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THE LOCAL h-VECTOR OF THE CLUSTER SUBDIVISION OF A SIMPLEX

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Abstract. The cluster complex $\Delta(\Phi)$ is an abstract simplicial complex, introduced by Fomin and Zelevinsky for a finite root system Φ. The positive part of $\Delta(\Phi)$ naturally defines a simplicial subdivision of the simplex on the vertex set of simple roots of Φ. The local h-vector of this subdivision, in the sense of Stanley, is computed and the corresponding γ-vector is shown to be nonnegative. Combinatorial interpretations to the entries of the local h-vector and the corresponding γ-vector are provided for the classical root systems, in terms of noncrossing partitions of types A and B. An analogous result is given for the barycentric subdivision of a simplex.

1. Introduction and results

Local h-vectors were introduced by Stanley [27] as a fundamental tool in his theory of face enumeration for subdivisions of simplicial complexes. Given a (finite, topological) simplicial subdivision Γ of the abstract simplex 2^V on an n-element vertex set V, the local h-polynomial $\ell_V(\Gamma, x)$ is defined as an alternating sum of the h-polynomials of the restrictions of Γ to the faces of 2^V (see Section 2 for all relevant definitions). The local h-vector of Γ is the sequence of coefficients $\ell_V(\Gamma) = (\ell_0, \ell_1, \ldots, \ell_n)$, where $\ell_V(\Gamma, x) = \ell_0 + \ell_1 x + \cdots + \ell_n x^n$.

The importance of local h-vectors stems from their appearance in the locality formula [27, Theorem 3.2], which expresses the h-polynomial of a simplicial subdivision of a pure simplicial complex Δ as a sum of local contributions, one for each face of Δ. Several fundamental properties of local h-vectors, including symmetry for all topological subdivisions, nonnegativity for quasi-geometric subdivisions and unimodality for regular (geometric) subdivisions, were proven in [27].

The local h-vector of the barycentric subdivision of a simplex affords an elegant interpretation [27, Proposition 2.4] in terms of the combinatorics of permutations. The focus of this paper is on another example of subdivision of the simplex with remarkable combinatorial properties, termed as the *cluster subdivision*. This is the simplicial subdivision of the simplex on the vertex set of simple roots of a finite root system Φ which is naturally defined by the positive part of the cluster complex $\Delta(\Phi)$ [18] (see the discussion below). Our main results compute the local h-vector of the cluster subdivision, providing combinatorial interpretations for the classical root systems in terms of the combinatorics of noncrossing partitions (for the deep connections between cluster combinatorics and noncrossing partitions see, for instance, [3, 9, 24]).

Before proceeding further, we recall the following notation and terminology from [2]. Let Γ be a simplicial subdivision of an $(n-1)$-dimensional simplex 2^V. Since $\ell_V(\Gamma, x)$ has

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symmetric coefficients, there exists [19, Proposition 2.1.1] a unique polynomial $\xi_V(\Gamma, x) = \xi_0 + \xi_1 x + \cdots + \xi_{\lfloor n/2 \rfloor} x^{\lfloor n/2 \rfloor}$ such that

$$\ell_V(\Gamma, x) \;=\; \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i x^i (1+x)^{n-2i}.$$

Following [2, Section 5], we will refer to $\xi_V(\Gamma, x)$ as the *local γ-polynomial* of Γ (with respect to V) and to the sequence $\xi_V(\Gamma) = (\xi_0, \xi_1, \ldots, \xi_{\lfloor n/2 \rfloor})$ as the *local γ-vector* of Γ (with respect to V). As explained in [2], these concepts play a role in the theory of face enumeration for flag homology spheres and their flag simplicial subdivisions.

We will show that $\xi_V(\Gamma, x)$ has nonnegative coefficients for all cluster subdivisions by providing combinatorial interpretations in terms of noncrossing partitions, or by explicit computation. Since cluster subdivisions are geometric and flag, this result provides evidence for a conjecture by the first author [2, Conjecture 5.4], stating that $\xi_V(\Gamma, x)$ has nonnegative coefficients for a family of simplicial subdivisions of the simplex which includes all flag geometric subdivisions. We will also provide combinatorial interpretations to the coefficients of $\xi_V(\Gamma, x)$ for the barycentric subdivision of the simplex.

The remainder of this section states the main results of this paper in more precise form. Their proofs are given in Sections 3 and 4, after some of the relevant background on simplicial complexes, cluster complexes, simplicial subdivisions and noncrossing partitions is recalled in Section 2. Remarks and related open problems are included in Section 5.

1.1. **Cluster subdivisions.** Let Φ be a finite root system of rank n, equipped with a positive system Φ^+ and corresponding simple system $\Pi = \{\alpha_i : i \in I\}$, where I is an n-element index set. The cluster complex $\Delta(\Phi)$ was introduced by Fomin and Zelevinsky in the context of algebraic Y-systems [18]. It is an abstract simplicial complex on the vertex set $\Phi^+ \cup (-\Pi)$, consisting of the positive roots and the negative simple roots, which is homeomorphic to the $(n-1)$-dimensional sphere. When Φ is crystallographic, the combinatorics of $\Delta(\Phi)$ encodes the exchange of clusters in the corresponding cluster algebra of finite type [17]. An overview of cluster complexes and their connection to cluster algebras can be found in [16]. The restriction $\Delta_+(\Phi)$ of $\Delta(\Phi)$ on the vertex set Φ^+, known as the positive part of $\Delta(\Phi)$, is homeomorphic to the $(n-1)$-dimensional ball.

The complex $\Delta_+(\Phi)$ has the structure of a (geometric) simplicial subdivision of the simplex 2^Π on the vertex set Π (see Section 2.4). The restriction of this subdivision to the face $\{\alpha_i : i \in J\}$ of 2^Π indexed by $J \subseteq I$ is the complex $\Delta_+(\Phi_J)$, where Φ_J is the standard parabolic root subsystem of Φ corresponding to J (so that $\Phi_I = \Phi$). We will refer to this subdivision as the cluster subdivision associated to Φ and will denote it by $\Gamma(\Phi)$. We will write

$$\ell_I(\Gamma(\Phi), x) \;=\; \sum_{i=0}^{n} \ell_i(\Phi) x^i \tag{1}$$

for the local h-polynomial of $\Gamma(\Phi)$ and $\ell_I(\Phi) = (\ell_0(\Phi), \ell_1(\Phi), \ldots, \ell_n(\Phi))$ for the corresponding local h-vector. The relevant definitions lead (see Section 2.4) to the formula

$$\ell_I(\Gamma(\Phi), x) \;=\; \sum_{J \subseteq I} (-1)^{|I \smallsetminus J|} \, h(\Delta_+(\Phi_J), x), \tag{2}$$

where $h(\Delta_+(\Phi_J), x)$ is the h-polynomial of $\Delta_+(\Phi_J)$. The results of [27], mentioned earlier, imply that $\ell_I(\Gamma(\Phi), x)$ has nonnegative and symmetric coefficients for every root system Φ.

The h-polynomial of $\Delta_+(\Phi)$ admits several combinatorial interpretations [4, Corollary 1.4 and Theorem 1.5] [3, Corollaries 7.4 and 7.5] in terms of order ideals of roots, hyperplane regions, Weyl group orbits on a finite torus, lattice points and noncrossing partitions. It was computed explicitly for all irreducible (crystallographic) root systems in [4, Section 6]. We denote by $\mathrm{NC}^A(n)$ and $\mathrm{NC}^B(n)$ the set of noncrossing partitions of the set $\{1, 2, \ldots, n\}$ and that of B_n-noncrossing partitions, respectively, and refer to Section 2.5 for the relevant background and any undefined terminology. Our first result determines the local h-polynomial of $\Gamma(\Phi)$ as follows.

Theorem 1.1. *Let $\ell_I(\Gamma(\Phi), x) = \sum_{i=0}^{n} \ell_i(\Phi) x^i$ be the local h-polynomial of the cluster subdivision $\Gamma(\Phi)$, associated to an irreducible root system Φ of rank n and Cartan-Killing type $\mathcal{X}$. Then $\ell_i(\Phi)$ is equal to:*

- *the number of partitions $\pi \in \mathrm{NC}^A(n)$ with i blocks, such that every singleton block of π is nested, if $\mathcal{X} = A_n$,*
- *the number of partitions $\pi \in \mathrm{NC}^B(n)$ with no zero block and i pairs $\{B, -B\}$ of nonzero blocks, such that every positive singleton block of π is nested, if $\mathcal{X} = B_n$,*
- *$n-2$ times the number of partitions $\pi \in \mathrm{NC}^A(n-1)$ with i blocks, if $\mathcal{X} = D_n$.*

Moreover, $\ell_I(\Gamma(\Phi), x)$ is equal to

$$\begin{cases} (m-2)x, & \text{if } \mathcal{X} = I_2(m) \\ 8x + 8x^2, & \text{if } \mathcal{X} = H_3 \\ 42x + 124x^2 + 42x^3, & \text{if } \mathcal{X} = H_4 \\ 10x + 29x^2 + 10x^3, & \text{if } \mathcal{X} = F_4 \\ 7x + 63x^2 + 125x^3 + 63x^4 + 7x^5, & \text{if } \mathcal{X} = E_6 \\ 16x + 204x^2 + 644x^3 + 644x^4 + 204x^5 + 16x^6, & \text{if } \mathcal{X} = E_7 \\ 44x + 748x^2 + 3380x^3 + 5472x^4 + 3380x^5 + 748x^6 + 44x^7, & \text{if } \mathcal{X} = E_8. \end{cases}$$

We will write $\xi_I(\Phi) = (\xi_0(\Phi), \xi_1(\Phi), \ldots, \xi_{\lfloor n/2 \rfloor}(\Phi))$ for the local γ-vector of $\Gamma(\Phi)$, so that

$$\ell_I(\Gamma(\Phi), x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i(\Phi)\, x^i (1+x)^{n-2i}. \tag{3}$$

Our second result computes the numbers $\xi_i(\Phi)$ (hence, via equation (3), the numbers $\ell_i(\Phi)$ as well) explicitly.

Theorem 1.2. *Let Φ be an irreducible root system of rank n and Cartan-Killing type $\mathcal{X}$ and let $\xi_i(\Phi)$ be the integers uniquely defined by (3). Then $\xi_0(\Phi) = 0$ and*

$$\xi_i(\Phi) \ = \ \begin{cases} \dfrac{1}{n-i+1}\dbinom{n}{i}\dbinom{n-i-1}{i-1}, & \text{if } \mathcal{X} = A_n \\[2ex] \dbinom{n}{i}\dbinom{n-i-1}{i-1}, & \text{if } \mathcal{X} = B_n \\[2ex] \dfrac{n-2}{i}\dbinom{2i-2}{i-1}\dbinom{n-2}{2i-2}, & \text{if } \mathcal{X} = D_n \end{cases}$$

for $1 \le i \le \lfloor n/2 \rfloor$. Moreover,

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i(\Phi) x^i \ = \ \begin{cases} (m-2)x, & \text{if } \mathcal{X} = I_2(m) \\ 8x, & \text{if } \mathcal{X} = H_3 \\ 42x + 40x^2, & \text{if } \mathcal{X} = H_4 \\ 10x + 9x^2, & \text{if } \mathcal{X} = F_4 \\ 7x + 35x^2 + 13x^3, & \text{if } \mathcal{X} = E_6 \\ 16x + 124x^2 + 112x^3, & \text{if } \mathcal{X} = E_7 \\ 44x + 484x^2 + 784x^3 + 120x^4, & \text{if } \mathcal{X} = E_8. \end{cases}$$

The proof of Theorem 1.2, given in Section 3, shows that when $\mathcal{X} = A_n$ (respectively, $\mathcal{X} = B_n$), the numbers $\xi_i(\Phi)$ enumerate partitions $\pi \in \mathrm{NC}^A(n)$ (respectively, partitions $\pi \in \mathrm{NC}^B(n)$ with no zero block) which have no singleton block, by the number of blocks; see Propositions 3.1 and 3.2.

When Φ is crystallographic, the cluster complex $\Delta(\Phi)$ can be realized as the boundary complex of a simplicial convex polytope [11]. One may deduce from this statement that $\Gamma(\Phi)$ is a regular (geometric) subdivision of the simplex 2^Π. Thus [27, Theorem 5.2] implies that the local h-vector of $\Gamma(\Phi)$ is unimodal, i.e., $\ell_0(\Phi) \le \ell_1(\Phi) \le \cdots \le \ell_{\lfloor n/2 \rfloor}(\Phi)$. The following corollary of Theorem 1.2 provides a stronger statement.

Corollary 1.3. *For every root system Φ the local γ-vector of $\Gamma(\Phi)$ is nonnegative, i.e., we have $\xi_i(\Phi) \ge 0$ for every index i.*

1.2. **Barycentric subdivisions.** Let V be an n-element set. We denote by $\mathrm{sd}(2^V)$ the (first) barycentric subdivision of the simplex 2^V and by $\mathcal{S}_n$ the set of permutations of $\{1, 2, \dots, n\}$. We recall that for $w \in \mathcal{S}_n$, a descent of w is an index $1 \le i \le n-1$ such that $w(i) > w(i+1)$; an excedance of w is an index $1 \le i \le n$ such that $w(i) > i$. The local h-polynomial of $\mathrm{sd}(2^V)$ was computed in [27, Proposition 2.4] as

$$\ell_V(\mathrm{sd}(2^V), x) \ = \ \sum_{w \in \mathcal{D}_n} x^{\mathrm{ex}(w)}, \tag{4}$$

where $\mathcal{D}_n$ is the set of derangements (permutations with no fixed points) in $\mathcal{S}_n$ and $\mathrm{ex}(w)$ is the number of excedances of $w \in \mathcal{S}_n$. We will provide similar combinatorial interpretations to the local γ-polynomial of $\mathrm{sd}(2^V)$ after we introduce some more terminology.

For $w \in \mathcal{S}_n$, an *ascending run* (or simply, a run) of w is a maximal string $\{i, i+1, \dots, j\}$ of integers, such that $w(i) < w(i+1) < \cdots < w(j)$. A *double descent* of w is an index $2 \le i \le n-1$ such that $w(i-1) > w(i) > w(i+1)$; a *double excedance* of w is an index $1 \le i \le n$ such that $w(i) > i > w^{-1}(i)$. A *left to right maximum* of w is an index $1 \le j \le n$ such that $w(i) < w(j)$ for all $1 \le i < j$.

Theorem 1.4. *Let $(\xi_0, \xi_1, \dots, \xi_{\lfloor n/2 \rfloor})$ be the local γ-vector of the barycentric subdivision* $\mathrm{sd}(2^V)$ *of the $(n-1)$-dimensional simplex 2^V. Then ξ_i is equal to each of the following:*

- (i) *the number of permutations $w \in \mathcal{S}_n$ with i runs and no run of length one,*
- (ii) *the number of derangements $w \in \mathcal{D}_n$ with i excedances and no double excedance,*
- (iii) *the number of permutations $w \in \mathcal{S}_n$ with i descents and no double descent, such that every left to right maximum of w is a descent.*

In particular, we have $\xi_i \ge 0$ for all $0 \le i \le \lfloor n/2 \rfloor$.

For the first few values of n we have:

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i x^i \;=\; \begin{cases} x, & \text{if } \ n = 2, 3 \\ x + 5x^2, & \text{if } \ n = 4 \\ x + 18x^2, & \text{if } \ n = 5 \\ x + 47x^2 + 61x^3, & \text{if } \ n = 6 \\ x + 108x^2 + 479x^3, & \text{if } \ n = 7 \\ x + 233x^2 + 2414x^3 + 1385x^4, & \text{if } \ n = 8 \\ x + 486x^2 + 9970x^3 + 19028x^4, & \text{if } \ n = 9. \end{cases}$$

The right-hand side of (4) is known as the *derangement polynomial* of order n; see, for instance, [12, Section 1], where some of its basic properties are summarized. Theorem 1.4 gives a combinatorial proof of the unimodality of this polynomial, thus answering a question of Brenti [8]; see the third comment in Section 5. Moreover, it implies that for given n, the sum of the coefficients ξ_i is equal to the total number of permutations in $\mathcal{S}_n$ with no ascending run of length one. Such permutations have been considered (in a more general context) and enumerated by Gessel [20, Chapter 5].

We should point out that the nonnegativity of the numbers ξ_i follows from the fact that the derangement polynomials are (symmetric and) real-rooted [31]. Alternatively, this can be deduced from [2, Proposition 6.1], which proves the nonnegativity of the local γ-vector for a family of flag simplicial subdivisions which can be obtained from the trivial subdivision of a simplex by successive stellar subdivisions.

2. Subdivisions, clusters and noncrossing partitions

This section begins by recalling basic definitions on simplicial complexes, simplicial subdivisions and their enumerative invariants. Cluster complexes are then reviewed and cluster subdivisions are formally defined. The section ends with a brief discussion of noncrossing partitions of types A and B. More information on these topics can be found in [5, 16, 25, 29] and references therein. Throughout this paper, $|S|$ denotes the cardinality, and 2^S the set of all subsets, of a finite set S.

2.1. **Simplicial complexes.** Given a finite set Ω, an (abstract) simplicial complex on the ground set Ω is a collection Δ of subsets of Ω such that $F \subseteq G \in \Delta$ implies $F \in \Delta$. The elements of Δ are called *faces*. The dimension of a face F is defined as one less than the cardinality of F. The dimension of Δ is the maximum dimension of a face and is denoted by $\dim(\Delta)$. Faces of Δ of dimension zero are called *vertices*. A face of Δ which is maximal with respect to inclusion is called a *facet*. The simplicial complex Δ is said to be *pure* if all its facets have the same dimension. The *link* of the face $F \in \Delta$ is the subcomplex of Δ defined as $\mathrm{link}_\Delta(F) = \{G \smallsetminus F : G \in \Delta, \, F \subseteq G\}$. The *restriction* of Δ on the ground set $\Omega_0 \subseteq \Omega$ is the subcomplex of Δ consisting of those faces which are contained in Ω_0.

Suppose that Ω_1 and Ω_2 are two disjoint finite sets. The (simplicial) *join* $\Delta_1 * \Delta_2$ of two collections Δ_1 and Δ_2 of subsets of Ω_1 and Ω_2, respectively, is the collection whose elements are the sets of the form $F_1 \cup F_2$, where $F_1 \in \Delta_1$ and $F_2 \in \Delta_2$. The join of two (or more) simplicial complexes is again a simplicial complex.

Every simplicial complex Δ has a geometric realization $\|\Delta\|$ [5, Section 9], uniquely defined up to homeomorphism. All topological properties of Δ we mention in the sequel will refer to those of $\|\Delta\|$. In particular, we say that Δ is a simplicial (topological) ball if $\|\Delta\|$ is homeomorphic to a ball. The *boundary* of a simplicial d-dimensional ball Δ is the subcomplex $\partial\Delta$, consisting of all subsets of those $(d-1)$-dimensional faces of Δ which are contained in a unique facet of Δ. The *interior* of this ball is the set $\Delta \smallsetminus \partial\Delta$; the *interior faces* are the elements of $\Delta \smallsetminus \partial\Delta$. For example, the (abstract) simplex 2^V, consisting of all subsets of an n-element set V, is a simplicial $(n-1)$-dimensional ball whose only interior face is V. The join of two (or more) simplicial balls is a simplicial ball whose interior is equal to the join of the interiors of these balls.

2.2. **Simplicial subdivisions.** Given a finite set V, a (finite, topological) *simplicial subdivision* [27, Section 2] of the abstract simplex 2^V is a simplicial complex Γ together with a map $\sigma : \Gamma \to 2^V$, such that the following hold for every $F \subseteq V$: (a) the set $\Gamma_F := \sigma^{-1}(2^F)$ is a subcomplex of Γ which is a simplicial ball of dimension $\dim(F)$; and (b) the interior of Γ_F is equal to $\sigma^{-1}(F)$. The set $\sigma(E)$ is called the *carrier* of the face $E \in \Gamma$. The complex Γ_F is called the *restriction* of Γ to $F \subseteq V$. The subdivision Γ is called *quasi-geometric* [27, Definition 4.1 (a)] if there do not exist $E \in \Gamma$ and face $F \in 2^V$ of dimension smaller than $\dim(E)$, such that the carrier of every vertex of E is contained in F. Moreover, Γ is called *geometric* [27, Definition 4.1 (b)] if there exists a geometric realization of Γ which geometrically subdivides a geometric realization of 2^V.

Suppose that Γ is a simplicial subdivision of the simplex 2^V and Γ' is a simplicial subdivision of the simplex $2^{V'}$, where V and V' are disjoint sets. The join $\Gamma * \Gamma'$ naturally becomes a simplicial subdivision of the simplex $2^V * 2^{V'} = 2^{V \cup V'}$ if one defines the carrier of a face $E \cup E' \in \Gamma * \Gamma'$ as the union of the carriers of $E \in \Gamma$ and $E' \in \Gamma'$. Given faces $F \subseteq V$ and $F' \subseteq V'$, the restriction of $\Gamma * \Gamma'$ to the face $F \cup F'$ of this simplex is then equal to $\Gamma_F * \Gamma'_{F'}$.

2.3. **Face enumeration.** A fundamental enumerative invariant of a simplicial complex Δ is the h-polynomial, defined by

$$h(\Delta, x) \; = \; \sum_{F \in \Delta} x^{|F|}(1-x)^{d-|F|},$$

where $\dim(\Delta) = d-1$. For the join of two simplicial complexes Δ_1 and Δ_2 we have $h(\Delta_1 * \Delta_2, x) = h(\Delta_1, x)h(\Delta_2, x)$.

The local h-vector of a simplicial subdivision of a simplex was defined in [27, Definition 2.1] as follows.

Definition 2.1. Let V be an n-element set and Γ be a simplicial subdivision of the simplex 2^V. The polynomial $\ell_V(\Gamma, x) = \ell_0 + \ell_1 x + \cdots + \ell_n x^n$ defined by

$$\ell_V(\Gamma, x) \; = \sum_{F \subseteq V} (-1)^{n-|F|} \, h(\Gamma_F, x) \tag{5}$$

is the *local h-polynomial* of Γ (with respect to V). The sequence $\ell_V(\Gamma) = (\ell_0, \ell_1, \ldots, \ell_n)$ is the *local h-vector* of Γ (with respect to V).

The local h-vector $\ell_V(\Gamma) = (\ell_0, \ell_1, \ldots, \ell_n)$ was shown to be symmetric (meaning that $\ell_i = \ell_{n-i}$ holds for $0 \le i \le n$) for every simplicial subdivision Γ of 2^V [27, Theorem 3.3] and to have nonnegative entries for every quasi-geometric simplicial subdivision Γ of 2^V [27, Corollary 4.7]. Moreover (see [27, Example 2.3]), $\ell_0 = 0$ and ℓ_1 is equal to the number of interior vertices of Γ, for $n \ge 1$.

We recall from the introduction that, given a simplicial subdivision Γ of an $(n-1)$-dimensional simplex 2^V, the local γ-polynomial $\xi_V(\Gamma, x) = \xi_0 + \xi_1 x + \cdots + \xi_{\lfloor n/2 \rfloor} x^{\lfloor n/2 \rfloor}$ of Γ (with respect to V) is uniquely defined by

$$\ell_V(\Gamma, x) \;=\; (1+x)^n \; \xi_V \left(\Gamma, \frac{x}{(1+x)^2} \right) \;=\; \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i x^i (1+x)^{n-2i}. \tag{6}$$

The following lemma will be used in the proof of Corollary 1.3.

Lemma 2.2. *Let V and V' be disjoint finite sets. For all simplicial subdivisions Γ of 2^V and Γ' of $2^{V'}$ we have $\ell_{V \cup V'}(\Gamma * \Gamma', x) = \ell_V(\Gamma, x)\, \ell_{V'}(\Gamma', x)$ and $\xi_{V \cup V'}(\Gamma * \Gamma', x) = \xi_V(\Gamma, x)\, \xi_{V'}(\Gamma', x)$.*

Proof. Let $n = |V|$ and $n' = |V'|$. Using the defining equation (5), we find that

$$\begin{aligned}
\ell_{V \cup V'}(\Gamma * \Gamma', x) \;&=\; \sum_{F \subseteq V} \sum_{F' \subseteq V'} (-1)^{|V \cup V'| - |F \cup F'|} \, h((\Gamma * \Gamma')_{F \cup F'}, x) \\
&=\; \sum_{F \subseteq V} \sum_{F' \subseteq V'} (-1)^{n+n'-|F|-|F'|} \, h(\Gamma_F * \Gamma'_{F'}, x) \\
&=\; \sum_{F \subseteq V} \sum_{F' \subseteq V'} (-1)^{n-|F|} \, h(\Gamma_F, x) \; (-1)^{n'-|F'|} \, h(\Gamma'_{F'}, x) \\
&=\; \ell_V(\Gamma, x)\, \ell_{V'}(\Gamma', x).
\end{aligned}$$

This result and (6) imply that $\xi_{V \cup V'}(\Gamma * \Gamma', x) = \xi_V(\Gamma, x)\, \xi_{V'}(\Gamma', x)$. $\square$

2.4. **Cluster complexes and subdivisions.** Let Φ be a finite root system of rank n. As in the introduction, we will fix a positive system Φ^+ with corresponding simple system $\Pi = \{\alpha_i : i \in I\}$, where I is an n-element index set, and set $\Phi_{\ge -1} := \Phi^+ \cup (-\Pi)$. For $J \subseteq I$, the standard parabolic root subsystem Φ_J is endowed with the induced positive system $\Phi_J^+ = \Phi^+ \cap \Phi_J$ and corresponding simple system $\Pi_J = \{\alpha_i : i \in J\}$.

The cluster complex $\Delta(\Phi)$ is a simplicial complex on the vertex set $\Phi_{\ge -1}$. Its faces are the sets consisting of mutually compatible elements of $\Phi_{\ge -1}$, where compatibility is a symmetric binary relation on $\Phi_{\ge -1}$ defined in [18, Section 3]. We refer the reader to [18] [16, Section 4.3] for the precise definition of compatibility and collect the properties of $\Delta(\Phi)$ and its restriction $\Delta_+(\Phi)$ on the vertex set Φ^+ which will be important for us, in the following proposition. Part (ii) is implicit in [18, Section 3] (see Lemma 3.12 and the proof of Theorem 1.10 there) and [6, Section 8]. The other parts follow directly from the results of [18, Section 3].

Proposition 2.3. (i) *The cluster complex $\Delta(\Phi)$ is homeomorphic to an $(n-1)$-dimensional sphere.*
(ii) *The complex $\Delta_+(\Phi)$ is homeomorphic to an $(n-1)$-dimensional ball.*
(iii) *For $J \subseteq I$ we have $\mathrm{link}_{\Delta(\Phi)}(-\Pi_J) = \Delta(\Phi_J)$.*
(iv) *For $J \subseteq I$, the restriction of $\Delta(\Phi)$ to the vertex set $(\Phi_J)_{\ge -1}$ is equal to $\Delta(\Phi_J)$ and that of $\Delta_+(\Phi)$ to the vertex set Φ_J^+ is equal to $\Delta_+(\Phi_J)$.*
(v) *If Φ is a direct product $\Phi_1 \times \Phi_2$, then $\Delta(\Phi) = \Delta(\Phi_1) * \Delta(\Phi_2)$ and $\Delta_+(\Phi) = \Delta_+(\Phi_1) * \Delta_+(\Phi_2)$.* □

The following result of [4] will be needed in Section 3 in order to compute the right-hand side of (2).

Lemma 2.4. ([4, Proposition 6.1]) *For the h-polynomial of $\Delta_+(\Phi)$ we have*

$$h(\Delta_+(\Phi), x) \;=\; \begin{cases} \displaystyle\sum_{i=0}^{n} \frac{1}{i+1}\binom{n}{i}\binom{n-1}{i} x^i, & \textit{if } \mathcal{X} = A_n \\ \displaystyle\sum_{i=0}^{n} \binom{n}{i}\binom{n-1}{i} x^i, & \textit{if } \mathcal{X} = B_n \\ \displaystyle\sum_{i=0}^{n} \left(\binom{n}{i}\binom{n-2}{i} + \binom{n-2}{i-2}\binom{n-1}{i}\right) x^i, & \textit{if } \mathcal{X} = D_n, \end{cases}$$

where $\mathcal{X}$ is the Cartan-Killing type of Φ. □

We now formally define the cluster subdivision $\Gamma(\Phi)$. Given a positive root $\alpha \in \Phi^+$, there is a unique set $J \subseteq I$ such that α is a positive linear combination of the elements of Π_J. We call Π_J the *support* of α and for $E \in \Delta_+(\Phi)$, we denote by $\sigma(E)$ the union of the supports of the elements of E. Equivalently, $\sigma(E)$ is the smallest set $\Pi_J \subseteq \Pi$ such that $\alpha \in \Phi_J^+$ for every $\alpha \in E$.

Proposition 2.5. *The map $\sigma : \Delta_+(\Phi) \mapsto 2^{\Pi}$ defines a simplicial subdivision $\Gamma(\Phi)$ of the simplex 2^{Π}, whose local h-polynomial is given by (2).*

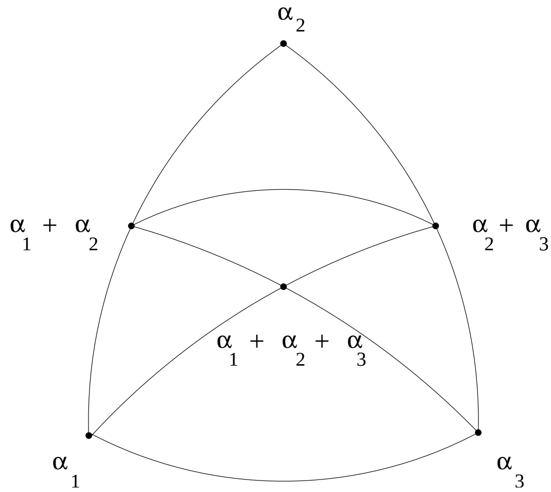

FIGURE 1. The cluster subdivision of type A_3.

Proof. It suffices to show that for every $J \subseteq I$: (a) $\sigma^{-1}(2^{\Pi_J})$ is a subcomplex of $\Delta_+(\Phi)$ which is homeomorphic to a ball of dimension $|J| - 1$; (b) $\sigma^{-1}(\Pi_J)$ is the interior of this ball; and (c) $\sigma^{-1}(2^{\Pi_J}) = \Delta_+(\Phi_J)$. Indeed, (a) and (b) confirm that σ defines a simplicial subdivision of the simplex 2^Π and (c) ensures that the restriction of this subdivision to the face Π_J of 2^Π is equal to $\Delta_+(\Phi_J)$. Equation (2) is a consequence of the last statement and Definition 2.1.

Part (c) follows from the definition of the map σ and Proposition 2.3 (iv) and part (a) follows from (c) and Proposition 2.3 (ii). To verify (b), we may assume that $J = I$. We need to show that the boundary of $\Delta_+(\Phi)$ is equal to the union of the subcomplexes $\Delta_+(\Phi_J)$, where J runs through the proper subsets of I. For that, it suffices to show that an $(n-2)$-dimensional face, say E, of $\Delta_+(\Phi)$ is contained in a unique facet of $\Delta_+(\Phi)$ if and only if $E \in \Delta_+(\Phi_J)$ for some $(n-1)$-element set $J \subseteq I$. This is a consequence of parts (i) and (iii) of Proposition 2.3. Indeed, part (i) implies that E is contained in exactly two facets of $\Delta(\Phi)$. Part (iii) implies that at most one of these contains a negative simple root and that this is the case if and only if $E \in \Delta_+(\Phi_J)$ for some $(n-1)$-element set $J \subseteq I$. □

Example 2.6. The complex $\Delta_+(\Phi)$ and cluster subdivision $\Gamma(\Phi)$ are drawn on Figure 1 for the root system Φ of type A_3. The simple roots $\alpha_1, \alpha_2, \alpha_3$ have been labeled so that α_1 is orthogonal to α_3.

The subdivision $\Gamma(\Phi)$ triangulates the 2-dimensional simplex 2^Π into five 2-dimensional simplices, which are the facets of $\Delta_+(\Phi)$. There is one interior vertex, namely $\alpha_1+\alpha_2+\alpha_3$. The supports of $\alpha_1 + \alpha_2$ and $\alpha_2 + \alpha_3$ are equal to $\{\alpha_1, \alpha_2\}$ and $\{\alpha_2, \alpha_3\}$, respectively. The restriction of $\Gamma(\Phi)$ on the face $\{\alpha_1, \alpha_2\}$ of 2^Π is a subdivision of a 1-dimensional simplex with one interior vertex, namely $\alpha_1 + \alpha_2$. □

Remark 2.7. One can define a cluster complex, and hence a corresponding cluster subdivision, for every orientation of the Dynkin diagram of Φ [22]; see also [24, Section 7] (the cluster complex of [18] [16, Section 4.3], treated here, corresponds to the alternating orientation). By [22, Proposition 3.4] (see also [24, Proposition 7.3]) and the results of [22, Section 6], the h-vector of the positive part of the cluster complex and the local h-vector of the corresponding cluster subdivision do not depend on the orientation chosen. □

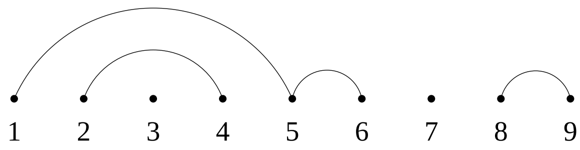

FIGURE 2. The noncrossing partition $\{\{1,5,6\},\{2,4\},\{3\},\{7\},\{8,9\}\}$.

We conclude this section with the following lemma, which will be used in the proof of Corollary 1.3.

Lemma 2.8. *If Φ is a direct product $\Phi_1 \times \Phi_2$, then $\Gamma(\Phi) = \Gamma(\Phi_1) * \Gamma(\Phi_2)$.*

Proof. This statement follows from Proposition 2.3 (v) and the definitions of the cluster subdivision and the join of two simplicial subdivisions. $\square$

2.5. **Noncrossing partitions.** This section summarizes those concepts and results from the theory of noncrossing partitions which are involved in the statements and proofs of Theorems 1.1 and 1.2.

The set of noncrossing partitions of $\{1,2,\dots,n\}$, which we will denote by $\mathrm{NC}^A(n)$, was introduced and studied by Kreweras [21]. It consists of all set partitions π of $\{1,2,\dots,n\}$ with the following property: if $a < b < c < d$ are such that a, c are contained in a block B of π and b, d are contained in a block B' of π, then $B = B'$. An example of a noncrossing partition for $n = 9$ is shown on Figure 2. Among several other fundamental results, Kreweras [21, Section 4] showed that the cardinality of $\mathrm{NC}^A(n)$ is equal to the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$ and that

$$\sum_{\pi \in \mathrm{NC}^A(n)} x^{n-|\pi|} \;=\; \sum_{i=0}^{n} \frac{1}{i+1}\binom{n}{i}\binom{n-1}{i} x^i. \tag{7}$$

We will say that a singleton block $\{b\}$ of $\pi \in \mathrm{NC}^A(n)$ is *nested* if some block of π contains elements a and c such that $a < b < c$; otherwise we say that $\{b\}$ is *nonnested.* For the example of Figure 2 the singleton block $\{3\}$ is nested, while $\{7\}$ is not. Clearly, a partition $\pi \in \mathrm{NC}^A(n)$ with nonnested singleton block $\{b\}$ is determined by its restrictions to $\{1,2,\dots,b-1\}$ and $\{b+1,\dots,n\}$, which are again noncrossing partitions.

Noncrossing partitions of type B were defined by Reiner [25] as follows. A set partition π of $\{1,2,\dots,n\} \cup \{-1,-2,\dots,-n\}$ is called a *B_n-partition* if the following conditions hold: (a) if B is a block of π, then $-B$ (the set obtained by negating the elements of B) is also a block of π; and (b) there at most one block of π (called the *zero block*, if present) which contains both i and $-i$ for some $i \in \{1,2,\dots,n\}$. Such a partition can be represented pictorially [1, Section 2] by placing the integers $1,2,\dots,n,-1,-2,\dots,-n$ (in this order) along a line and drawing arcs above the line between i and j whenever i and j lie in the same block B of π and no other element between them does. The B_n-partition π is called *noncrossing* if no two arcs in this diagram cross. An example for $n = 7$ appears in Figure 3. The set of noncrossing B_n-partitions will be denoted by $\mathrm{NC}^B(n)$.

We will be interested in the enumeration of noncrossing B_n-partitions with no zero block, by the number of blocks. Although we have not been able to locate the following statement explicitly in the literature, its proof follows easily from that of [1, Theorem 2.3].

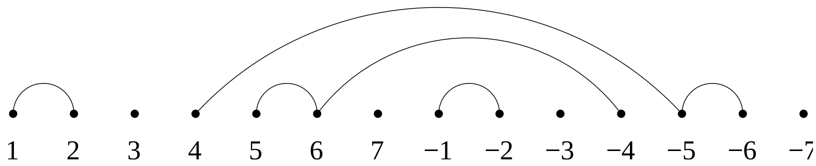

FIGURE 3. A B_7-noncrossing partition.

Lemma 2.9. *The number of partitions $\pi \in \mathrm{NC}^B(n)$ which have no zero block and a total of k pairs $\{B, -B\}$ of nonzero blocks is equal to $\binom{n}{k}\binom{n-1}{k-1}$.*

Proof. The proof of [1, Theorem 2.3], given in [1, Section 4], shows that the partitions $\pi \in \mathrm{NC}^B(n)$ which have no zero block and a total of k pairs $\{B, -B\}$ of nonzero blocks are in one-to-one correspondence with pairs (S, f), where S is a k-element subset of $\{1, 2, \dots, n\}$ and $f : S \to \{1, 2, \dots\}$ is a map whose values sum to n. Since there are $\binom{n}{k}$ ways to choose S and, for any such choice, there are $\binom{n-1}{k-1}$ ways to choose f, the result follows. □

We will say that a singleton block $\{b\}$ of $\pi \in \mathrm{NC}^B(n)$ is *nested* if some block of π contains an element which precedes b and one which succeeds b in the linear ordering $1, 2, \dots, n, -1, -2, \dots, -n$; otherwise we say that $\{b\}$ is *nonnested.* The example of Figure 3 has the nonnested positive singleton block $\{3\}$ and the nested positive singleton block $\{7\}$. A partition $\pi \in \mathrm{NC}^B(n)$ with nonnested positive singleton block $\{b\}$ is determined by its restrictions to $\{1, 2, \dots, b-1\}$ and $\{b+1, \dots, n\} \cup \{-b-1, \dots, -n\}$, which are noncrossing partitions of types A and B, respectively.

3. PROOFS FOR CLUSTER SUBDIVISIONS

This section provides proofs for Theorems 1.1 and 1.2 and Corollary 1.3. As part of the proof for the types A_n and B_n, combinatorial interpretations similar to those of Theorem 1.1 for the numbers $\ell_i(\Phi)$ are provided for the numbers $\xi_i(\Phi)$.

As in previous sections, $\Phi = \Phi_I$ will be a finite root system of rank n. We will denote by $\mathcal{D}(\Phi)$ the Dynkin diagram of Φ and identify the vertex set of $\mathcal{D}(\Phi)$ with the n-element index set I. We will first treat the root systems of types A_n, B_n and D_n.

3.1. **The root system A_n.**

The following proposition is the main result of this section. Note that noncrossing partitions with no singleton block and given number of blocks, which appear there, were considered and enumerated by Kreweras [21, Section 5].

Proposition 3.1. *For the root system Φ of type A_n the following hold:*

- *$\ell_i(\Phi)$ is equal to the number of partitions $\pi \in \mathrm{NC}^A(n)$ with i blocks, such that every singleton block of π is nested,*
- *$\xi_i(\Phi)$ is equal to the number of partitions $\pi \in \mathrm{NC}^A(n)$ which have no singleton block and a total of i blocks.*

Moreover, we have the explicit formula

$$\xi_i(\Phi) = \begin{cases} 0, & \text{if } i = 0 \\ \dfrac{1}{n-i+1}\dbinom{n}{i}\dbinom{n-i-1}{i-1}, & \text{if } 1 \le i \le \lfloor n/2 \rfloor. \end{cases} \tag{8}$$

For the first few values of n we have

$$\sum_{i=0}^{n} \ell_i(\Phi) x^i \ = \ \begin{cases} 0, & \text{if } \ n = 1 \\ x, & \text{if } \ n = 2 \\ x + x^2, & \text{if } \ n = 3 \\ x + 4x^2 + x^3, & \text{if } \ n = 4 \\ x + 8x^2 + 8x^3 + x^4, & \text{if } \ n = 5 \\ x + 13x^2 + 29x^3 + 13x^4 + x^5, & \text{if } \ n = 6 \\ x + 19x^2 + 73x^3 + 73x^4 + 19x^5 + x^6, & \text{if } \ n = 7 \\ x + 26x^2 + 151x^3 + 266x^4 + 151x^5 + 26x^6 + x^7, & \text{if } \ n = 8 \end{cases}$$

and

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i(\Phi) x^i \ = \ \begin{cases} 0, & \text{if } \ n = 1 \\ x, & \text{if } \ n = 2, 3 \\ x + 2x^2, & \text{if } \ n = 4 \\ x + 5x^2, & \text{if } \ n = 5 \\ x + 9x^2 + 5x^3, & \text{if } \ n = 6 \\ x + 14x^2 + 21x^3, & \text{if } \ n = 7 \\ x + 20x^2 + 56x^3 + 14x^4, & \text{if } \ n = 8. \end{cases}$$

The Dynkin diagram $\mathcal{D}(\Phi)$ is a path on the vertex set I. For notational convenience we set $I = \{1, 2, \dots, n\}$, where i and $i+1$ are adjacent in $\mathcal{D}(\Phi)$ for $1 \le i \le n-1$.

Proof of Proposition 3.1. We need to compute the right-hand side of (2), so we focus on $h(\Delta_+(\Phi_J), x)$. Lemma 2.4 and Equation (7) show that

$$h(\Delta_+(\Phi_I), x) \ = \ \sum_{\pi \in \mathrm{NC}^A(n)} x^{n - |\pi|}. \tag{9}$$

For general $J \subseteq I$ we have a direct product decomposition $\Phi_J = \Phi_1 \times \cdots \times \Phi_k$ into irreducible subsystems $\Phi_1, \dots, \Phi_k$. The Dynkin diagrams of $\Phi_1, \dots, \Phi_k$ are the connected components of the diagram obtained from $\mathcal{D}(\Phi)$ by deleting the vertices in $I \smallsetminus J$. Since $\mathcal{D}(\Phi)$ is a path with no multiple edges, each Φ_i is again a root system of type A. Denoting by p_i the rank of Φ_i and using Proposition 2.3 (v) and Equation (9) we find that

$$\begin{aligned} h(\Delta_+(\Phi_J), x) \ &= \ h(\Delta_+(\Phi_1) * \cdots * \Delta_+(\Phi_k), x) \ = \ \prod_{i=1}^{k} h(\Delta_+(\Phi_i), x) \\ &= \ \prod_{i=1}^{k} \sum_{\pi \in \mathrm{NC}^A(p_i)} x^{p_i - |\pi|} \ = \ \sum_{\pi \in \mathrm{NC}^A(J)} x^{n - |\pi|}, \end{aligned}$$

where $\mathrm{NC}^A(J)$ denotes the set of partitions $\pi \in \mathrm{NC}^A(n)$ such that $\{a\}$ is a nonnested singleton of π for every $a \in I \smallsetminus J$. The previous computation and (2) imply that

$$\sum_{i=0}^{n} \ell_i(\Phi) x^i \;=\; \sum_{J \subseteq I} (-1)^{|I \smallsetminus J|} \sum_{\pi \in \mathrm{NC}^A(J)} x^{n-|\pi|}. \tag{10}$$

A simple application of the principle of inclusion-exclusion shows that the right-hand side of (10) is equal to the sum of $x^{n-|\pi|}$, where π runs through those partitions in $\mathrm{NC}^A(n)$ which have no nonnested singleton block. This result and the fact that $\ell_i(\Phi) = \ell_{n-i}(\Phi)$ yield the desired interpretation for $\ell_i(\Phi)$.

To prove the interpretation claimed for $\xi_i(\Phi)$ we need to show that

$$\sum_{i=0}^{n} \ell_i(\Phi) x^i \;=\; \sum_{i=0}^{\lfloor n/2 \rfloor} m_i \, x^i (1+x)^{n-2i}, \tag{11}$$

where m_i is the number of partitions $\pi \in \mathrm{NC}^A(n)$ with a total of i blocks, none of which is a singleton. Let us denote by $\mathrm{NC}^A_0(n)$ the subset of $\mathrm{NC}^A(n)$ consisting of those noncrossing partitions, every singleton block of which is nested. We define an equivalence relation on $\mathrm{NC}^A(n)$ by declaring two partitions π_1 and π_2 equivalent if there is a one-to-one correspondence, say f, from the set of nonsingleton blocks of π_1 to the set of nonsingleton blocks of π_2 such that for every nonsingleton block B of π_1 the sets B and $f(B)$ have the same minimum and the same maximum element. For example, the partition in Figure 2 is equivalent to a total of four noncrossing partitions, namely itself, $\{\{1,5,6\},\{2,3,4\},\{7\},\{8,9\}\}$, $\{\{1,6\},\{2,3,4\},\{5\},\{7\},\{8,9\}\}$ and $\{\{1,6\},\{2,4\},\{3\},\{5\},\{7\},\{8,9\}\}$.

We leave it to the reader to check that this relation restricts to an equivalence relation on $\mathrm{NC}^A_0(n)$ and that each equivalence class within $\mathrm{NC}^A_0(n)$ contains a unique partition π_0 having no singleton block. Moreover, for the equivalence class $O(\pi_0)$ of such a partition $\pi_0 \in \mathrm{NC}^A_0(n)$ we have

$$\sum_{\pi \in O(\pi_0)} x^{|\pi|} \;=\; x^i (1+x)^{n-2i},$$

where i is the number of blocks of π_0. Summing the previous equation over all elements $\pi_0 \in \mathrm{NC}^A_0(n)$ which have no singleton block we get (11).

Finally, (8) is a consequence of the equality $\xi_i(\Phi) = m_i$ and the results of [21, p. 344], which enumerate noncrossing partitions with no singleton block and given number of blocks. □

3.2. **The root system B_n.** This section proves the following statement on the case $\mathcal{X} = B_n$.

Proposition 3.2. *For the root system Φ of type B_n the following hold:*

- *$\ell_i(\Phi)$ is equal to the number of partitions $\pi \in \mathrm{NC}^B(n)$ with no zero block and i pairs $\{B, -B\}$ of nonzero blocks, such that every positive singleton block of π is nested,*
- *$\xi_i(\Phi)$ is equal to the number of partitions $\pi \in \mathrm{NC}^B(n)$ which have no zero block, no singleton block and a total of i pairs $\{B, -B\}$ of nonzero blocks.*

Moreover, we have the explicit formula

$$\xi_i(\Phi) \;=\; \begin{cases} 0, & \textit{if } \; i = 0 \\ \\ \binom{n}{i}\binom{n-i-1}{i-1}, & \textit{if } \; 1 \le i \le \lfloor n/2 \rfloor. \end{cases} \tag{12}$$

For the first few values of n we have

$$\sum_{i=0}^{n} \ell_i(\Phi) x^i \;=\; \begin{cases} 2x, & \text{if } \; n = 2 \\ 3x + 3x^2, & \text{if } \; n = 3 \\ 4x + 14x^2 + 4x^3, & \text{if } \; n = 4 \\ 5x + 35x^2 + 35x^3 + 5x^4, & \text{if } \; n = 5 \\ 6x + 69x^2 + 146x^3 + 69x^4 + 6x^5, & \text{if } \; n = 6 \\ 7x + 119x^2 + 427x^3 + 427x^4 + 119x^5 + 7x^6, & \text{if } \; n = 7 \end{cases}$$

and

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i(\Phi) x^i \;=\; \begin{cases} 2x, & \text{if } \; n = 2 \\ 3x, & \text{if } \; n = 3 \\ 4x + 6x^2, & \text{if } \; n = 4 \\ 5x + 20x^2, & \text{if } \; n = 5 \\ 6x + 45x^2 + 20x^3, & \text{if } \; n = 6 \\ 7x + 84x^2 + 105x^3, & \text{if } \; n = 7 \\ 8x + 140x^2 + 336x^3 + 70x^4, & \text{if } \; n = 8. \end{cases}$$

The Dynkin diagram $\mathcal{D}(\Phi)$ is a path on the vertex set $I = \{1, 2, \dots, n\}$ with one double edge. We will assume that i and $i+1$ are adjacent in $\mathcal{D}(\Phi)$ for $1 \le i \le n-1$ and that the double edge connects vertices $n-1$ and n.

Proof of Proposition 3.2. A proof which parallels that of Proposition 3.1 can be given as follows. We denote by $\mathrm{NC}^B_+(n)$ the set of partitions $\pi \in \mathrm{NC}^B(n)$ which do not have a zero block. To compute the right-hand side of (2), we consider $h(\Delta_+(\Phi_J), x)$ for $J \subseteq I$. Lemmas 2.4 and 2.9, together with some straightforward computations, show that

$$h(\Delta_+(\Phi_I), x) \;=\; \sum_{\pi \in \mathrm{NC}^B_+(n)} x^{n - \|\pi\|}, \tag{13}$$

where $\|\pi\|$ stands for the number of pairs $\{B, -B\}$ of (nonzero) blocks of π. For general $J \subseteq I$ we claim that

$$h(\Delta_+(\Phi_J), x) \;=\; \sum_{\pi \in \mathrm{NC}^B_+(J)} x^{n - \|\pi\|}, \tag{14}$$

where $\mathrm{NC}^B_+(J)$ denotes the set of partitions $\pi \in \mathrm{NC}^B_+(n)$ such that $\{a\}$ is a nonnested (positive) singleton block of π for every $a \in I \smallsetminus J$. Given (14), the first statement follows by an application of inclusion-exclusion, as in the type A_n case.

The proof of (14) proceeds without essential change if $n-1$ or n does not belong to J. Otherwise we have $\{n-1,n\} \subseteq J$ and the argument in the proof of Proposition 3.1 should be modified as follows. Let b denote the maximum element of $I \smallsetminus J$. Then $\{b+1,\dots,n\}$ is the vertex set of the Dynkin diagram of one of the irreducible components, say Φ_k, of Φ_J. This component is of type B, while each of $\Phi_1,\dots,\Phi_{k-1}$ is of type A. Moreover, given $\pi \in \mathrm{NC}^B_+(J)$, the restriction of π on $\{b+1,\dots,n\} \cup \{-b-1,\dots,-n\}$ is a noncrossing partition of type B, while that on the vertex set of the Dynkin diagram of each of $\Phi_1,\dots,\Phi_{k-1}$ is a noncrossing partition of type A. Thus (14) follows by the computation in proof of Proposition 3.1 and the use of (9) and (13).

For the second statement, we need to replace the equivalence relation on $\mathrm{NC}^A(n)$ by one on $\mathrm{NC}^B_+(n)$, defined as follows. Suppose that $\pi \in \mathrm{NC}^B_+(n)$ has a nested positive singleton block $\{b\}$. Then there is a unique block $B \in \pi$ such that replacing the blocks $B, -B, \{b\}$ and $\{-b\}$ of π by the unions $B \cup \{b\}$ and $(-B) \cup \{-b\}$ results in a noncrossing partition $\pi' \in \mathrm{NC}^B_+(n)$. The required equivalence relation on $\mathrm{NC}^B_+(n)$ is defined as the finest equivalence relation under which π and π' are equivalent for all such pairs (π, b). For example, the partition in Figure 3 is equivalent to exactly one other noncrossing partition, of which $\{5,6,7,-4\}$ is a block. The proof then proceeds as in the type A_n case with only trivial adjustments; the details are left to the reader.

Finally, to deduce the explicit formula (12) we argue as in the proof of Lemma 2.9. The proof of [1, Theorem 2.3] shows that the partitions $\pi \in \mathrm{NC}^B(n)$ which have no zero block, no singleton block and a total of i pairs $\{B,-B\}$ of nonzero blocks are in one-to-one correspondence with pairs (S,f), where S is an i-element subset of $\{1,2,\dots,n\}$ and $f : S \to \{2,3,\dots\}$ is a function whose values sum to n. Clearly, the number of such pairs is given by the right-hand side of (12) and the proof follows. $\square$

3.3. **The root system D_n.** This section proves the following part of Theorems 1.1 and 1.2.

Proposition 3.3. *For the root system Φ of type D_n we have:*

$$\ell_i(\Phi) \;=\; (n-2)\cdot \#\{\pi \in \mathrm{NC}^A(n-1) : |\pi| = i\}$$

$$= \begin{cases} 0, & \text{if } \ i=0 \\[2ex] \dfrac{n-2}{i}\dbinom{n-1}{i-1}\dbinom{n-2}{i-1}, & \text{if } \ 1 \le i \le n, \end{cases}$$

and

$$\xi_i(\Phi) \;=\; \begin{cases} 0, & \text{if } \ i=0 \\[2ex] \dfrac{n-2}{i}\dbinom{2i-2}{i-1}\dbinom{n-2}{2i-2}, & \text{if } \ 1 \le i \le \lfloor n/2 \rfloor. \end{cases}$$

For the first few values of n we have

$$\sum_{i=0}^{n} \ell_i(\Phi) x^i \ = \ \begin{cases} 2x + 6x^2 + 2x^3, & \text{if } \ n = 4 \\ 3x + 18x^2 + 18x^3 + 3x^4, & \text{if } \ n = 5 \\ 4x + 40x^2 + 80x^3 + 40x^4 + 4x^5, & \text{if } \ n = 6 \\ 5x + 75x^2 + 250x^3 + 250x^4 + 75x^5 + 5x^6, & \text{if } \ n = 7 \end{cases}$$

and

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i(\Phi) x^i \ = \ \begin{cases} 2x + 2x^2, & \text{if } \ n = 4 \\ 3x + 9x^2, & \text{if } \ n = 5 \\ 4x + 24x^2 + 8x^3, & \text{if } \ n = 6 \\ 5x + 50x^2 + 50x^3, & \text{if } \ n = 7 \\ 6x + 90x^2 + 180x^3 + 30x^4, & \text{if } \ n = 8. \end{cases}$$

One can easily deduce from Proposition 3.3 a combinatorial interpretation to the numbers $\xi_i(\Phi)$; see also [23, Section 11.3]. We are not aware, however, of one which is analogous to those in Propositions 3.1 and 3.2 for types A_n and B_n.

The following notation and enumerative result will be used in the proof of Proposition 3.3. We will write

$$C_n(x) \ := \ \sum_{\pi \in \mathrm{NC}^A(n)} x^{|\pi|-1} \ = \ \sum_{\pi \in \mathrm{NC}^A(n)} x^{n-|\pi|} \ = \ \sum_{i=0}^{n} \frac{1}{i+1} \binom{n}{i} \binom{n-1}{i} x^i$$

and

$$F(x,t) \ := \ \sum_{n \ge 1} C_n(x)\, t^n \ = \ t + (1+x)\, t^2 + (1+3x+x^2)\, t^3 + \cdots \tag{15}$$

Then (see, for instance, [23, Equation (11)] and [28, Exercise 6.36]) we have

$$F(x,t) \ = \ xt\, F^2(x,t) + (1+x)t\, F(x,t) + t. \tag{16}$$

We will label the vertices of the Dynkin diagram $\mathcal{D}(\Phi)$ so that i and $i+1$ are adjacent in $\mathcal{D}(\Phi)$ for $1 \le i \le n-3$, while $n-2$ is adjacent to both $n-1$ and n.

Proof of Proposition 3.3. Let us write $\ell_n(x) := \ell_I(\Gamma(\Phi), x) = \sum_{i=0}^{n} \ell_i(\Phi) x^i$ for $n \ge 4$. The proposed formula for $\ell_i(\Phi)$ is equivalent to the equation

$$\ell_n(x) \ = \ (n-2) \cdot x\, C_{n-1}(x). \tag{17}$$

The formula for $\xi_i(\Phi)$ follows from that and the known explicit formula (see [23, Proposition 11.14]) for the γ-polynomial associated to $C_n(x)$. Thus, it suffices to prove (17).

We begin by rewriting the right-hand side of (2) in the following way. For $1 \le r \le n$, we will denote by $\mathcal{J}_r$ the collection of all subsets $J \subseteq I$ which contain $\{1, 2, \dots, r-1\}$ but do not contain r. Using Proposition 2.3 (v) and the type A_n case of Lemma 2.4, we find that

$$\sum_{J \in \mathcal{J}_r} (-1)^{|I \smallsetminus J|} \, h(\Delta_+(\Phi_J), x) \; = \; \begin{cases} -\,\ell_{n-1}(x), & \text{if } \; r = 1 \\ -\,C_{r-1}(x)\,\ell_{n-r}(x), & \text{if } \; 2 \le r \le n-3 \\ 0, & \text{if } \; r = n-2, \\ C_{n-2}(x) - C_{n-1}(x), & \text{if } \; r = n-1, \\ -\,C_{n-1}(x), & \text{if } \; r = n. \end{cases}$$

As a result, (2) can be rewritten as

$$\ell_n(x) \; = \; h(\Delta_+(\Phi_I), x) \, - \, \ell_{n-1}(x) \, - \sum_{r=2}^{n-3} C_{r-1}(x)\,\ell_{n-r}(x) \, + \, C_{n-2}(x) \, - \, 2C_{n-1}(x).$$

Thus, using induction on n, it suffices to prove that

$$\begin{aligned} h(\Delta_+(\Phi_I), x) &= (n-2)xC_{n-1}(x) \, + \, (n-3)xC_{n-2}(x) \\ &\quad + \sum_{r=2}^{n-3} (n-r-2)xC_{r-1}(x)C_{n-r-1}(x) \, - \, C_{n-2}(x) \, + \, 2C_{n-1}(x) \end{aligned} \tag{18}$$

for $n \ge 4$. Let $R_n(x)$ denote the right-hand side of (18) and $S_n(x)$ denote the sum which appears there. It follows directly from (15) that

$$\sum_{n \ge 4} S_n(x) t^n \; = \; xt^3 F(x,t) \, \frac{\partial F}{\partial t}(x,t) \, - \, xt^2 F^2(x,t).$$

Using (16), as well as the equation which results from that by differentiation with respect to t, we can rewrite the previous equation as

$$\sum_{n \ge 4} 2S_n(x) t^n \; = \; 2(1+x)t^2 F(x,t) \, + \, 2t^2 \, - \, 3tF(x,t) \, + \, (t^2 - t^3 - xt^3) \, \frac{\partial F}{\partial t}(x,t).$$

Equating the coefficients of t^n in the two sides above, we conclude that

$$2S_n(x) \; = \; (n-4)C_{n-1}(x) \, - \, (n-4)(1+x)C_{n-2}(x)$$

and hence that

$$R_n(x) \; = \; (n-2)xC_{n-1}(x) \, + \, \frac{n}{2} C_{n-1}(x) \, + \, (\frac{n}{2} - 1)(x-1)C_{n-2}(x).$$

Equation (18) follows from the formula for $h(\Delta_+(\Phi_I), x)$, given by the type D_n case of Lemma 2.4, and the previous expression for $R_n(x)$ by straightforward computation. This completes the proof of the proposition. $\square$

Proof of Theorems 1.1 and 1.2. The cases $\mathcal{X} \in \{A_n, B_n, D_n\}$ are covered by Propositions 3.1, 3.2 and 3.3. For $\mathcal{X} \in \{F_4, E_6, E_7, E_8\}$ the proposed formulas follow from (2) by explicit computation, based on the formulas for $h(\Delta_+(\Phi), x)$ given in [4, Section 6]. It remains to comment on the cases of types $I_2(m)$, H_3 and H_4.

For types $I_2(m)$ and H_3, it follows from the theory of local h-vectors (see parts (c) and (d) of [27, Example 2.3]) that $\xi_I(\Gamma(\Phi), x) = tx$, where t is the number of interior vertices of $\Gamma(\Phi)$. We have $t = m-2$ for $\mathcal{X} = I_2(m)$ and $t = 8$ for $\mathcal{X} = H_3$ (see [6, Figure 1] or

Remark 1 in Section 5) and the proposed formulas follow. Finally, let $\mathcal{X} = H_4$. From (2), (3) and the fact that $\xi_0(\Phi) = \ell_0(\Phi) = 0$ we get

$$\xi_1(\Phi)x(1+x)^2 + \xi_2(\Phi)x^2 \ = \ \sum_{J \subseteq I} (-1)^{|I \smallsetminus J|} \ h(\Delta_+(\Phi_J), x).$$

Setting $x = 1$ in the previous equality we get

$$4\xi_1(\Phi) + \xi_2(\Phi) \ = \ \sum_{J \subseteq I} (-1)^{|I \smallsetminus J|} \ N^+(\Phi_J), \tag{19}$$

where $N^+(\Psi)$ denotes the number of facets of $\Delta_+(\Psi)$ (i.e., the number of positive clusters for the root system Ψ). The right-hand side of (19) can be easily computed by hand (it equals 208), using [18, Proposition 3.9] and Proposition 2.3 (v). Since $\xi_1(\Phi) = \ell_1(\Phi)$ is equal to the number of interior vertices of $\Gamma(\Phi)$, we have $\xi_1(\Phi) = 42$ (see Remark 1 in Section 5). It follows from (19) that $\xi_2(\Phi) = 40$. $\square$

Proof of Corollary 1.3. Theorem 1.2 shows that the statement holds when Φ is irreducible. The general case then follows from Lemmas 2.2 and 2.8. $\square$

4. Proof of Theorem 1.4

We first review two of the tools from the combinatorics of permutations which will be used in the proof of Theorem 1.4. Throughout this section, we will denote by $\mathcal{E}_n$ the set of permutations in $\mathcal{S}_n$ for which every left to right maximum is a descent.

Descents and excedances. Given a permutation $w \in \mathcal{S}_n$, we may write w in cycle form so that each cycle begins with its largest element and the cycles of w are arranged in the increasing order of their largest elements (this is the standard representation of w, discussed on [26, p. 17]). We denote by $\phi(w)$ the sequence (or word) which is obtained after removing the parentheses from the cycles of w, considered as a permutation in $\mathcal{S}_n$. For instance, if $n = 9$ and $w = (5\,2\,4)(6\,1)(8)(9\,7\,3)$ in standard cycle form, then $\phi(w) = (5, 2, 4, 6, 1, 8, 9, 7, 3)$ is the permutation in $\mathcal{S}_9$ which maps 1 to 5, 2 to itself, 3 to 4 etc. The following properties hold (recall that $\mathcal{D}_n$ denotes the set of derangements in $\mathcal{S}_n$):

(a) the map $\phi : \mathcal{S}_n \to \mathcal{S}_n$ is bijective,
(b) $\phi(\mathcal{D}_n) = \mathcal{E}_n$,
(c) for $w \in \mathcal{S}_n$ and $1 \le i \le n$ we have $w(i) < i$ if and only if i is a descent of $\phi(w)$.

We will denote by $\hat{\phi} : \mathcal{D}_n \to \mathcal{E}_n$ the bijective map induced by ϕ on the set $\mathcal{D}_n$.

The Foata-Schützenberger-Strehl action. We will need the following variant of the Foata-Schützenberger-Strehl action on permutations; see, for instance, [14, Section V.1] [13, 15]. Up to date expositions and several applications of this construction can be found in [7, 23].

We let $w = (w_1, w_2, \dots, w_n)$ be a permutation in $\mathcal{E}_n$, where $w_i = w(i)$ for $1 \le i \le n$, and set $w_0 = 0$ and $w_{n+1} = n + 1$. A *double ascent* of w is an index $1 \le i \le n$ such that $w_{i-1} < w_i < w_{i+1}$. Given a double ascent or a double descent i of w, we define the permutation $\psi_i(w) \in \mathcal{S}_n$ as follows: If i is a double ascent of w, then $\psi_i(w)$ is the permutation obtained from w by moving w_i between w_j and w_{j+1}, where j is the largest index satisfying $1 \le j < i$ and $w_j > w_i > w_{j+1}$ (note that such an index exists, since

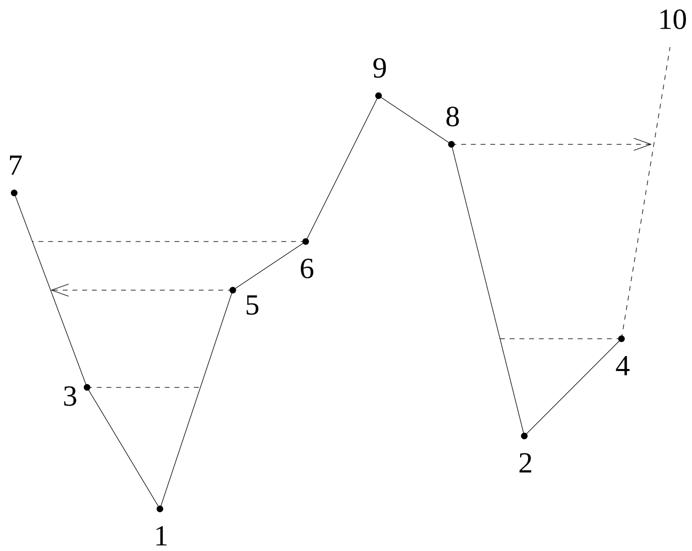

FIGURE 4. The permutation $w = (7, 3, 1, 5, 6, 9, 8, 2, 4) \in \mathcal{E}_9$.

$w \in \mathcal{E}_n$ and hence i is not a left to right maximum of w). Similarly, if i is a double descent of w, then $\psi_i(w)$ is the permutation obtained from w by moving w_i between w_j and w_{j+1}, where j is the smallest index satisfying $i < j \le n$ and $w_j < w_i < w_{j+1}$ (note that such an index exists, since $w_{n+1} = n + 1$). For instance, for the example of Figure 4 we have $\psi_4(w) = (7, 5, 3, 1, 6, 9, 8, 2, 4)$ and $\psi_7(w) = (7, 3, 1, 5, 6, 9, 2, 4, 8)$. Since the values at left to right maxima are unchanged when passing from w to $\psi_i(w)$, we have $\psi_i(w) \in \mathcal{E}_n$ in both cases.

We call two permutations in $\mathcal{E}_n$ equivalent (under the Foata-Schützenberger-Strehl action on $\mathcal{E}_n$) if one can be obtained by applying a sequence of maps of the form ψ_i to the other. We leave it to the reader to check that this defines an equivalence relation on $\mathcal{E}_n$ and that each equivalence class contains a unique element having no double descent. Moreover, if $w \in \mathcal{E}_n$ has no double descent and k double ascents, then the equivalence class $O(w)$ of w has 2^k elements and exactly $\binom{k}{j}$ of them have j descents more than w, so that

$$\sum_{u \in O(w)} x^{\mathrm{des}(u)} \; = \; x^{\mathrm{des}(w)} (1 + x)^k \; = \; x^{\mathrm{des}(w)} (1 + x)^{n - 2\mathrm{des}(w)}. \tag{20}$$

Proof of Theorem 1.4. Starting from (4) we find that

$$\ell_V(\Gamma, x) \; = \; \sum_{u \in \mathcal{D}_n} x^{\mathrm{ex}(u)} \; = \; \sum_{u \in \mathcal{D}_n} x^{\mathrm{ex}(u^{-1})} \; = \; \sum_{u \in \mathcal{D}_n} x^{n - \mathrm{ex}(u)} \; = \; \sum_{u \in \mathcal{E}_n} x^{\mathrm{des}(u)},$$

where the last equality uses property (c) for the map $\hat{\phi} : \mathcal{D}_n \to \mathcal{E}_n$. Summing (20) over all equivalence classes of the Foata-Schützenberger-Strehl action on $\mathcal{E}_n$ we get

$$\sum_{u \in \mathcal{E}_n} x^{\mathrm{des}(u)} \; = \; \sum_{w \in \hat{\mathcal{E}}_n} x^{\mathrm{des}(w)} (1 + x)^{n - 2\mathrm{des}(w)},$$

where $\hat{\mathcal{E}}_n$ denotes the set of permutations $w \in \mathcal{E}_n$ with no double descent. From the previous equalities and (6) we conclude that ξ_i is equal to the number of permutations $w \in \hat{\mathcal{E}}_n$ with $\mathrm{des}(w) = i$, so we have derived interpretation (iii) in the theorem. The latter and property (c), applied to the map $\hat{\phi} : \mathcal{D}_n \to \mathcal{E}_n$, imply that ξ_i is also equal to the number of derangements $w \in \mathcal{D}_n$ with $n - i$ excedances and no index j satisfying $w(j) < j < w^{-1}(j)$. Passing to the inverse permutation w^{-1} leads to interpretation (ii) of the theorem.

Finally, to check the equality between (i) and (ii), we work with descending (instead of ascending) runs. We observe that the map $\hat{\phi} : \mathcal{D}_n \to \mathcal{E}_n$ induces a bijection from the set of derangements $w \in \mathcal{D}_n$ with no double excedance onto the set of permutations in $\mathcal{S}_n$ with no descending run of length one. Moreover, the number of excedances of such w is equal to the number of descending runs of $\hat{\phi}(w)$ and the proof follows. $\square$

5. Remarks

1. It follows from the results of [27, Section 2] (see also our discussion in Section 2.3) that $\ell_1(\Phi) = \xi_1(\Phi)$ is equal to the number of interior vertices of $\Gamma(\Phi)$. These vertices are exactly the positive roots of Φ with support equal to Π (i.e., the positive roots which do not belong to any proper parabolic root subsystem Φ_J). The number of these roots was computed by Chapoton [10] and admits an elegant, uniform formula; see [10, Proposition 1.1]. It would be interesting to find uniform interpretations or formulas for $\ell_i(\Phi)$ or $\xi_i(\Phi)$ for other values of i. We are not aware of a simple closed form expression for $\ell_i(\Phi)$ in the type A_n and B_n cases.

2. It is natural to inquire for a more conceptual proof of Proposition 3.3, in the spirit of those of Propositions 3.1 and 3.2.

3. The unimodality of the derangement polynomials was first proved by Brenti [8, Corollary 1], who also asked for a combinatorial proof [8, p. 1140]. Such a proof was given by Stembridge [30, Corollary 2.2]. Theorem 1.4 provides another combinatorial proof (for a stronger statement). Since the barycentric subdivision $\mathrm{sd}(2^V)$ is a regular subdivision of 2^V, the unimodality of the derangement polynomials also follows from (4) and [27, Theorem 5.2].

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