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A WORD HOPF ALGEBRA BASED ON THE SELECTION/QUOTIENT PRINCIPLE

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Abstract. We define a Hopf algebra structure on the vector space spanned by packed words using a selection/quotient coproduct. We show that this algebra is free on its irreducible packed words. We also compute the Hilbert series of this Hopf algebra, and we investigate its primitive elements.

1. Introduction

In algebraic combinatorics, one often associates algebraic structures with various sets of combinatorial objects. Such structures are, for example, Hopf algebras on trees, graphs, tableaux, matroids, words, etc.

A first type of combinatorial Hopf algebra structure is constructed using the selection/quotient principle. This simply means that the comultiplication is of the form

$$\Delta(S) = \sum_{\substack{A\subseteq S+ \\ \text{conditions}}} S[A] \otimes S/_A, \tag{1.1}$$

where $S[A]$ is a substructure of S and $S/_A$ is a quotient.

Examples of such Hopf algebras are the Connes–Kreimer Hopf algebra of Feynman graphs, underlying the combinatorics of perturbative renormalization in quantum field theory [2] or in non-commutative Moyal quantum field theory [10], [11] (the interested reader may also consult [12], [14] for some short reviews of these algebras). For the sake of completeness, let us also mention that similar Hopf algebraic structures have been proposed for quantum gravity spin-foam models, see [7], [13].

Matroid theory has been introduced by Whitney in [15]. A structure of matroid Hopf algebra is defined in [8], where the product is given by the direct sum operation and the coproduct is given by the selection/quotient principle mentioned above.

A second type of combinatorial Hopf algebra structure relies on the selection/complement principle. This means that the comultiplication is of the form

$$\Delta(S) = \sum_{\substack{A\subseteq S+ \\ \text{conditions}}} S[A] \otimes [S-A]. \tag{1.2}$$

Examples of such Hopf algebras are the Loday–Ronco Hopf algebra of planar binary trees [6] or the Hopf algebra of matrix quasi-symmetric functions **MQSym** [3].

It would thus be interesting to find Hopf algebraic structures on words with a comultiplication not of type (1.2), but of type (1.1). This is the issue we address in this paper.

In this article, we introduce a new Hopf algebraic structure, which we call WMat, on the set of packed words with product given by the shifted concatenation and coproduct given by such a selection/quotient principle. Each letter $x_j, j \geq 1$, can be seen as the

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infinite column vector with 1 at the j^{th} place and all other entries zero. WMat has the zero column, x_0, as a special element. In graph theory, it would correspond to the self-loop. Using the notion of irreducible element, we prove that WMat is free as an algebra.

2. Algebra structure

2.1. **Definitions.** Let X be an infinite totally ordered alphabet $\{x_i\}_{i\geq 0}$ and X^* be the set of words with letters in the alphabet X.

A word w of length $n = |w|$ is a mapping $i \mapsto w[i]$ from $\{1, 2, \ldots, |w|\}$ to X. For a letter $x_i \in X$, the partial degree $|w|_{x_i}$ is the number of times the letter x_i occurs in the word w. We have

$$|w|_{x_i} = \sum_{j=1}^{|w|} \delta_{w[j],x_i}. \tag{2.1}$$

For a word $w \in X^*$, we define the alphabet $Alph(w)$ as the set of its letters, while $IAlph(w)$ denotes the set of indices in $Alph(w)$. In symbols,

$$Alph(w) = \{x_i : |w|_{x_i} \neq 0\} \ ; \ IAlph(w) = \{i \in \mathbb{N} : |w|_{x_i} \neq 0\}. \tag{2.2}$$

The upper bound $sup(w)$ is the supremum of $IAlph(w)$, i.e.,

$$sup(w) = sup_{\mathbb{N}}(IAlph(w)). \tag{2.3}$$

We denote the empty word by 1_{X^*}, and we put $sup(1_{X^*}) = 0$.

Let us define the substitution operators. Let $w = x_{i_1} \ldots x_{i_m}$ and $\phi : IAlph(w) \longrightarrow \mathbb{N}$, with $\phi(0) = 0$. We then have

$$S_\phi(x_{i_1} \ldots x_{i_m}) = x_{\phi(i_1)} \ldots x_{\phi(i_m)}. \tag{2.4}$$

Next we define the pack operator of a word w. Let $\{j_1, j_2, \ldots, j_k\} = IAlph(w) \setminus \{0\}$ with $j_1 < j_2 < \cdots < j_k$ and define ϕ_w as

$$\phi_w(i) = \begin{cases} m, & \text{if } i = j_m, \\ 0, & \text{if } i = 0. \end{cases} \tag{2.5}$$

The corresponding packed word, denoted by $pack(w)$, is $S_{\phi_w}(w)$. A word $w \in X^*$ is said to be *packed* if $w = pack(w)$.

Example 2.1. Let $w = x_1x_1x_5x_0x_4$. Then we have $pack(w) = x_1x_1x_3x_0x_2$.

Remark 2.2. The presence of the letter x_0 dramatically influences the picture since there is an infinite number of distinct packed words of weight m (here, the weight is the sum of the indices), which are obtained by adding multiple copies of the letter x_0.

Example 2.3. The packed words of weight 2 are $x_0^{k_1}x_1x_0^{k_2}x_1x_0^{k_3}$, with $k_1, k_2, k_3 \geq 0$.

The operator $pack : X^* \longrightarrow X^*$ is idempotent ($pack \circ pack = pack$). By linear extension, it defines a projector. The image, $pack(X^*)$, is the set of packed words.

Let u, v be two words. We define the shifted concatenation $*$ by

$$u * v = uT_{sup(u)}(v), \tag{2.6}$$

where, for $t \in \mathbb{N}$, $T_t(w)$ denotes the image of w by S_ϕ for $\phi(n) = n + t$ if $n > 0$ and $\phi(0) = 0$ (that is, all letters are reindexed except x_0). It is straightforward to check that, in the case the words are packed, the result of a shifted concatenation is a packed word.

Definition 2.4. *Let k be a field. We define the vector space $\mathcal{H} = span_k(pack(X^*))$. This space is endowed with a product (on the words) given by*

$$\begin{aligned} \mu : &\mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}, \\ &u \otimes v \longmapsto u * v. \end{aligned}$$

Remark 2.5. The product above is similar to the shifted concatenation for permutations (see [3]). Moreover, if u and v are two words in X^*, then $sup(u * v) = sup(u) + sup(v)$.

Proposition 2.6. *$(\mathcal{H}, \mu, 1_{X^*})$ is an associative algebra with unit.*

Proof. Let u, v, w be three words in $\mathcal{H}$. We have

$$(u * v) * w = (uT_{sup(u)}(v))T_{sup(u*v)}(w) = u(T_{sup(u)}(v)(T_{sup(u)+sup(v)})(w)) = u * (v * w). \tag{2.7}$$

Thus, $(\mathcal{H}, \mu)$ is associative. On the other hand, for all $u \in pack(X^*)$, we have

$$u * 1_{X^*} = uT_{sup(u)}(1_{X^*}) = u1_{X^*} = u \ ,$$

and

$$1_{X^*} * u = (1_{X^*})T_{sup(1_{X^*})}(u) = (1_{X^*})u = u \ .$$

Now observe that $pack(1_{X^*}) = 1_{X^*}$. This is in fact obvious from the fact that $1_{X^*} = 1_{\mathcal{H}}$. We conclude that $(\mathcal{H}, \mu, 1_{X^*})$ is an associative algebra with unit. □

As already announced in the introduction, we call this algebra WMat.

Remark 2.7. The product is non-commutative, for example: $x_1 * x_1x_1 \neq x_1x_1 * x_1$.

Let $w = x_{k_1} \ldots x_{k_n}$ be a word and $I \subseteq \{1, 2, \ldots, n\}$. The subword $w[I]$ is defined as $x_{k_{i_1}} \ldots x_{k_{i_l}}$, where $I = \{i_1, i_2, \ldots, i_l\}$ with $i_1 < i_2 < \cdots < i_l$. A further notation that we shall use from here on is $I + J$ for the disjoint union of the two sets I and J.

Lemma 2.8. *Let u, v be two words, and $I \subset \{1, 2, \ldots, |u|\}$ and $J \subset \{|u| + 1, |u| + 2, \ldots, |u| + |v|\}$. Then we have*

$$pack(u * v[I + J]) = pack(u[I]) * pack(v[J']), \tag{2.8}$$

where J' is the set $\{i - |u| : i \in J\}$.

Proof. By direct computation, we have

$$pack(u * v[I + J]) = pack(uT_{sup(u)}(v)[I + J]) = pack(u[I]T_{sup(u)}(v)[J]). \tag{2.9}$$

Furthermore, we have $sup(u[I]) \leq sup(u)$, and this, together with (2.9), leads to

$$pack(u[I]T_{sup(u)}(v)[J]) = pack(u[I]T_{sup(u[I])}(v[J'])) = pack(u[I]) * pack(v[J']). \tag{2.10}$$

□

Theorem 2.9. *Let $k\langle X\rangle$ be equipped with the shifted concatenation. The mapping pack*

$$k\langle X\rangle \xrightarrow{pack} \mathcal{H} \tag{2.11}$$

is a homomorphism of associative algebras with unit.

Proof. By using Lemma 2.8 with $I = \{1, 2, \ldots, |u|\}$ and $J = \{|u|+1, |u|+2, \ldots, |u|+|v|\}$, the assertion of the theorem follows immediately. □

2.2. WMat **is a free algebra.** WMat is, by construction, the algebra of the monoid $pack(X^*)$, therefore to check that WMat is a free algebra, it is sufficient to show that $pack(X^*)$ is a free monoid on its letters. In the following diagram, a free monoid is a pair $(F(X), j_X)$ where $F(X)$ is a monoid, $j_X : X \to F(X)$ is a mapping such that for all $M \in Mon$ and all $f : X \to M$ there exists a unique $f_X \in Hom(F(X), M)$ such that $f = f_x \circ j_X$.

Set	Monoids

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "j_X"'] & \mathcal{M} \\ & F(X) \arrow[u, dotted, "f_X"'] \end{tikzcd}$$

Here we use an "internal" characterization of free monoids in terms of irreducible elements.

Definition 2.10. *A packed word w in $pack(X^*)$ is called an irreducible word if and only if it cannot be written in the form $w = u * v$, where u and v are two non trivial packed words.*

Example 2.11. The word $x_1x_1x_1$ is an irreducible word. The word $x_1x_1x_2$ is a reducible word because it can be written as $x_1x_1x_2 = x_1x_1 * x_1$.

Proposition 2.12. *If w is a packed word, then w can be written uniquely as $w = v_1 * v_2 * \cdots * v_n$, where v_i are non-trivial irreducible words, $1 \le i \le n$.*

Proof. The i^{th} position of word w is called an admissible cut if $sup(w[1 \dots i]) = inf(w[i+1 \dots |w|]) - 1$ or $sup(w[i+1 \dots |w|]) = 0$, where $inf(w)$ is the infimum of $IAlph(w)$.

Since the length of word is finite, we may write $w = v_1 * v_2 * \dots v_n$, with n maximal and v_i non-trivial, $1 \le i \le n$.

We assume that one word can be written in two ways

$$w = v_1 * v_2 * \cdots * v_n \tag{2.12}$$

and

$$w = v'_1 * v'_2 * \cdots * v'_m. \tag{2.13}$$

Denoting by k the first number such that $v_k \neq v'_k$, without loss of generality we may suppose that $|v_k| < |v'_k|$. From Equation (2.12), the k^{th} position is an admissible cut of w. From Equation (2.13), the k^{th} position is not an admissible cut of w. Thus, we obtain a contradiction. Hence, we have $n = m$ and $v_i = v'_i$ for all $1 \le i \le n$. □

We thus conclude that $pack(X^*)$ is free as monoid with the packed words as a basis.

3. Bialgebra structure

We now give the definition of the coproduct and prove that the coassociativity property holds.

Definition 3.1. *For $A \subset X$, we define $w/A = S_{\phi_A}(w)$ with*

$$\phi_A(i) = \begin{cases} i, & \text{if } x_i \notin A, \\ 0, & \text{if } x_i \in A. \end{cases}$$

For a word u, we define ${}^{w}/_{u} = {}^{w}/_{Alph(u)}$.

Definition 3.2. *The coproduct of $\mathcal{H}$ is given by*

$$\Delta(w) = \sum_{I+J=\{1,\ldots,|w|\}} pack(w[I]) \otimes pack(w[J]/_{w[I]}), \quad \text{for all } w \in \mathcal{H}. \tag{3.1}$$

Example 3.3. We have

$$\Delta(x_1x_2x_1) = x_1x_2x_1 \otimes 1_{X^*} + x_1 \otimes x_1x_0 + x_1 \otimes x_1^2 + x_1 \otimes x_0x_1 + x_1x_2 \otimes x_0 \\ + x_1^2 \otimes x_1 + x_2x_1 \otimes x_0 + 1_{X^*} \otimes x_1x_2x_1.$$

Next, we prove coassociativity.

Let $I = \{i_1, i_2, \ldots, i_n\}$ and $I_1 = \{j_1, j_2, \ldots, j_k\} \subseteq \{1, 2, \ldots, |I|\}$, and let α be the mapping

$$\begin{aligned} \alpha : I &\longrightarrow \{1, 2, \ldots, n\}, \\ i_s &\longmapsto s. \end{aligned} \tag{3.2}$$

Lemma 3.4. *Let $w \in X^*$ be a word, I be a subset of $\{1, 2, \ldots, |w|\}$, and $I_1 \subset \{1, 2, \ldots, |I|\}$. Then we have*

$$pack(w[I])[I_1] = S_{\phi_{w[I]}}(w[I_1']), \tag{3.3}$$

where I_1' is $\alpha^{-1}(I_1)$ and $\phi_{w[I]}$ is the packing map of $w[I]$ that is given in (2.5).

Proof. Using the definition of the packing map $\phi_{w[I]}$, we can directly check that Equation (3.3) holds. ☐

Lemma 3.5. *Let $w, w_1, w_2 \in X^*$ be words and ϕ be a strictly increasing map from $IAlph(w)$ to $\mathbb{N}$. Then*

$$pack(S_\phi(w)) = pack(w) \tag{3.4}$$

and

$$S_\phi({}^{w_1}/_{w_2}) = {}^{S_\phi(w_1)}/_{S_\phi(w_2)}. \tag{3.5}$$

Proof. In order to prove (3.4), we observe that

$$pack(S_\phi(w)) = S_{\phi_0}(S_\phi(w)) = S_{\phi_0 \circ \phi}(w), \tag{3.6}$$

where ϕ_0 is the packing map which is given in (2.5). Note that both ϕ and ϕ_0 are strictly increasing maps.

We write $I = IAlph(w) = \{j_1, j_2, \ldots, j_k\}$, $j_1 < j_2 < \cdots < j_k$, and $j_i' = \phi(j_i)$, $i = 1, 2, \ldots, k$, so that the image set $\phi(I)$ is given by $\phi(I) = \{j_1', j_2', \ldots, j_k'\}$ with $j_1' < j_2' < \cdots < j_k'$. From the definition of ϕ_0, we have $\phi_0(j_i') = i = \phi_0(j_i)$. This leads to

$$S_{\phi_0 \circ \phi}(w) = S_{\phi_0}(w) = pack(w), \tag{3.7}$$

from which (3.4) follows in combination with (3.6).

Now we turn to the proof of (3.5). Let $I_2 = Alph(w_2)$ and $I_2' = Alph(S_\phi(w_1))$. Using this notation, the left-hand side of (3.5) becomes

$$S_\phi({}^{w_1}/_{w_2}) = S_\phi(S_{\phi_{I_2}}(w_1)) = S_{\phi \circ \phi_{I_2}}(w_1), \tag{3.8}$$

and the right-hand side becomes

$$S_\phi(w_1)/_{S_\phi(w_2)} = S_{\phi_{I_2'}}(S_\phi(w_1)) = S_{\phi_{I_2'}\circ\phi}(w_1). \tag{3.9}$$

With $x_i \in Alph(w_1)$, there are two cases:

1. If $x_i \in I_2$, then $\phi_{I_2}(i) = 0$ and $\phi \circ \phi_{I_2}(i) = 0$ since ϕ is a strictly increasing map. On the other hand, $\phi(i) \in I_2'$, and this implies $\phi_{I_2'} \circ \phi(i) = \phi_{I_2'}(\phi(i)) = 0$.
2. If $x_i \notin I_2$, then $\phi_{I_2}(i) = i$ and $\phi \circ \phi_{I_2}(i) = \phi(i)$. On the other hand, since ϕ is a strictly increasing map, we have $\phi(i) \notin I_2'$, and $\phi_{I_2'} \circ \phi(i) = \phi(i)$.

Thus, we have $\phi \circ \phi_{I_2}(i) = \phi_{I_2'} \circ \phi(i)$. Using this result and (3.8) and (3.9), we arrive at (3.5). □

Lemma 3.6. *Let w be a word in $\mathcal{H}$, and I, J, K be three mutually disjoint subsets of $\{1, 2, \ldots, |w|\}$. Then*

$$\frac{w[K]/_{w[I]}}{w[J]/_{w[I]}} = w[K]/_{w[I+J]}. \tag{3.10}$$

Proof. Using Lemma 3.5, we have

$$\frac{w[K]/_{w[I]}}{w[J]/_{w[I]}} = S_{\phi_I}(w[K])/_{S_{\phi_I}(w[K])} = S_{\phi_I}(w[K]/_{w[J]})$$
$$= S_{\phi_I}(S_{\phi_J}(w[K])) = S_{\phi_I\circ\phi_J}(w[K]) =^{w[K]} /_{w[I+J]}.$$

□

Proposition 3.7. *The vector space $\mathcal{H}$ endowed with the coproduct* (3.1) *is a coassociative coalgebra with co-unit. The co-unit is given by*

$$\epsilon(w) = \begin{cases} 1, & \textit{if } w = 1_{\mathcal{H}}, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. We first prove the coassociativity of the coproduct (3.1), that is,

$$(\Delta \otimes Id) \circ \Delta(w) = (Id \otimes \Delta) \circ \Delta(w). \tag{3.11}$$

The left-hand side (3.11) can be written in the form

$$\begin{aligned}(\Delta \otimes Id) \circ \Delta(w) = &\sum_{I+J=\{1,\ldots,|w|\}} \Bigg(\sum_{I_1+I_2=\{1,\ldots,|I|\}} pack(pack(w[I])[I_1]) \\ &\otimes pack(^{pack(w[I][I_2])}/_{pack(w[I][I_1])}) \Bigg) \otimes pack(^{w[J]}/_{w[I]}) \\ = &\sum_{I+J=\{1,\ldots,|w|]} \Bigg(\sum_{I_1'+I_2'=I} pack(S_\phi(w[I_1'])) \otimes pack(^{S_\phi(w[I_2'])}/_{S_\phi(w[I_1'])}) \Bigg) \otimes pack(^{w[J]}/_{w[I]}) \\ s = &\sum_{I_1'+I_2'+J=\{1,\ldots,|w|\}} pack(w[I_1']) \otimes pack(^{w[I_2]}/_{w[I_1']}) \otimes pack(^{w[J]}/_{w[I_1+I_2]}). \end{aligned} \tag{3.12}$$

The right-hand side of (3.11) can be written as

$$
\begin{aligned}
&(Id \otimes \Delta) \circ \Delta(w) \\
&= \sum_{I+J=\{1,\ldots,|w|\}} pack(w[I]) \otimes \Bigg(\sum_{J_1+J_2=\{1,\ldots,|J|\}} pack(pack({}^{w[J]}/{}_{w[I]}))[J_1]) \\
&\qquad \otimes pack({}^{pack({}^{w[J]}/{}_{w[I]})[J_2]}/{}_{pack({}^{w[J]}/{}_{w[I]})[J_1]})) \Bigg) \\
&= \sum_{I+J=\{1,\ldots,|w|\}} pack(w[I]) \otimes \Bigg(\sum_{J_1'+J_2'=J} pack({}^{w[J_1']}/{}_{w[I]}) \otimes pack(S_\phi({}^{{}^{w[J_2']}/{}_{w[I]}}/{}_{{}^{w[J_1']}/{}_{w[I]}})) \Bigg) \\
&= \sum_{I+J=\{1,\ldots,|w|\}} pack(w[I]) \otimes \Bigg(\sum_{J_1'+J_2'=J} pack({}^{w[J_1']}/{}_{w[I]}) \otimes pack({}^{w[J_2']}/{}_{w_{[I+J_1']}}) \Bigg) \\
&= \sum_{I+J_1'+J_2'=\{1,\ldots,|w|\}} pack(w[I]) \otimes pack({}^{w[J_1']}/{}_{w[I]}) \otimes pack({}^{w[J_2']}/{}_{w_{[I+J_1']}}). \qquad (3.13)
\end{aligned}
$$

Comparing (3.12) and (3.13), we conclude that the coproduct (3.1) is coassociative.

We now claim that

$$(\epsilon \otimes Id) \circ \Delta(w) = (Id \otimes \epsilon) \circ \Delta(w), \qquad (3.14)$$

for all words $w \in \mathcal{H}$. In order to establish this claim, we rewrite the left-hand side of (3.14) as

$$
\begin{aligned}
(\epsilon \otimes Id) \circ \Delta(w) &= (\epsilon \otimes Id) \Bigg(\sum_{I+J=\{1,\ldots,|w|\}} pack(w[I]) \otimes pack(w[J]/w[I]) \Bigg) \\
&= \sum_{I+J=\{1,\ldots,|w|\}} \epsilon(pack(w[I])) \otimes pack(w[J]/w[I]) = 1_{\mathcal{H}} \otimes pack(w) = pack(w),
\end{aligned}
$$

and the right-hand side as

$$
\begin{aligned}
(Id \otimes \epsilon) \circ \Delta(w) &= (Id \otimes \epsilon) \Bigg(\sum_{I+J=\{1,\ldots,|w|\}} pack(w[I]) \otimes pack(w[J]/w[I]) \Bigg) \\
&= \sum_{I+J=\{1,\ldots,|w|\}} pack(w[I]) \otimes \epsilon(pack(w[J]/w[I])) = pack(w) \otimes 1_{\mathcal{H}} = pack(w).
\end{aligned}
$$

Consequently, the triple $(\mathcal{H}, \Delta, \epsilon)$ is a coassociative coalgebra with co-unit. □

Remark 3.8. This coalgebra is not cocommutative. For example, we have

$$
\begin{aligned}
T_{12} \circ \Delta(x_1^2) &= T_{12}(x_1^2 \otimes 1_{\mathcal{H}} + 2x_1 \otimes x_0 + 1_{\mathcal{H}} \otimes x_1^2) \\
&= x_1^2 \otimes 1_{\mathcal{H}} + 2x_0 \otimes x_1 + 1_{\mathcal{H}} \otimes x_1^2 \neq \Delta(x_1^2),
\end{aligned}
$$

where the operator T_{12} is given by $T_{12}(u \otimes v) = v \otimes u$.

Lemma 3.9. *Let u, v be two words. Furthermore, let $I_1 + J_1 = \{1, 2, \ldots, |u|\}$ and $I_2 + J_2 = \{|u|+1, |u|+2, \ldots, |u|+|v|\}$. Then*

$$pack({}^{u*v[J_1+J_2]}/{}_{u*v[I_1+I_2]}) = pack({}^{u[J_1]}/{}_{u[I_1]}) * pack({}^{v[J_2']}/{}_{v[I_2']}), \qquad (3.15)$$

where I_2' is the set $\{k - |u| : k \in I_2\}$ and J_2' is the set $\{k - |u| : k \in J_2\}$.

Proof. We have

$$
\begin{aligned}
pack({}^{u*v[J_1+J_2]}/_{u*v[I_1+I_2]}) &= pack(S_{\phi_{I_1+I_2}}(u*v[J_1+J_2])) \\
&= pack(S_{\phi_{I_1+I_2}}(uT_{sup(u)}(v)[J_1+J_2])) \\
&= pack(S_{\phi_{I_1}+\phi_{I_2}}(u[J_1]T_{sup(u)}(v)[J_2])) \\
&= pack(S_{\phi_{I_1}}S_{\phi_{I_2}}(u[J_1]T_{sup(u)}(v[J_2']))) \\
&= pack(S_{\phi_{I_1}}(u[J_1])S_{\phi_{I_2}}(T_{sup(u)}(v[J_2']))) \\
&= pack({}^{u[J_1]}/_{u[I_1]})T_{sup({}^{u[J_1]}/_{u[I_1]})}pack(S_{\phi_{I_2'}}(v[J_2'])) \\
&= pack(({}^{u[J_1]}/_{u[I_1]}) * pack({}^{u[J_2']}/_{u[I_2']}).
\end{aligned}
$$

□

Proposition 3.10. *For any words u, v in $\mathcal{H}$, we have*

$$\Delta(u * v) = \Delta(u) *^{\otimes 2} \Delta(v). \tag{3.16}$$

Proof. We have

$$
\begin{aligned}
\Delta(u*v) = &\sum_{\substack{I_1+I_2=I \\ J_1+J_2=J \\ I_1,J_1\subset\{1,\dots,|u|\} \\ I_2,J_2\subset\{|u|+1,\dots,|u|+|v|\}}} (pack(u*v[I_1+I_2]))\otimes\big(pack({}^{(u*v[J_1+J_2]}/_{(u*v[I_1+I_2]})\big) \\
= &\sum_{\substack{I_1+J_1=\{1,\dots,|u|\} \\ I_2'+J_2'=\{1,\dots,|v|\}}} \big(pack(u[I_1])\otimes pack({}^{u[J_1]}/_{u[I_1]})\big) * \Big(pack(v[I_2])\otimes pack({}^{u[J_2']}/_{u[I_2']})\Big) \\
= &\left(\sum_{I_1+J_1=\{1,\dots,|u|\}} pack(u[I_1])\otimes pack({}^{u[J_1]}/_{u[I_1]})\right) \\
&\qquad * \left(\sum_{I_2'+J_2'=\{1,\dots,|v|\}} pack(v[I_2])\otimes pack({}^{u[J_2']}/_{u[I_2']})\right) \\
= &\Delta(u) *^{\otimes 2} \Delta(v).
\end{aligned}
$$

□

Since $\mathcal{H}$ is graded by word length, we have the following theorem.

Theorem 3.11. *$(\mathcal{H}, *, 1_{\mathcal{H}}, \Delta, \epsilon)$ is a Hopf algebra.*

Proof. This follows from the above results. □

For $w \neq 1_{\mathcal{H}}$, the antipode is given by the recursion

$$S(w) = -w - \sum_{I+J=\{1,\dots,|w|\}, I,J\neq\emptyset} S(pack(w[I])) * pack({}^{w[J]}/_{w[I]}). \tag{3.17}$$

4. Hilbert series of the Hopf algebra WMat

In this section, we compute the number of packed words with length n and supremum k. It is the same as the number of cyclically ordered partitions of an n-element set. Using the formula for Stirling numbers of the second kind (see [1]), we obtain an explicit formula for the number of packed words with length n, which we denote by d_n.

Definition 4.1. *The Stirling number of the second kind $S(n,k)$ counts the number of set partitions of an n-element set into precisely k blocks. They are given by the recurrence*

$$S(n+1,k) = S(n,k-1) + kS(n,k), \quad \text{for } 0 < k \leq n,$$

and initial conditions $S(0,0) = 1$ and $S(n,0) = 0$ for $n > 0$.

We may define a word without x_0 by its positions, meaning that, if a word $w = x_{i_1}x_{i_2}\ldots x_{i_n}$ has length n and alphabet $IAlph(w) = \{1,2,\ldots,k\}$, then this word can be determined from the list $\{S_1,S_2,\ldots,S_k\}$, where S_i is the set of positions of x_i in the word w, with $1 \leq i \leq k$. It is straightforward to check that $(S_i)_{1\leq i\leq k}$ is a partition of $\{1,2,\ldots,n\}$.

We may divide the set of packed words with length n and supremum k into two parts: *"pure"* packed words (which have no x_0 in their alphabet), denoted by $pack^+_{n,k}(X)$, and packed words which have x_0 in their alphabet, denoted by $pack^0_{n,k}(X)$. It is obvious that

$$d(n,k) = \#pack^+_{n,k}(X) + \#pack^0_{n,k}(X). \tag{4.1}$$

We now compute the cardinality of these two sets separately.

Consider a word $w \in pack^+_{n,k}(X)$. Clearly, we have $IAlph(w) = \{1,2,\ldots,k\}$. This word is determined by $\{S_1,S_2,\ldots,S_k\}$, where S_i is the set of positions of x_i, for $1 \leq i \leq k$. It is obvious that:

(1) $S_i \neq \emptyset$, for all $i \in \{1,2,\ldots,k\}$;
(2) $\bigsqcup_{1\leq i\leq k} S_i = \{1,2,\ldots,n\}$.

Note that (1) and (2) hold even for $w = 1_{\mathcal{H}}$.

Thus, the cardinality of packed words with length n and supremum k equals

$$d^+(n,k) = \#pack^+_{n,k}(X) = S(n,k)k!. \tag{4.2}$$

Similarly, a word $w \in \#pack^0_{n,k}(X)$ can be determined by $\{S_0,S_1,S_2,\ldots,S_k\}$ where S_i is the set of positions of x_i, for $0 \leq i \leq k$. In this case, we have

$$d^0(n,k) = \#pack^0_{n,k}(X) = S(n,k+1)(k+1)!. \tag{4.3}$$

The number of packed words with length n and supremum k is then the sum of these two numbers, namely

$$d(n,k) = d^+(n,k) + d^0(n,k) = S(n,k)k! + S(n,k+1)(k+1)! = S(n+1,k+1)k!. \tag{4.4}$$

Using *Maple*, we computed the first few values of $d(n,k)$. They are displayed in Table 1. Note that the values of Table 1 correspond to those of the triangular array $A028246$ in [9].

Remark 4.2. Formulas (4.2) and (4.3) imply that the packed words of length n and supremum k without, respectively with, x_0 are in bijection with the ordered partitions of $\{1,2,\ldots,n\}$ into k blocks, respectively into $k+1$ blocks. Therefore formula (4.4) implies that the set of packed words of length n with supremum k is in bijection with the circularly ordered partitions of $n+1$ elements into $k+1$ blocks.

The number d_n of all packed words of length n is then given by

$$d_n = \sum_{k=0}^{n} d(n,k) = \sum_{k=0}^{n} S(n+1,k+1)k!. \tag{4.5}$$

Using again *Maple*, we computed the values listed in Table 2.

	k								
n	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0
2	1	3	2	0	0	0	0	0	0
3	1	7	12	6	0	0	0	0	0
4	1	15	50	60	24	0	0	0	0
5	1	31	180	390	360	120	0	0	0
6	1	63	602	2100	3360	2520	720	0	0
7	1	127	1932	10206	25200	31920	20160	5040	0
8	1	255	6050	46620	166824	317520	332640	181440	40320

TABLE 1. Values of $d(n,k)$ given by the explicit formula (4.4) and computed with *Maple*.

n	0	1	2	3	4	5	6	7	8	9	10
d_n	1	2	6	26	150	1082	9366	94586	1091670	14174522	204495126

TABLE 2. Some value of d_n by the formula (4.5).

The number of packed words is the sequence *A*000629 in [9], where it is also mentioned that this sequence corresponds to the sequence of ordered Bell numbers multiplied by two (except for the 0th term).

The ordinary and exponential generating functions of our sequence are also given in [9], while the ordinary generating function is given by the formula $\sum_{n\geq 0}\frac{2^n n! x^n}{\prod_{k=0}^n(1+kx)}$. The exponential generating function is given by $\frac{e^x}{2-e^x}$. Below we provide the proof of the latter fact.

First, recall that the exponential generating function of the ordered Bell numbers (see, for example, page 109 of [4]) is

$$\frac{1}{2-e^x}=\sum_{n\geq 0}\sum_{k=0}^{n}S(n,k)k!\frac{x^n}{n!}. \tag{4.6}$$

By differentiating both sides of this equation with respect to x, we obtain

$$\frac{e^x}{2-e^x}=\sum_{n\geq 1}\sum_{k=1}^{n}S(n,k)k!\frac{x^{n-1}}{(n-1)!}. \tag{4.7}$$

Combining (4.5) and (4.7), we get

$$\frac{e^x}{2-e^x}=\sum_{n\geq 0}\sum_{k=0}^{n}S(n+1,k+1)k!\frac{x^n}{n!}=\sum_{n\geq 0}d_n\frac{x^n}{n!}, \tag{4.8}$$

as claimed in [9].

We now investigate the combinatorics of irreducible packed words (see Definition 2.10). First, we notice that there are still infinitely many irreducible packed words of weight m, and again they are obtained by adding multiple copies of the letter x_0.

Example 4.3. The word $x_1x_0^kx_1x_0^kx_1$ (with k an arbitrary positive integer) is an irreducible packed word of weight 3.

Let us denote by i_n the number of irreducible packed words of length n. Then we have

$$i_n = \sum_{\substack{j_1+\cdots+j_k=n \\ j_l\neq 0}} (-1)^{k+1} d_{j_1}\ldots d_{j_k}. \tag{4.9}$$

Using *Maple*, we computed the first values of i_n. They are displayed in Table 3.

n	0	1	2	3	4	5	6	7	8	9	10
i_n	1	2	2	10	66	538	5170	59906	704226	9671930	145992338

TABLE 3. The ten first values of the number of irreducible packed words

Note that this sequence does not appear in Sloane's On-Line Encyclopedia of Integer Sequences [9].

5. Primitive elements of WMat

We emphasize that this Hopf algebra, although graded, is not cocommutative and thus the primitive elements do not generate the whole algebra but only the sub-Hopf algebra on which Δ is cocommutative (the largest subalgebra on which the Cartier–Quillen–Milnor–Moore theorem holds).

We denote by $Prim(\mathrm{WMat})$ the algebra generated by the primitive elements of $\mathcal{H}$.

We recall the following result.

Lemma 5.1. *Let $V^{(1)}$ and $V^{(2)}$ be two graded vector spaces,*

$$V^{(i)} = \oplus_{n\geq 0} V_n^{(i)},\ i=1,2. \tag{5.1}$$

Furthermore, let $\phi \in Hom^{gr}(V^{(1)}, V^{(2)})$, that is, $\phi(V_n^{(1)}) \subseteq V_n^{(2)})$ for all $n \geq 0$. Then $Ker(\phi)$ is graded.

We then obtain the following property of $Prim(\mathrm{WMat})$.

Proposition 5.2. *$Prim(\mathrm{WMat})$ is a Lie subalgebra of* WMat*, graded by the word's length.*

Proof. We define the mapping

$$\begin{aligned} \Delta^+ : \qquad & \mathrm{WMat} \longrightarrow \mathrm{WMat}\otimes\mathrm{WMat} \\ & \begin{cases} 1_{\mathcal{H}} & \longmapsto 0, \\ h & \longmapsto \Delta(h) - 1_{\mathcal{H}}\otimes h - h\otimes 1_{\mathcal{H}}. \end{cases} \end{aligned} \tag{5.2}$$

This mapping is graded. Using Lemma 5.1, we infer $Prim(\mathrm{WMat}) = Ker(\Delta^+)$. Thus, the subalgebra $Prim(\mathrm{WMat})$ is graded. □

Let us now compute the dimensions of the first few spaces $Prim(\mathrm{WMat})_n$:

- For $n=1$, a basis is formed by the primitive elements x_0 and x_1. Then one can check that the primitive elements of length 1 have the form ax_0+bx_1, with a and b scalars.
- For $n=2$, a basis is formed by the primitive elements $x_0x_1-x_1x_0$ and $x_1x_2-x_2x_1$. Then one can check that all the primitive elements of length 2 have the form $a(x_0x_1-x_1x_0)+b(x_1x_2-x_2x_1)$, with a and b scalars. This is seen by explicitly solving a system of 4 equations with $d_2=6$ variables.

Nevertheless, the explicit calculations quickly become lengthy. For example, for $n = 3$, we have to solve a system of 22 equations with 26 variables.

To conclude, let us give some a posteriori explanations on the choice of the name "WMat" for our algebraic structure. We have defined here a Hopf algebra on some set of words with a selection/quotient coproduct rule in the spirit of graphs [2] and matroid Hopf algebras [8]. We could thus call our algebra WGraph or WMat. We prefer the name WMat, since matroids are more general structures than graphs.

On the other hand, the "W" in our name simply refers to the fact that we have a Hopf algebra with basis indexed by a subset X of the free monoid $\mathbb{N}^*$. Let us say that, at this point, there is no polynomial realization of WMat (as is is the case for $WSym$ or $WQSym$ or the Connes–Kreimer Hopf algebras on trees; see [5] and the references therein).

This actually seems to us to be an important perspective for future work related to the new Hopf algebra proposed in this paper.

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