0)}(\mathrm{red}(\sigma_{2j+1} \ldots \sigma_{2n+1})) = n-j+k-j+1$. Thus it follows that, for $n \geq k+2$,

$$B_{2n+1}^{=n+k} = \sum_{j=1}^{k+1} \binom{2n}{2j-1} B_{2j-1}(1) B_{2(n-j)+1}^{=(n-j)+k-j+1}. \tag{3.5}$$

Now define $q_k(n) = \frac{B_{2n+1}^{=n+k}}{(2n)!!}$ for $n \geq k+1$. Note that $B_{2k+3}^{(k+1)+k} = (2k+2)B_{2k+1}(1)$ since for a permutation $\tau = \tau_1 \ldots \tau_{2k+3} \in UD_{2k+3}$ to have $\mathrm{mmp}^{(1,0,0,0)}(\tau) = 2k+1$, it must satisfy $\tau_{2k+2} = 2k+3$, and, hence, we have $2k+2$ choices for τ_{2k+3} and $B_{2k+1}(1)$ choices for $\tau_1 \ldots \tau_{2k+1}$. Thus, $q_k(k+1) = \frac{(2k+2)B_{2k+1}(1)}{(2k+2)!!} = \frac{B_{2k+1}(1)}{(2k)!!}$.

We may rewrite (3.5) as

$$\begin{aligned} q_k(n)(2n)!! &= (2n)q_k(n-1)(2n-2)!! \\ &\quad + \sum_{j=2}^{k+1} \frac{\prod_{i=0}^{2j-2}(2n-j)}{(2j-1)!} B_{2j-1}(1) q_{k-j+1}(n-j)(2n-2j)!!. \end{aligned} \tag{3.6}$$

Dividing both sides of (3.6) by $(2n)!!$, we obtain

$$q_k(n) - q_k(n-1) = \sum_{j=2}^{k+1} \frac{B_{2j-1}(1) \prod_{s=1}^{j-1}(2n-2s-1)}{(2j-1)!} q_{k-j+1}(n-j). \tag{3.7}$$

Hence, for $n \geq k+2$, we have

$$\begin{aligned} q_k(n) - q_k(k+1) &= \sum_{t=k+2}^{n} q_k(t) - q_k(t-1) \\ &= \sum_{t=k+2}^{n} \sum_{j=1}^{k} \frac{B_{2j+1}(1) 2^j \prod_{s=1}^{j-1}(2n-2s-1)}{(2j+1)!} q_{k-j}(t-j-1). \end{aligned}$$

It follows that, for $n \geq k+1$,

$$q_k(n) = \frac{B_{2k+1}(1)}{(2k)!!} + \sum_{j=1}^{k} \sum_{t=k+2}^{n} \frac{B_{2j+1}(1)2^j \prod_{s=1}^{j-1}(2n-2s-1)}{(2j+1)!} q_{k-j}(t-j-1). \tag{3.8}$$

This proves (3.4).

Since $q_s(x)$ is a polynomial for $s < k$, it is easy to see that

$$\sum_{t=k+2}^{n} \frac{B_{2j+1}(1) \prod_{s=1}^{j-1}(2n-2s-1)}{(2j+1)!} q_{k-j}(t-j-1)$$

is a polynomial in n for $j = 1, \ldots, k$. Thus $q_k(n)$ is a polynomial in n. $\square$

One may use *Mathematica* and (3.4) to compute the first few expressions for $q_k(n)$. For example, we have computed that

$$\begin{aligned} q_0(n) &= 1, \\ q_1(n) &= \frac{n^2-1}{3}, \\ q_2(n) &= \frac{(n-2)(n-1)(5n^2+n-3)}{90}, \text{ and} \\ q_3(n) &= \frac{35n^6 - 84n^5 - 193n^4 + 345n^3 + 140n^2 - 81n + 198}{5670}. \end{aligned}$$

3.3. **Recursions on down-up permutations.** Similar results hold for down-up permutations.

Theorem 5. *There are sequences of polynomials $r_0(x), r_1(x), \ldots$ and $s_0(x), s_1(x), \ldots$ such that, for all $k \geq 0$, we have*

$$C_{2n}^{=n-1+k} = r_k(n)(2n-2)!! \text{ for all } n \geq k+1, \tag{3.9}$$

and

$$D_{2n+1}^{=n-1+k} = s_k(n)(2n-1)!! \text{ for all } n \geq k+1. \tag{3.10}$$

Proof. By Theorem 2, we have $C_{2n}^{=n-1} = (2n-2)!!$ and $D_{2n+1}^{=n} = (2n-1)!!$ for all $n \geq 1$. Thus we may let $r_0(x) = s_0(x) = 1$.

For a permutation $\sigma = \sigma_1 \ldots \sigma_{2n} \in DU_{2n}$ to have $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = n-1+k$, we must have $2n \in \{\sigma_1, \sigma_3, \ldots, \sigma_{2k+1}\}$. If $\sigma_{2j+1} = 2n$ where $j \in \{0, 1, \ldots, k\}$, then there are $\binom{2n-1}{2j}$ ways to pick the elements of $\sigma_1 \ldots \sigma_{2j}$ and $C_{2j}(1)$ ways to order them. Then $\mathrm{red}(\sigma_{2j+2} \ldots \sigma_{2n}) \in UD_{2(n-j-1)+1}$ and it must have $n-1+k-(2j)$ matches of $MMP(1,0,0,0)$. Thus we have $B_{2(n-j-1)+1}^{=(n-j-1)+k-j}$ ways to order $\sigma_{2j+2} \ldots \sigma_{2n}$. It follows that, for $n \geq k+1$,

$$C_{2n}^{=n-1+k} = \sum_{j=0}^{k} \binom{2n-1}{2j} C_{2j}(1) B_{2(n-j-1)+1}^{=n-j-1+k-j}. \tag{3.11}$$

But $C_{2j}(1) = A_{2j}(1)$ and $B_{2(n-j-1)+1}^{=n-j-1+k-j} = (2(n-j-1))!!q_{k-j}(n-j-1)$. Thus, for $n \geq k+1$, we have

$$C_{2n}^{=n-1+k} = \sum_{j=0}^{k} \binom{2n-1}{2j} A_{2j}(1)(2(n-j-1))!!q_{k-j}(n-j-1)$$
$$= (2n-2)!! \sum_{j=0}^{k} \frac{A_{2j}(1)\prod_{s=1}^{j}(2n+1-2s)}{(2j)!}(2(n-j-1))!!q_{k-j}(n-j-1).$$

Thus $C_{2n}^{=n-1+k} = (2n-2)!!r_k(n)$, where

$$r_k(n) = \sum_{j=0}^{k} \frac{A_{2j}(1)\prod_{s=1}^{j}(2n+1-2s)}{(2j)!}q_{k-j}(n-j-1). \tag{3.12}$$

A similar argument shows that, for $n \geq k+1$,

$$D_{2n+1}^{=n+k} = \sum_{j=0}^{k} \binom{2n}{2j} C_{2j}(1) A_{2(n-j)}^{=n-j+k-j}.$$

Since $A_{2(n-j)}^{=n-j+k-j} = (2(n-j)-1)!!p_{k-j}(n-j)$, we obtain

$$D_{2n+1}^{=n+k} = \sum_{j=0}^{k} \binom{2n}{2j} A_{2j}(1)(2(n-j)-1)!!p_{k-j}(n-j)$$
$$= (2n-1)!! \sum_{j=0}^{k} \frac{A_{2j}(1)\prod_{s=1}^{j}(2n+2-2s)}{(2j)!}p_{k-j}(n-j).$$

Thus $D_{2n+1}^{=n+k} = (2n-2)!!s_k(n)$, where

$$s_k(n) = \sum_{j=0}^{k} \frac{A_{2j}(1)\prod_{s=1}^{j}(2n+2-2s)}{(2j)!}p_{k-j}(n-j). \tag{3.13}$$

□

One may use (3.12) and (3.13) to compute $r_k(n)$ and $s_k(n)$ for the first few values of k. For example, we have

$$r_0(n) = 1,$$
$$r_1(n) = \frac{2n^2+2n-3}{6},$$
$$r_2(n) = \frac{20n^4+24n^3-128n^2-12n+45}{360},$$

and

$$r_3(n) = \frac{280n^6+168n^5-4820n^4+3168n^3+8734n^2-6702n+2835}{45360}.$$

Similarly, we have

$$s_0(n) = 1,$$
$$s_1(n) = \frac{n(n+2)}{3},$$
$$s_2(n) = \frac{n(5n^3+16n^2-68n+47)}{90},$$

and

$$s_3(n) = \frac{n(35n^5+126n^4-340n^3-417n^2+656n-60)}{5760}.$$

4. Conclusion

In this paper, we have shown that one can find the generating functions for the distribution of the quadrant marked mesh patterns $MMP(1,0,0,0)$, $MMP(0,1,0,0)$, $MMP(0,0,1,0)$, and $MMP(0,0,0,1)$ in both up-down and down-up permutations by proving simple recursions based on the position of the largest element in a permutation. As noted in Subsection 2.5, these recursions of simple type no longer hold for the distribution of the quadrant marked mesh patterns $MMP(k,0,0,0)$, $MMP(0,k,0,0)$, $MMP(0,0,k,0)$, and $MMP(0,0,0,k)$ in both up-down and down-up permutations when $k \geq 2$. However, our results allow us to derive generating functions for the distribution of other quadrant marked mesh patterns in up-down and down-up permutations. More specifically, for any $a,b,c,d \in \{\emptyset\} \cup \mathbb{N}$, let

$$A^{(a,b,c,d)}(x,t) = 1 + \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} \sum_{\sigma \in UD_{2n}} x^{\mathrm{mmp}^{(a,b,c,d)}(\sigma)},$$
$$B^{(a,b,c,d)}(x,t) = \sum_{n\geq 0} \frac{t^{2n+1}}{(2n+1)!} \sum_{\sigma \in UD_{2n+1}} x^{\mathrm{mmp}^{(1,0,\emptyset,0)}(\sigma)},$$
$$C^{(a,b,c,d)}(x,t) = 1 + \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} \sum_{\sigma \in DU_{2n}} x^{\mathrm{mmp}^{(a,b,c,d)}(\sigma)},$$

and

$$D^{(a,b,c,d)}(x,t) = \sum_{n\geq 0} \frac{t^{2n+1}}{(2n+1)!} \sum_{\sigma \in DU_{2n+1}} x^{\mathrm{mmp}^{(a,b,c,d)}(\sigma)}.$$

Then our results allow us to find the distribution of $MMP(\emptyset,0,0,0)$, $MMP(0,\emptyset,0,0)$, $MMP(0,0,\emptyset,0)$, and $MMP(0,0,0,\emptyset)$ over up-down and down-up permutations. More precisely, it is clear that for any $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we have

$$\mathrm{mmp}^{(1,0,0,0)}(\sigma) + \mathrm{mmp}^{(\emptyset,0,0,0)}(\sigma) = n, \tag{4.1}$$

since any σ_i either matches $MMP(1,0,0,0)$ or $MMP(\emptyset,0,0,0)$ in σ but cannot match both $MMP(1,0,0,0)$ and $MMP(\emptyset,0,0,0)$ in σ. For any set $A \subseteq S_n$, let $A^r = \{\sigma^r : \sigma \in A\}$, $A^c = \{\sigma^c : \sigma \in A\}$, and $A^{rc} = \{(\sigma^r)^c : \sigma \in A\}$. Then, clearly,

$$\sum_{\sigma \in A} x^{\mathrm{mmp}^{(\emptyset,0,0,0)}(\sigma)} = \sum_{\sigma \in A^r} x^{\mathrm{mmp}^{(0,\emptyset,0,0)}(\sigma)} = \sum_{\sigma \in A^c} x^{\mathrm{mmp}^{(0,0,0,\emptyset)}(\sigma)} = \sum_{(\sigma \in A^{rc}} x^{\mathrm{mmp}^{(0,0,\emptyset,0)}(\sigma)}.$$

It then follows from (1.1) and (4.1) that

$$A(tx, \frac{1}{x}) = A^{(\emptyset,0,0,0)}(t,x) = C^{(0,\emptyset,0,0)}(t,x) = C^{(0,0,0,\emptyset)}(t,x) = A^{(0,0,\emptyset,0)}(t,x).$$

Using the fact that

$$\sigma \in UD_{2n+1} \iff \sigma^r \in UD_{2n+1} \iff \sigma^c \in DU_{2n+1} \iff (\sigma^r)^c \in DU_{2n+1}, \tag{4.2}$$

it follows that

$$B(tx, \frac{1}{x}) = B^{(\emptyset,0,0,0)}(t,x) = B^{(0,\emptyset,0,0)}(t,x) = D^{(0,0,0,\emptyset)}(t,x) = D^{(0,0,\emptyset,0)}(t,x).$$

Similar reasoning will show that

$$C(tx, \frac{1}{x}) = C^{(\emptyset,0,0,0)}(t,x) = A^{(0,\emptyset,0,0)}(t,x) = A^{(0,0,0,\emptyset)}(t,x) = C^{(0,0,\emptyset,0)}(t,x)$$

and

$$D(tx, \frac{1}{x}) = D^{(\emptyset,0,0,0)}(t,x) = D^{(0,\emptyset,0,0)}(t,x) = B^{(0,0,0,\emptyset)}(t,x) = B^{(0,0,\emptyset,0)}(t,x).$$

Moreover, our techniques can be used to study the distribution of other quadrant marked mesh patterns in up-down and down-up permutations. For example, in [8], we have proved similar recursions based on the position of the smallest element in a permutation to study the distribution of the quadrant marked mesh patterns $MMP(1,0,\emptyset,0)$, $MMP(0,1,0,\emptyset)$, $MMP(\emptyset,0,1,0)$, and $MMP(0,\emptyset,0,1)$ in both up-down and down-up permutations. In this case, the recursions are a bit more subtle, and the corresponding generating functions are not always as simple as in the results of this paper. For example, we have shown that

$$A^{(1,0,\emptyset,0)}(t,x) = (\sec(t))^x,$$

$$B^{(1,0,\emptyset,0)}(t,x) = \frac{\sin(t)\cos(t)(1-x+\sec(t))}{x+(1-x)\cos(t)} \times \left((1-x)\ {}_2F_1\left(\frac{1}{2},\frac{1+x}{2};\frac{3}{2};\sin\left(t^2\right)\right) + x\ {}_2F_1\left(\frac{1}{2},\frac{2+x}{2};\frac{3}{2};\sin\left(t^2\right)\right)\right),$$

where ${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$ and $(x)_n = x(x-1)\cdots(x-n+1)$ if $n \geq 1$ and $(x)_0 = 1$.

There are several directions for further research that are suggested by the results of this paper. First, one may study the distribution in up-down and down-up permutations of other quadrant marked meshed patterns $MMP(a,b,c,d)$ in the case where $a,b,c,d \in \{\emptyset,0,1\}$. More generally, one may study the distribution of quadrant marked mesh patterns on other classes of pattern-restricted permutations such as 2-*stack-sortable permutations* or *vexillary permutations* (see [6] for definitions of these) and many other permutation classes having nice properties. Finally, we conjecture that the polynomials $A_{2n}(x)$, $B_{2n+1}(x)$, $C_{2n}(x)$, and $D_{2n+1}(x)$ are unimodal for all $n \geq 1$. This is certainly true for small values of n.

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