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Doubly-refined enumeration of Alternating Sign Matrices and determinants of 2-staircase Schur functions

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Abstract. We prove a determinantal identity concerning Schur functions for 2-staircase diagrams $\lambda = (\ell n+\ell', \ell n, \ell(n-1)+\ell', \ell(n-1), \dots, \ell+\ell', \ell, \ell', 0)$. When $\ell = 1$ and $\ell' = 0$ these functions are related to the partition function of the 6-vertex model at the combinatorial point and hence to enumerations of Alternating Sign Matrices. A consequence of our result is an identity concerning the doubly-refined numbers of Alternating Sign Matrices.

1. Introduction

1.1. Alternating Sign Matrices. An *alternating sign matrix* (ASM) is a square matrix with entries in $\{-1, 0, +1\}$, such that along each row and along each column, if one forgets the 0's, the $+1$'s and -1's alternate, and the sum of the entries of each row and of each column is equal to 1. It is a famous combinatorial result that the number of such matrices of size n is

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = 1, 2, 7, 42, 429, \dots \tag{1.1}$$

After having been a conjecture for several years [**12**], this was first proven by Zeilberger in [**17**], and a simpler proof was given by Kuperberg [**9**], using a connection with the 6-Vertex Model of statistical mechanics, and an appropriate multivariate extension of the mere counting function A_n. A vivid account can be found in [**1**].

It follows easily from the definition that an alternating sign matrix has exactly one $+1$ in its first (and last) row (and column). Thus we have a sensible four-variable refined statistics, for these four positions in $\{1, \dots, n\}^4$, together with their projections on a smaller number of variables. After the dihedral symmetry of the square is taken into account, what remains is a single one-variable statistics (exhibiting a round formula), and two doubly-refined statistics: one, $\mathcal{A}^n_{ij}$, for the first and last row (or the rotated case), and one, $\mathcal{B}^n_{ij}$, for the first row and column (or the three rotated cases), see Figure 1, left. Matrices $\mathcal{A}^n$ of these doubly-refined alternating sign matrix numbers $\mathcal{A}^n_{ij}$

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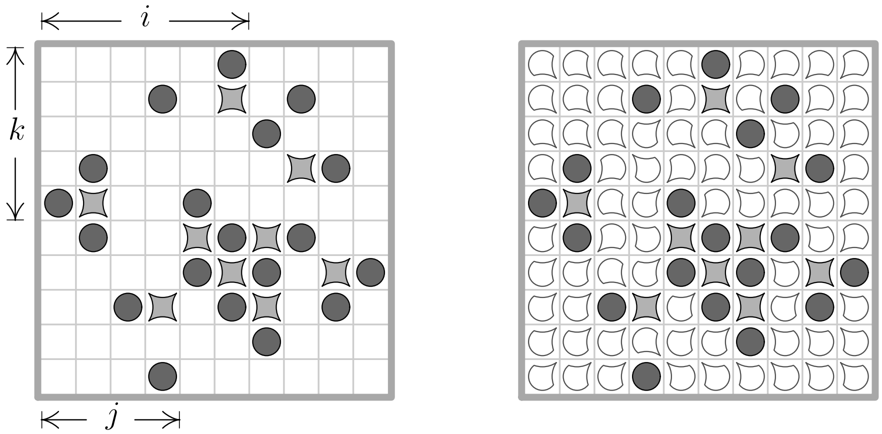

FIGURE 1. Left: a typical alternating sign matrix of size $n = 10$ (empty cells, disks and diamonds stand for 0, $+1$ and -1 entries, respectively). This matrix contributes to the statistics $\mathcal{A}^n_{ij}$ and $\mathcal{B}^n_{ik}$, with $(i, j, k) = (6, 4, 5)$. Right: empty cells are replaced by scale-shaped tiles, as to produce a valid tiling (i.e., concavities of neighbouring arcs do match). The direction of the tip specifies if the cell is of type NW, NE, SE or SW.

for $n = 1, 2, 3, 4, 5$ are given by

$$\mathcal{A}^1 = \begin{pmatrix} 1 \end{pmatrix}; \qquad \mathcal{A}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \mathcal{A}^3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix};$$

$$\mathcal{A}^4 = \begin{pmatrix} 0 & 2 & 3 & 2 \\ 2 & 4 & 5 & 3 \\ 3 & 5 & 4 & 2 \\ 2 & 3 & 2 & 0 \end{pmatrix}; \qquad \mathcal{A}^5 = \begin{pmatrix} 0 & 7 & 14 & 14 & 7 \\ 7 & 21 & 33 & 30 & 14 \\ 14 & 33 & 41 & 33 & 14 \\ 14 & 30 & 33 & 21 & 7 \\ 7 & 14 & 14 & 7 & 0 \end{pmatrix}.$$

Of course, by definition $\sum_{i,j} \mathcal{A}^n_{ij} = A_n$, i.e., $1, 2, 7, 42, 429, \ldots$ for the cases above. A simple argument (if an alternating sign matrix contains a $+1$ in a corner, then the rows and columns not containing that $+1$ form an alternating sign matrix of size one less) implies that the sum of the entries along the first (and last) row (and column) gives A_{n-1}, i.e., $1, 1, 2, 7, 42, \ldots$, and that the bottom-left and top-right entries are A_{n-2}, i.e., $1, 1, 1, 2, 7, \ldots$ These simple identities are *linear*. There exist also *quadratic* relations, of Plücker nature, relating these doubly-refined enumerations to A_n and the (singly-) refined enumerations (see, e.g., [**16, 2**]).

Let us now evaluate the *determinant* of these matrices:

$$\det(\mathcal{A}^2) = -1 = -1^{-1}, \qquad \det(\mathcal{A}^3) = 1 = 2^0,$$
$$\det(\mathcal{A}^4) = -7 = -7^1, \qquad \det(\mathcal{A}^5) = 1764 = 42^2, \quad \ldots$$

These small numerics suggest a relation that we prove in this paper.

Theorem 1.

$$\det(\mathcal{A}^n) = (-A_{n-1})^{n-3}. \tag{1.2}$$

This relation is *non-linear.* Its degree is neither fixed, nor bounded. What is fixed is what we could call "co-degree", namely the system size, minus the degree (in analogy to the definition of co-dimension of a subspace). Relations of this nature seem to be a novelty for the subject at hand.

Our proof of the theorem above will result as corollary of a much more general result on certain Schur functions. To see why these two topics are connected, we have to recall Kuperberg's solution of the Alternating Sign Matrix conjecture.

1.2. ASM, the 6-Vertex Model, and Schur functions. It follows from the connection with the 6-Vertex Model, that the generating function for a certain weighted enumeration of alternating sign matrices is given by a closed determinantal formula. Given an ASM $B = \{B_{ij}\}_{1\leq i,j\leq n}$ and i, j such that $B_{ij} = 0$, we say that (i, j) is a *north-west* (NW) site (respectively NE, SE, SW) if, forgetting the zeroes, the next $+1$ entry along the same column is in north direction, and along the same row is in west direction (and analogously for the other three cases) — see the right part of Figure 1. Consider some complex-valued function $\mu_n(B)$ defined on $n \times n$ ASMs, and call

$$Z_n = \sum_B \mu_n(B)$$

the corresponding generating function (in statistical mechanics $\mu_n(B)$ is a generalized *Gibbs measure* — an ordinary measure if it is real-positive and normalized — and Z_n is the *partition function*).

When $\mu_n(B)$ has the following factorized form, parametrized by $2n + 1$ variables $(x_1, \dots, x_n, y_1, \dots, y_n, q) = (\vec{x}, \vec{y}, q)$,

$$\mu_n(B; \vec{x}, \vec{y}, q) = \prod_{1\leq i,j\leq n} w_{i,j}(B)\,; \tag{1.3a}$$

$$w_{i,j}(B) = \begin{cases} (q - q^{-1})\sqrt{x_i y_j} & B_{ij} = \pm 1; \\ q^{-1}x_i - qy_j & B_{ij} = 0, \quad (i, j) \text{ is NW or SE}; \\ -x_i + y_j & B_{ij} = 0, \quad (i, j) \text{ is NE or SW}; \end{cases} \tag{1.3b}$$

integrability methods, and a recursion due to Korepin [**7**], allowed Izergin [**6**] to establish a determinantal expression for the generating function $Z_n(\vec{x}, \vec{y}, q) = \sum_B \mu_n(B; \vec{x}, \vec{y}, q)$. In particular, this function is symmetric under $\mathfrak{S}_n \times \mathfrak{S}_n$ acting on row- and column-parameters x_i and y_j.

The evaluation of A_n is recovered if we set $q = \exp(\frac{2\pi i}{3})$, $x_i = q^{-1}$ for all i and $y_j = q$ for all j, as in this case the local weights $w_{i,j}$ become all equal to $\sqrt{-3}$, independently of B, and thus $\mu_n(B)$ becomes constant (i.e., the *uniform measure*, up to an overall factor).

Later on, it has been discovered [**16, 14**] that the value $q = \exp(\frac{2\pi i}{3})$ (sometimes called the *combinatorial point*) has a special combinatorial property: $Z_n(\vec{x}, \vec{y}, q)$ becomes fully symmetric under $\mathfrak{S}_{2n}$ (acting on the $2n$-tuple of qx_i's and $q^{-1}y_j$'s together); more precisely, it is proportional to the Schur function associated to the Young diagram $\lambda_n = (n - 1, n - 1, n - 2, n - 2, \dots, 1, 1, 0, 0)$ (see Appendix A for the definition), evaluated at $\{qx_1, \dots, qx_n, q^{-1}y_1, \dots, q^{-1}y_n\}$ (see Figure 2, left, for a picture of this Young diagram). One consequence is that we have

$$A_n = 3^{-\binom{n}{2}} s_{\lambda_n}(1, 1, \dots, 1), \tag{1.4}$$

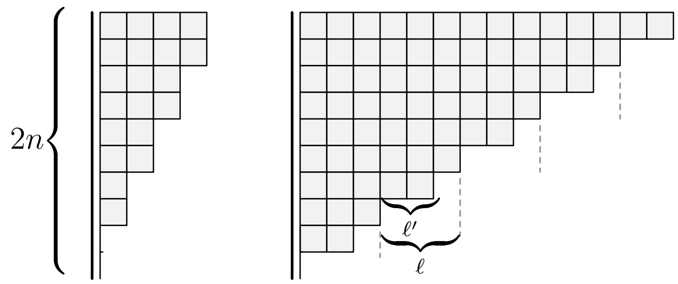

FIGURE 2. Left: the Young diagram λ_n, for $n = 5$. Right: the Young diagram $\lambda_{n,\ell,\ell'}$, for $n = 5$, $\ell = 3$ and $\ell' = 2$.

and also the refined enumerations introduced above are related to specializations of this Schur function, in which some parameters are left as indeterminates.

In particular for the $\mathcal{A}^n_{ij}$'s, defining the generating function

$$\mathcal{A}_n(u,v) = \sum_{1\leq i,j\leq n} \mathcal{A}^n_{ij}\, u^{i-1} v^{n-j} \,,$$

one finds

$$\mathcal{A}_n(u,v) = 3^{-\binom{n}{2}} (q^2(q+u)(q+v))^{n-1} s_{\lambda_n}\big(\tfrac{1+qu}{q+u}, \tfrac{1+qv}{q+v}, 1, \dots, 1\big) \,. \tag{1.5}$$

(The rational function $\frac{1+qu}{q+u}$ originates from the ratio of $w_{ij}(B)$ in the two last cases of (1.3b).)

A detailed analysis of the doubly-refined enumeration formula (1.5) restated in terms of multiple contour integrals, and the proof of a relation with a doubly-refined enumeration formula for totally-symmetric self-complementary plane partitions in a cube of size $2n$, can be found in [**5**].

1.3. On the determinants of Schur functions. In this section we state a theorem concerning the determinant of a matrix whose entries are Schur functions s_{λ_n}. Not surprisingly, as these functions are related to ASM enumerations, e.g., through equations (1.4) and (1.5), this property will show up to be the structure behind Theorem 1, and conceivably, it has an interest by itself. For this reason, in this paper we pursue the task of stating and proving a much wider version of the aforementioned property, than the one that would suffice for Theorem 1. This leads us to introduce a wider family of Young diagrams.

We define the *2-staircase diagram* $\lambda_{n,\ell,\ell'}$, for $n \geq 1$, $0 \leq \ell' \leq \ell$, as

$$\lambda_{n,\ell,\ell'} = \big((n-1)\ell + \ell', (n-1)\ell, (n-2)\ell + \ell', (n-2)\ell, \dots, \ell', 0\big);$$

i.e., $(\lambda_{n,\ell,\ell'})_{2j-1} = (n-j)\ell + \ell'$ and $(\lambda_{n,\ell,\ell'})_{2j} = (n-j)\ell$ (see Figure 2, right). We call the associated Schur polynomial, $s_{\lambda_{n,\ell,\ell'}}(z_1, \dots, z_{2n})$, a *2-staircase Schur function.*

The name comes from the fact that this family of diagrams generalizes the well-known family of *staircase diagrams* $\mu_{n,\ell}$

$$\mu_{n,\ell} = \big((n-1)\ell, (n-2)\ell, \dots, \ell, 0\big) \,. \tag{1.6}$$

The Schur functions s_{λ_n} are thus particular cases of 2-staircase Schur functions, corresponding to $\ell = 1$ and $\ell' = 0$.

The polynomials $s_{\lambda_{n,\ell,\ell'}}$ have been considered recently by Alain Lascoux. In particular, in [**11**, Lemma 13] they are shown to coincide with the specialization at $q = \exp(\frac{2\pi i}{\ell+2})$ of a certain natural extension of *Gaudin functions*.

In an apparently unrelated context we see the appearance of the polynomials $s_{\lambda_{n,\ell,\ell'}}$, for $\ell' = 0$ only. This context, analysed by Paul Zinn-Justin in [**18**], is the study of the solution of the qKZ equation related to the spin $\ell/2$ representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}(2)})$ with $q = \exp(\frac{2\pi i}{\ell+2})$. It is shown that, by taking the scalar product of the solution of the qKZ equation with a natural reference state, one obtains $s_{\lambda_{n,\ell,0}}$.

As anticipated, our Theorem 1 will be a corollary of the following result, of independent interest, which exhibits a remarkable factorization of a determinant of 2-staircase Schur functions:

Theorem 2. *Let $N = \ell(n-1) + \ell' + 1$. Let $\{x_i, y_i\}_{1 \le i \le N}$ be indeterminates, let $f(\vec{z}, w_1, w_2)$ stand for $f(z_1, \ldots, z_{2n-2}, w_1, w_2)$, and, for an ordered N-tuple $\vec{x} = (x_1, x_2, \ldots, x_N)$, let $\Delta(\vec{x}) = \prod_{i<j}(x_i - x_j)$ denote the usual Vandermonde determinant. Then*

$$\begin{aligned}(1.7)\quad \det\Big(s_{\lambda_{n,\ell,\ell'}}(\vec{z}, x_i, y_j)\Big)_{1 \le i,j \le N} \\ = c(n,\ell,\ell')\,\Delta(\vec{x})\Delta(\vec{y})\left(\prod_{i=1}^{2n-2} z_i^{\ell'(\ell+1)}\right) s^{\ell}_{\mu_{2n-2,\ell+1}}(\vec{z})\; s^{\ell(n-2)+\ell'-1}_{\lambda_{n-1,\ell,\ell'}}(\vec{z})\,.\end{aligned}$$

The quantity $c(n,\ell,\ell')$ has its values in $\{0, \pm 1\}$. More precisely,

$$(1.8)\qquad c(n,\ell,\ell') = \begin{cases} (-1)^{(n-1)\binom{\ell+1}{2} + \binom{\ell'+1}{2}} & \textit{if } n = 1 \textit{ or } \gcd(\ell+2, \ell'+1) = 1, \\ 0 & \textit{if } n > 1 \textit{ and } \gcd(\ell+2, \ell'+1) \neq 1. \end{cases}$$

Observe that, as is well known, the staircase Schur function $s_{\mu_{2n-2,\ell+1}}$ can be further factorized. Let us recall the definition of the (bivariate homogeneous) Chebyshev polynomials (of the second kind)

$$(1.9)\qquad U_h(x,y) = \frac{x^{h+1} - y^{h+1}}{x - y} = x^h + x^{h-1}y + \cdots + y^h = \prod_{i=1}^{h}(x - \zeta^i y)\,,$$

where ζ is a primitive $(h+1)$-st root of unity. One can write (cf. equation (A.6))

$$(1.10)\qquad s_{\mu_{N,h}}(\vec{z}) = \prod_{1 \le i < j \le N} U_h(z_i, z_j)\,.$$

As Schur functions have several determinant representations (see Appendix A), the left-hand side quantity of the theorem is a "determinant of determinants", a structure in linear algebra that is sometimes called a *compound determinant* [**13**, Ch. VI]. As we will see, the theory of compound determinants will play a crucial role in our proof.

Results of the form of the one in Theorem 2, or at least approaches to quantities as in the left-hand side of equation (1.7), already exist in the literature, although mostly with partitions of comparatively simpler structure. Cf. [**11**], where also a general approach is outlined. In particular, equations (23) and (24) in [**11**] have a form of striking similarity with our theorem above, while involving a rectangular partition $r^p \equiv (r, r, \ldots, r)$ (r is repeated p times), and the basic 1-staircase partition $(r, r-1, r-2, \ldots, 1, 0)$, respectively, and the unnumbered third equation after Corollary 9 of [**11**] (for which, however, no factorization is stated) has a similar structure to what will be the matrix of our analysis, with the only difference that it presents a Chebyshev polynomial in the denominator instead of in the numerator.

Theorem 2 is easily seen to hold for $n = 1$ and any (ℓ, ℓ'). This could seem a good base for an induction. However we use inductive arguments only for the minor task of determining the overall constant $c(n, \ell, \ell')$ in Section 4.2. On the other hand, in Section 4.1 we prove divisibility results, by a method reminiscent of the "exhaustion of factors" method described in Krattenthaler's survey [**8**].

Note however that the factors $s_{\lambda_{n-1,\ell,\ell'}}$ are polynomials of 'large' degree, $\ell n(n-1) + \ell' n$, with no factorizations as long as $\gcd(\ell+2, \ell'+1) = 1$ (we give a partial proof of this statement in Proposition 5 below — a full proof is not hard to achieve). Thus, in a sense, the tools we develop in Section 3 should be regarded as an extension of the exhaustion of factor method to the case in which we have an infinite family of determinantal identities, and some of the factors have an *unbounded* degree, scaling with the size parameter associated to the family.

Finally, let us add a few words on notations: throughout the paper, if $\vec{z}$ is a vector of length n (the length will be clear by the context), we write $f(\vec{z})$ as a shortcut for $f(z_1, \dots, z_n)$, and $f(\vec{z}, w_1, w_2, \dots)$ as a shortcut for $f(z_1, \dots, z_n, w_1, w_2, \dots)$. We also write $f(\vec{z}_{\setminus i_1 \cdots i_k}, w_1, w_2, \dots)$ if the variables z_{i_1}, …, z_{i_k} are dropped from the list $(z_1, \dots, z_n)$.

The paper is organized as follows. In Section 2 we show how to derive Theorem 1 from Theorem 2 specialized to $\ell = 1$ and $\ell' = 0$. In Section 3 we present some preparatory lemmas for the proof of Theorem 2, which is presented in Section 4. Appendix A collects some basic definitions and facts on Schur functions, while in Appendix B we introduce an even larger class of staircase Schur functions, and study some of their properties.

2. Derivation of Theorem 1 from Theorem 2

For a polynomial $f(x, y)$, denote by $[x^i y^j] f(x, y)$ the coefficient of the monomial $x^i y^j$. We first state a simple but useful lemma.

Lemma 1. *Let $P(u, v)$ be a polynomial in two indeterminates, of degree at most $n-1$ in each variable. Set $P = \big([u^{i-1} v^{j-1}] P(u, v)\big)_{1 \le i,j \le n}$. Furthermore, let u_i, v_j be indeterminates. Then*

$$\det \big(P(u_i, v_j)\big)_{1 \le i,j \le n} = \Delta(\vec{u}) \Delta(\vec{v}) \det P \,. \tag{2.1}$$

Proof. Let $V(\vec{u})$ denote the Vandermonde matrix $(u_j^{i-1})_{1 \le i,j \le n}$. Then

$$\det V(\vec{u}) = (-1)^{n(n-1)/2} \Delta(\vec{u}),$$

and the matrix $\big(P(u_i, v_j)\big)_{1 \le i,j \le n}$ is the product $V(\vec{u})^{\mathrm{T}} P \, V(\vec{v})$. □

This lemma implies that our Theorem 2 is equivalent to

$$\det\Big([x^iy^j]s_{\lambda_{n,\ell,\ell'}}(\vec{z},x,y)\Big)_{0\le i,j\le \ell(n-1)+\ell'} \\ = c(n,\ell,\ell')\left(\prod_{i=1}^{2n-2} z_i^{\ell'(\ell+1)}\right) s^{\ell}_{\mu_{2n-2,\ell+1}}(\vec{z})\; s^{\ell(n-2)+\ell'-1}_{\lambda_{n-1,\ell,\ell'}}(\vec{z}) \tag{2.2}$$

(of course, with $c(n,\ell,\ell')$ as in (1.8)).

Now we proceed to the proof of Theorem 1. With $\vec{u}=(u_1,\dots,u_n)$, we compute

$$\Delta\big(\{\tfrac{1+qu_i}{q+u_i}\}\big) = \Delta(\vec{u})\,(q^2-1)^{\binom{n}{2}}\prod_i (q+u_i)^{-(n-1)}. \tag{2.3}$$

It follows from Lemma 1, and equation (1.5), that

$$\begin{aligned}\Delta(\vec{u})\Delta(\vec{v})\det(\mathcal{A}^n_{ij}) &= \\ &= (-1)^{\binom{n}{2}}\det\left(\frac{(q^2(q+u_i)(q+v_j))^{n-1}}{3^{\binom{n}{2}}} s_{\lambda_n}\big(\tfrac{1+qu_i}{q+u_i},\tfrac{1+qv_j}{q+v_j},1,\dots,1\big)\right)_{1\le i,j\le n} \\ &= \left(\frac{-q^4}{3^n}\right)^{\binom{n}{2}}\prod_{i=1}^{n}\big((q+u_i)(q+v_i)\big)^{n-1}\det\Big(s_{\lambda_n}\big(\tfrac{1+qu_i}{q+u_i},\tfrac{1+qv_j}{q+v_j},1,\dots,1\big)\Big)_{1\le i,j\le n}.\end{aligned} \tag{2.4}$$

Applying Theorem 2 with $\ell=1$, $\ell'=0$, $x_i=\frac{1+qu_i}{q+u_i}$, and $y_j=\frac{1+qv_j}{q+v_j}$ to the determinant on the right-hand side, and then (2.3), we obtain

$$\Delta(\vec{u})\Delta(\vec{v})\det(\mathcal{A}^n_{ij}) = \Delta(\vec{u})\Delta(\vec{v})\,(-1)^{n-1+\binom{n}{2}}\left(\frac{(q-q^2)^2}{3^n}\right)^{\binom{n}{2}} \\ \times s_{\mu_{2n-2,2}}(1,1,\dots,1)\; s^{n-3}_{\lambda_{n-1}}(1,1,\dots,1) \tag{2.5}$$

It should be noted that $(q-q^2)^2=-3$. By the explicit evaluation of a staircase Schur function, equation (1.10), we have

$$s_{\mu_{2n-2,2}}(1,1,\dots,1) = 3^{\binom{2n-2}{2}}. \tag{2.6}$$

Theorem 1 follows from (1.4), (2.5), and (2.6). □

3. Preliminary results

3.1. On the minor expansion of a sum of matrices. Consider k $n\times n$ matrices $M^{(a)}=(M^{(a)}_{ij})_{1\le i,j\le n}$, $1\le a\le k$, where the $M^{(a)}_{ij}$ are indeterminates. Given an $n\times n$ matrix A and subsets I,J of $[n]:=\{1,2,\dots,n\}$, denote by $A_{I,J}$ the restriction of A to rows in I and columns in J. Denote by $\mathcal{I}=(I_1,\dots,I_k)$ an ordered k-tuple of subsets $I_a\subseteq[n]$ (possibly empty), forming a partition of $[n]$. For two such k-tuples $\mathcal{I}$ and $\mathcal{J}$, say that they are *compatible* if $|I_a|=|J_a|$ for all $a\in\{1,\dots,k\}$, and write $\mathcal{I}\sim\mathcal{J}$ in this case. Denote by $\epsilon(\mathcal{I},\mathcal{J})$ the sign of the permutation that reorders $(I_1,\dots,I_k)$ into $(J_1,\dots,J_k)$, with elements within the blocks in order. Then we have the following fact.

Proposition 1 (Minor expansion of a sum of matrices)**.**

$$\det\left(\sum_{a=1}^{k} M^{(a)}\right) = \sum_{\substack{\mathcal{I},\mathcal{J}\\ \mathcal{I}\sim\mathcal{J}}} \epsilon(\mathcal{I},\mathcal{J}) \prod_{a=1}^{k} \det M^{(a)}_{I_a,J_a}\,. \tag{3.1}$$

Proof. Consider the full expansion of the determinant

$$\begin{aligned}\det\left(\sum_{a=1}^{k} M^{(a)}\right) &= \sum_{\sigma\in\mathfrak{S}_n} \epsilon(\sigma) \prod_{i=1}^{n} \Big(\sum_{a=1}^{k} M^{(a)}_{i\,\sigma(i)}\Big)\\ &= \sum_{\sigma\in\mathfrak{S}_n} \sum_{b\in[k]^n} \epsilon(\sigma) \prod_{i=1}^{n} M^{(b(i))}_{i\,\sigma(i)}\,.\end{aligned}$$

To each pair (σ, b) in the linear combination above, we associate a pair $(\mathcal{I},\mathcal{J})$ of compatible partitions by

$$I_a = \{i \;:\; b(i) = a\}\,; \qquad\qquad J_a = \{j \;:\; b(\sigma^{-1}(j)) = a\}\,. \tag{3.2}$$

So $\mathcal{I}$ is determined by b alone, and all the permutations σ producing the same $\mathcal{J}$ can be written as the "canonical" permutation τ that reorders $(I_1,\dots,I_k)$ into $(J_1,\dots,J_k)$ with elements within the blocks in order, acting from the left on a permutation $\rho = \prod_a \rho_a \in \mathfrak{S}_{I_1}\times\cdots\times\mathfrak{S}_{I_k}$. The sign factorizes, $\epsilon(\sigma) = \epsilon(\tau)\prod_a \epsilon(\rho_a)$, and $\epsilon(\tau) = \epsilon(\mathcal{I},\mathcal{J})$ by definition, thus

$$\det\Big(\sum_{a=1}^{k} M^{(a)}\Big) = \sum_{\substack{\mathcal{I},\mathcal{J}\\ \mathcal{I}\sim\mathcal{J}}} \epsilon(\mathcal{I},\mathcal{J}) \prod_{a} \sum_{\rho_a\in\mathfrak{S}_{I_a}} \epsilon(\rho_a) \prod_{i\in I_a} M^{(a)}_{i\,\tau\circ\rho_a(i)}.$$

For each index a, the sum over the permutations ρ_a produces the determinant of the appropriate minor. □

3.2. Bazin–Reiss–Picquet Theorem. In this section we recall the Bazin–Reiss–Picquet Theorem [**13**, pp. 193–195, § 202–204]. Take a triple (m, n, p) of integers $m \geq n \geq p \geq 0$. Let $S_{n,p}$ be the set of subsets of $[n]$ of cardinality p. (Thus, $|S_{n,p}| = \binom{n}{p}$.) For a set $I \in S_{n,p}$, write $I = \{i_1,\dots,i_p\}$ for the ordered list of elements.

Consider the $m\times n$ matrices $A = (A_{ij})_{1\leq i\leq m,\,1\leq j\leq n}$ and $B = (B_{ij})_{1\leq i\leq m,\,1\leq j\leq n}$, and the $m\times(m-n)$ matrix $C = (C_{ij})_{1\leq i\leq m,\,1\leq j\leq m-n}$, where the A_{ij}, B_{ij}, and C_{ij} are indeterminates. Write $(X|Y)$ for the matrix resulting from taking all the columns of X, followed by all the columns of Y.

For a pair $(I, J) \in S_{n,p}\times S_{n,p}$, define $M^{I,J}$ as the matrix

$$M^{I,J}_{h,k} = \begin{cases} A_{h,k} & \text{if } k\leq n,\ k\notin I;\\ B_{h,j_\ell} & \text{if } k = i_\ell;\\ C_{h,k-n} & \text{if } n < k \leq m;\end{cases}$$

(that is, replace the columns I of $(A|C)$ by the columns J of B, in order). Define $D_{I,J} = \det M^{I,J}$. Choose a total ordering of $S_{n,p}$, and construct the matrix $D = \big(D_{I,J}\big)_{I,J\in S_{n,p}}$, which is of size $\binom{n}{p}$. Then the compound determinant $\det D$ does not depend on the chosen ordering, and it has the following factorization property.

Theorem 3 (Bazin–Reiss–Picquet)**.** *We have*

$$\det D = \det(A|C)^{\binom{n-1}{p}} \ \det(B|C)^{\binom{n-1}{p-1}} . \tag{3.3}$$

3.3. A divisibility corollary. A corollary of the Bazin–Reiss–Picquet Theorem is a divisibility result for a special family of determinants. Take $m \geq n \geq k \geq 0$. Consider m indeterminates z_i, n indeterminates y_j, and $2nk$ indeterminates u_i^a, v_i^a, with $1 \leq i \leq n$ and $1 \leq a \leq k$ (u_i^a, v_i^a may possibly be elements in the polynomial ring in z_i's and y_j's). Take m polynomial functions $f_j(x)$, and introduce the associated *Slater determinant*, that is, the antisymmetric polynomial

$$P(\vec{x}) = P(x_1, \ldots, x_m) = \det\big(f_j(x_i)\big)_{1 \leq i,j \leq m} .$$

A typical example is a shifted Vandermonde determinant,

$$P(x_1, \ldots, x_m) = \Delta_\lambda(x_1, \ldots, x_m),$$

for λ a partition of length at most m (see Appendix A).

Then we have the following fact.

Proposition 2. *The polynomial* $\det\Big(\sum_{a=1}^k u_i^a v_j^a \ P(\vec{z}_{\setminus i}, y_j)\Big)_{1\leq i,j\leq n}$ *is divisible by the polynomial* $\big(P(\vec{z})\big)^{n-k}$.

Proof. Apply the formula for the minor expansion of a sum of matrices given in Proposition 1, to get

$$\det\Big(\sum_{a=1}^k u_i^a v_j^a P(\vec{z}_{\setminus i}, y_j)\Big)_{1\leq i,j\leq n} = \sum_{\substack{\mathcal{I},\mathcal{J}\\ \mathcal{I}\sim\mathcal{J}}} \epsilon(\mathcal{I},\mathcal{J}) \prod_{\substack{1\leq a\leq k\\ i\in I_a}} u_i^a \prod_{\substack{1\leq a\leq k\\ j\in J_a}} v_j^a \prod_{a=1}^k \det\big(P(\vec{z}_{\setminus i}, y_j)\big)_{i\in I_a,\, j\in J_a} .$$

To each determinant of the form $\det\big(P(\vec{z}_{\setminus i}, y_j)\big)_{i\in I,\, j\in J}$ apply the Bazin–Reiss–Picquet Theorem with $(m,n,p) \to (m, |I|, 1)$, and get

$$\det\big(P(\vec{z}_{\setminus i}, y_j)\big)_{i\in I,\, j\in J} = P(\vec{z})^{|I|-1} P(\vec{w}(I,J)) ,$$

where

$$w_k(I,J) = \begin{cases} z_k & \text{if } k \notin I; \\ y_{j_\ell} & \text{if } k = i_\ell. \end{cases}$$

Thus we have

$$\det\Big(\sum_{a=1}^k u_i^a v_j^a P(\vec{z}_{\setminus i}, y_j)\Big)_{1\leq i,j\leq n} = P(\vec{z})^{n-k} \sum_{\substack{\mathcal{I},\mathcal{J}\\ \mathcal{I}\sim\mathcal{J}}} \epsilon(\mathcal{I},\mathcal{J}) \prod_{\substack{1\leq a\leq k\\ i\in I_a}} u_i^a \prod_{\substack{1\leq a\leq k\\ j\in J_a}} v_j^a \prod_{a=1}^k P(\vec{w}(I_a, J_a)) ,$$

and the quantity in the sum on the right-hand side is a polynomial. ∎

3.4. Vanishing and recursion properties of 2-staircase Schur functions. Here we gather some relevant facts about the family of 2-staircase Schur functions $s_{\lambda_{n,\ell,\ell'}}(\vec{z})$ introduced in (1.3). In this section we use q as synonym for $\exp(\frac{2\pi i}{\ell+2})$.

Proposition 3 (Wheel condition). *For distinct g, h and k in $\{0,\dots,\ell+1\}$, and distinct i, j and m in $\{1,\dots,2n\}$, we have*

$$s_{\lambda_{n,\ell,\ell'}}(\vec{z}_{\setminus ijm}, q^g w, q^h w, q^k w) = 0\,. \tag{3.4}$$

Proposition 4 (Recursion relation). *For k in $\{1,\dots,\ell+1\}$, and distinct i, j in $\{1,\dots,2n\}$, we have*

$$s_{\lambda_{n,\ell,\ell'}}(\vec{z}_{\setminus ij}, w, q^k w) = w^{\ell'} U_{\ell'}(1,q^k) \prod_{\substack{1\le m\le 2n\\ m\neq i,j}} \frac{U_{\ell+1}(z_m,w)}{z_m - q^k w}\; s_{\lambda_{n-1,\ell,\ell'}}(\vec{z}_{\setminus ij})\,. \tag{3.5}$$

Propositions 3 and 4 are occurrences, already known in the literature (cf., e.g., [**18**, Thm. 4]), of vanishing conditions (and related recursion properties) within a broad family, for which the name "wheel condition" is often used. There has been a recent interest in the investigation of the structure of the corresponding ideals, in the ring of symmetric polynomials (see, e.g., [**3, 4**]).

We prove the propositions above in Appendix B. More precisely, in the appendix we generalize 2-staircase Schur functions to the *m-staircase* case, and prove the appropriate generalizations of the propositions above, together with some further properties of potential future interest.

Notice that, if $\gcd(\ell'+1,\ell+2) = g > 1$, then there exists some k with $1 \le k \le \ell+1$ such that q^k is a root of $U_{\ell'}(1,x)$ (e.g., $k = (\ell+2)/g$). Then it follows from equation (3.5) that $s_{\lambda_{n,\ell,\ell'}}$ vanishes if $z_i = q^k z_j$, i.e., it is divisible by $z_i - q^k z_j$. On the other hand, if $\gcd(\ell'+1,\ell+2) = 1$, one has the following proposition.

Proposition 5. *Suppose* $\gcd(\ell'+1,\ell+2) = 1$ *and $n \ge 2$, then $s_{\lambda_{n,\ell,\ell'}}$ has no factors of the form $(z_i - \eta z_j)$, for any ij with $1 \le i,j \le 2n$ and $\eta \in \mathbb{C}$.*

Proof. We prove the statement by induction on n. The case $n = 2$ is done by direct inspection of $s_{\lambda_{n,\ell,\ell'}}$.[1] Now suppose the statement is true up to $n-1$ and assume that there exists $i,j \in \{1,\dots,2n\}$ and $\eta \in \mathbb{C}$ such that $(z_i - \eta z_j)$ divides $s_{\lambda_{n,\ell,\ell'}}$. Then take k and h distinct indices in $\{1,\dots,2n\}\backslash\{i,j\}$ (note that we need $n \ge 2$ at this point), and specialize $s_{\lambda_{n,\ell,\ell'}}|_{z_k = q z_h}$. The linear term $z_i - \eta z_j$ must divide also the specialized polynomial, and, using the recursion relation of Proposition 4, it must divide

[1] E.g., observe that, for $z_1 - \eta z_2$ to divide the Schur function, it should divide the shifted Vandermonde determinant in the numerator, with a higher power with respect to the ordinary Vandermonde determinant in the denominator. The case $\eta = 1$ is easily ruled out (even if we further specialize $z_3 = z$, $z_4 = 0$, we obtain $s_{\lambda_{2,\ell,\ell'}}(z,z,z,0) = z^{2(\ell+\ell')}(\ell+2)(\ell'+1)(\ell-\ell'+1)/2$, which is not identically zero as we have $\ell \ge 0$ and $0 \le \ell' \le \ell$). For $\eta \neq 1$ we cannot have simplifications with the Vandermonde determinant in the denominator, and it suffices to analyse the shifted Vandermonde determinant, which gives

$$\Delta_{\lambda_{2,\ell,\ell'}}(z,\eta z,0,1) = z^{\ell+\ell'+3}\big(((\eta z)^{\ell+2}-1)(\eta^{\ell'+1}-1) - ((\eta z)^{\ell'+1}-1)(\eta^{\ell+2}-1)\big)\,.$$

Again, this is not identically zero, as, because of the gcd hypothesis, $\eta^{\ell'+1}-1$ and $\eta^{\ell+2}-1$ cannot vanish simultaneously.

the corresponding right-hand side expression for (3.5). However, this expression is non-zero for the other variables z_m being generic (because the only potentially dangerous factor, $U_{\ell'}(1,q^k)$, may vanish only if $\gcd(\ell'+1,\ell+2)>1$), and the factors of the form $z_k^{\ell'}$, and $U_{\ell+1}(z_m,z_k)$, for $m\neq k,h$, do not contain $z_i-\eta z_j$ as a factor. Thus $z_i-\eta z_j$ must divide $s_{\lambda_{n-1,\ell,\ell'}}$, in contradiction with the inductive hypothesis. □

4. Proof of Theorem 2

As outlined in the introduction, our strategy for proving Theorem 2 will be as follows: let us call $\psi_{n,\ell,\ell'}(z,x,y)$ the left-hand side of (1.7); first we identify several polynomial factors of $\psi_{n,\ell,\ell'}(z,x,y)$; then we show that these factors are relatively prime and that their product exhausts the degree of $\psi_{n,\ell,\ell'}(z,x,y)$; finally, we determine the overall constant factor. As in the previous subsection, also in this section we set $q=e^{\frac{2\pi i}{\ell+2}}$.

4.1. Polynomial factors of $\psi_{n,\ell,\ell'}(\vec{z},\vec{x},\vec{y})$. We start by identifying a polynomial factor of $\psi_{n,\ell,\ell'}(\vec{z},\vec{x},\vec{y})$ whose factorization involves only monomials and binomials. By Lemma 1, the polynomial $\psi_{n,\ell,\ell'}(\vec{z},\vec{x},\vec{y})$ is divisible by $\Delta(\vec{x})$ and $\Delta(\vec{y})$. Since the degree of $\psi_{n,\ell,\ell'}$ in each variable x_i or y_i separately is $(n-1)\ell+\ell'$, which is the same as the degree of $\Delta(\vec{x})\Delta(\vec{y})$, the quotient is a polynomial of degree zero in x_i and y_j (namely, it is the determinant of the matrix of coefficients in x and y of $s_{\lambda_{n,\ell,\ell'}}(\vec{z},x,y)$). Call $Q_{n,\ell,\ell'}(\vec{z})$ the resulting quotient

$$Q_{n,\ell,\ell'}(\vec{z})=\frac{\psi_{n,\ell,\ell'}(\vec{z},\vec{x},\vec{y})}{\Delta(\vec{x})\Delta(\vec{y})}. \tag{4.1}$$

We work out immediately the case of Theorem 2 corresponding to the second case of equation (1.8)

Proposition 6. *If* $\gcd(\ell'+1,\ell+2)>1$ *and* $n\geq 2$*, then* $Q_{n,\ell,\ell'}(\vec{z})=0$*.*

Proof. Say $\gcd(\ell'+1,\ell+2)=g>1$. It follows that the polynomials $U_{\ell'}(1,x)$ and $U_{\ell+1}(1,x)$ have a common root q^k, for $k=(\ell+2)/g$. We can exploit the fact that Q, defined in equation (4.1) as a rational function of the z, x and y's, is actually independent of the x and y's. In particular, we can choose $x_1=q^kz_1$ (and leave $x_2,\dots,x_n,y_1,\dots,y_n$ generic). Consider the matrix $(M_{ij})_{1\leq i,j\leq N}$ with $M_{ij}=s_{\lambda_{n,\ell,\ell'}}(\vec{z},x_i,y_j)$, whose determinant is $\psi_{n,\ell,\ell'}$. By applying the recursion relation of Proposition 4, we see that the row corresponding to x_1 vanishes identically. On the other hand, as the remaining x and y variables are generic, the Vandermonde factors are non-zero. As a consequence, $Q_{n,\ell,\ell'}(\vec{z})=0$. □

We proceed to find other factors of $Q_{n,\ell,\ell'}$, for the relevant case of equation (1.8).

Proposition 7. *For* $n\geq 2$*,* $s^{\ell}_{\mu_{2n-2,\ell+1}}(\vec{z})\left(\prod_{i=1}^{2n-2}z_i^{\ell'(\ell+1)}\right)$ *divides* $Q_{n,\ell,\ell'}(\vec{z})$*.*

Proof. Note that $Q_{n,\ell,\ell'}(\vec{z})$ is symmetric in the z_i's (as they enter only as simultaneous arguments of Schur functions). So, given the factorized form of s_μ, equation (1.10), it suffices to prove that Q is divisible by $z_1^{\ell'(\ell+1)}\prod_{m=2}^{2n-2}U^{\ell}_{\ell+1}(z_1,z_m)$. Using the independence from $\vec{x}$ and $\vec{y}$ of equation (4.1), we can choose to substitute $x_i=q^iz_1$ for $1\leq i\leq\ell+1$ and leave generic the other x_j's and all the y_j's (we have a sufficient number of x's since $(n-1)\ell+\ell'+1\geq\ell+1$ for $n\geq 2$).

By applying the recursion relation of Proposition 4 to the matrix entries M_{ij}, the first $\ell+1$ rows of M are simplified. Consider the matrix $\widetilde{M}$ that coincides with M in rows $i > \ell+1$, and otherwise is given by

$$\widetilde{M}_{ij} = \left(z_1^{\ell'} U_{\ell'}(1,q^i) \prod_{m=2}^{2n-2} \frac{U_{\ell+1}(z_m,z_1)}{z_m - q^i z_1} \right) \left(\frac{U_{\ell+1}(y_j,z_1)}{y_j - x_i} \, s_{\lambda_{n-1,\ell,\ell'}}(\vec{z}_{\setminus 1}, y_j) \right). \tag{4.2}$$

This matrix is a version of M in which we do *not* replace $x_i \to q^i z_1$ for all the occurrences of x_i in M_{ij}, but only for a subset. That is, we just have the property,

$$M_{ij} = \widetilde{M}_{ij}\Big|_{x_i = q^i z_1}, \qquad \text{for } 1 \leq i \leq \ell+1,$$

and thus $\det M = (\det \widetilde{M})|_{x_i = q^i z_1}$. We constructed $\widetilde{M}$ instead of M with specific intentions: the two factors in parenthesis in (4.2) are separately polynomials after replacing $x_i = q^i z_1$ (and, before the replacement, they are at most divisible by $y_j - x_i$); the factor on the left does not depend on index j (so it can be extracted from the i-th row of $\widetilde{M}$ when evaluating the determinant); finally, the dependence from i in the second factor is all due to x_i (so that the i-th and i'-th row of M are the same vector of functions, with different x argument, i.e., $\det \widetilde{M}$ is visibly divisible by $\Delta(x_1,\dots,x_{\ell+1})$).

The factors extracted from the rows give

$$\prod_{i=1}^{\ell+1} \left(z_1^{\ell'} U_{\ell'}(1,q^i) \prod_{2 \leq m \leq 2n-2} \frac{U_{\ell+1}(z_m,z_1)}{z_m - q^i z_1} \right),$$

that is, with some simplifications (including $\prod_{i=1}^{\ell+1} U_{\ell'}(1,q^i) = 1$ if $\gcd(\ell+2,\ell'+1) = 1$ and 0 otherwise),

$$z_1^{\ell'(\ell+1)} \prod_{m=2}^{2n-2} U_{\ell+1}^{\ell}(z_1,z_m)\,. \tag{4.3}$$

The divisibility of $\det \widetilde{M}$ by $\Delta(x_1,\dots,x_{\ell+1})$ implies that $\det \widetilde{M}/\Delta(x_1,\dots,x_N)$ has no factors $x_i - x_{i'}$ in the denominator with $1 \leq i < i' \leq \ell+1$, and thus no pure powers of z_1 at the denominator from the Vandermonde determinant, after the replacement $x_i = q^i z_1$ (indeed, all the potential factors in the denominator have the form $q^i z_1 - x_j$, with $j > \ell+1$, and $y_j - q^i z_1$, with $j \leq \ell+1$), thus they do not affect the claimed factor in (4.3). This completes the proof. $\square$

Now we complete the exhaustion of factors, by proving the following weaker form of Theorem 2.

Proposition 8. *For $n \geq 2$ and $\ell \geq 1$ we have*

$$Q_{n,\ell,\ell'}(\vec{z}) = c(n,\ell,\ell') \left(\prod_{i=1}^{2n-2} z_i^{\ell'(\ell+1)} \right) s^{\ell}_{\mu_{2n-2,\ell+1}}(\vec{z}) \, s^{\ell(n-2)+\ell'-1}_{\lambda_{n-1,\ell,\ell'}}(\vec{z})\,, \tag{4.4}$$

for some numerical constant $c(n,\ell,\ell')$.

Proof. As a consequence of Proposition 6, our claim is trivially true if $\gcd(\ell'+1,\ell+2) > 1$, as the constant in such a case is 0. Therefore it remains to analyse the case $\gcd(\ell'+1,\ell+2) = 1$.

We can again exploit the invariance in x and y of $Q_{n,\ell,\ell'}(\vec{z})$ from equation (4.1), in order to evaluate $\psi_{n,\ell,\ell'}(\vec{z},\vec{x},\vec{y})$ at a special set of values x and y. Our choice is to leave the y_j's generic, and specialize $x_i = q^{k_i} z_{m_i}$, for all the indices $i = 1,\dots,N$ (recall that we defined $N = \ell(n+1)+\ell'+1$), and $\{(k_i,m_i)\}$ being a whatever ordered subset of distinct pairs, of cardinality N, in the set of all valid pairs $\{1,\dots,\ell+1\}\times\{1,\dots,2n-2\}$. The difference of cardinality, $(\ell+2)(n-1)-\ell'-1$, is always positive in our range of interest $\ell \geq 1$, $0 \leq \ell' \leq \ell$, $n \geq 2$. Using Proposition 4, we have

$$\begin{aligned} M_{ij} &= s_{\lambda_{n,\ell,\ell'}}(\vec{z}, x_i = q^{k_i} z_{m_i}, y_j) \\ &= z_{m_i}^{\ell'} U_{\ell'}(1,q^{k_i}) \frac{U_{\ell+1}(y_j, z_{m_i})}{y_j - q^{k_i} z_{m_i}} \prod_{\substack{1\leq r\leq 2n-2 \\ r\neq m_i}} \frac{U_{\ell+1}(z_r, z_{m_i})}{z_r - q^{k_i} z_{m_i}} s_{\lambda_{n-1,\ell,\ell'}}(\vec{z}_{\setminus m_i}, y_j)\,. \end{aligned}$$

Let us adopt the representation (A.1) for the Schur polynomial (as the ratio of shifted Vandermonde determinant by a Vandermonde determinant), to get

$$\begin{aligned} M_{ij} &= \frac{z_{m_i}^{\ell'} U_{\ell'}(1,q^{k_i})}{\Delta(\vec{z}_{\setminus m_i}, y_j)} \frac{U_{\ell+1}(y_j, z_{m_i})}{y_j - q^{k_i} z_{m_i}} \prod_{\substack{1\leq r\leq 2n-2 \\ r\neq m_i}} \frac{U_{\ell+1}(z_r, z_{m_i})}{z_r - q^{k_i} z_{m_i}}\ \Delta_{\lambda_{n-1,\ell,\ell'}}(\vec{z}_{\setminus m_i}, y_j) \\ &= \frac{z_{m_i}^{\ell'} U_{\ell'}(1,q^{k_i})}{\Delta(\vec{z})} (-1)^{m_i-1} \Bigg(\prod_{r\neq m_i} \frac{(z_r - z_{m_i}) U_{\ell+1}(z_r, z_{m_i})}{z_r - q^{k_i} z_{m_i}} \Bigg) \Bigg(\prod_r \frac{1}{y_j - z_r} \Bigg) \\ &\quad \times \frac{(y_j - z_{m_i}) U_{\ell+1}(y_j, z_{m_i})}{y_j - q^{k_i} z_{m_i}}\ \Delta_{\lambda_{n-1,\ell,\ell'}}(\vec{z}_{\setminus m_i}, y_j) \\ &= \frac{z_{m_i}^{\ell'} U_{\ell'}(1,q^{k_i})}{\Delta(\vec{z})} \Bigg((-1)^{m_i-1} \prod_{r\neq m_i} U_{\ell+1}(z_r, q^{k_i} z_{m_i}) \Bigg) \Bigg(\prod_r \frac{1}{y_j - z_r} \Bigg) \\ &\quad \times U_{\ell+1}(y_j, q^{k_i} z_{m_i})\ \Delta_{\lambda_{n-1,\ell,\ell'}}(\vec{z}_{\setminus m_i}, y_j)\,, \end{aligned} \tag{4.5}$$

where in the last equality we made use of the relation

$$\frac{U_{\ell+1}(x, q^h y)}{x - q^k y} = \prod_{\substack{0\leq i\leq \ell+1 \\ i\neq h,k}} (x - q^i y) = \frac{U_{\ell+1}(x, q^k y)}{x - q^h y}\,. \tag{4.6}$$

In the last expression of equation (4.5), we recognize five factors: a factor independent of i and j, one depending on i alone, one depending on j alone, and one depending on both i and j, which is composed of $U_{\ell+1}(y_j, q^{k_i} z_{m_i})$, which is a homogeneous polynomial in y_j and z_{m_i} of degree $\ell+1$, and a shifted Vandermonde determinant. The first three factors are easily extracted when evaluating $\det M$, so we can write

$$\det M = \frac{A(\vec{z},\vec{y})}{B(\vec{z},\vec{y})\Delta(\vec{z})^N}\ \det \widehat{M} \tag{4.7}$$

with

$$\widehat{M}_{ij} = U_{\ell+1}(y_j, q^{k_i} z_{m_i}) \Delta_{\lambda_{n-1,\ell,\ell'}}(\vec{z}_{\setminus m_i}, y_j)\,; \tag{4.8}$$

$$A(\vec{z},\vec{y}) = \prod_i (-1)^{m_i-1} z_{m_i}^{\ell'} U_{\ell'}(1, q^{k_i}) \prod_{\substack{1\le i\le N\\ 1\le r\le 2n-2\\ r\neq m_i}} U_{\ell+1}(z_r, q^{k_i} z_{m_i})\,; \tag{4.9}$$

$$B(\vec{z},\vec{y}) = \prod_{\substack{1\le j\le N\\ 1\le r\le 2n-2}} (y_j - z_r)\,. \tag{4.10}$$

We now substitute the expression of equation (4.7) in (4.1), where we also do the replacement

$$\Delta(\vec{x}) \quad\longrightarrow\quad \Delta(q^{k_1} z_{m_1}, q^{k_2} z_{m_2}, \ldots)\,,$$

which leads to

$$Q_{n,\ell,\ell'}(\vec{z}) = \frac{A(\vec{z},\vec{y})}{B(\vec{z},\vec{y})} \frac{1}{\Delta(\vec{y})\Delta(q^{k_1} z_{m_1}, q^{k_2} z_{m_2}, \ldots)\Delta(\vec{z})^N}\ \det \widehat{M}\,. \tag{4.11}$$

Now, the matrix $\widehat{M}$ is in a form suitable for application of Proposition 2, the divisibility result discussed in Section 3.3, with $k = \ell+2$ and, for $0 \le a \le \ell+1$, $u_i^a v_j^a$ being the coefficient of the monomial $y_j^a z_{m_i}^{\ell+1-a}$ in the expansion of $U_{\ell+1}(y_j, q^{k_i} z_{m_i})$.

As a consequence, the polynomial $\Delta_{\lambda_{n-1,\ell,\ell'}}^{N-(\ell+2)}(\vec{z})$ divides $\det \widehat{M}$, and the exponent $N-(\ell+2) = \ell(n-1)+\ell'+1-(\ell+2) = \ell(n-2)+\ell'-1$ is exactly the desired one from the statement of Proposition 8 (and Theorem 2). So we can write

$$\det \widehat{M} = \Delta_{\lambda_{n-1,\ell,\ell'}}^{\ell(n-2)+\ell'-1}(\vec{z})\ R(\vec{z},\vec{y})$$

for R a polynomial. If we substitute this in (4.11), then we obtain

$$Q_{n,\ell,\ell'}(\vec{z}) = s_{\lambda_{n-1,\ell,\ell'}}^{\ell(n-2)+\ell'-1}(\vec{z})\ \frac{A(\vec{z},\vec{y})R(\vec{z},\vec{y})}{B(\vec{z},\vec{y})\Delta(\vec{y})\Delta(q^{k_1} z_{m_1}, q^{k_2} z_{m_2}, \ldots)\Delta(\vec{z})^{\ell+2}}\,. \tag{4.12}$$

Now, as $\gcd(\ell'+1, \ell+2) = 1$, we obtain two consequences from Proposition 5. First, observing that the denominator in (4.12) is completely factorized into linear terms (of the form $y_i - z_j$, or $z_i - q^k z_j$), $s_{\lambda_{n-1,\ell,\ell'}}(\vec{z})$ cannot be divisible by any of these factors, therefore it follows from equation (4.12) that $s_{\lambda_{n-1,\ell,\ell'}}^{\ell(n-2)+\ell'-1}(\vec{z})$ must divide $Q_{n,\ell,\ell'}(\vec{z})$.

Furthermore, we know from Proposition 7 that $s_{\mu_{2n-2,\ell+1}}^{\ell}(\vec{z}) \prod_{i=1}^{2n-2} z_i^{\ell'(\ell+1)}$ divides $Q_{n,\ell,\ell'}(\vec{z})$. Also this polynomial is factorized into linear terms, of the form z_i or $z_i - q^k z_j$, thus it is relatively prime with $s_{\lambda_{n-1,\ell,\ell'}}^{\ell(n-2)+\ell'-1}$. This shows that Proposition 8 holds, for $c(n,\ell,\ell')$ a polynomial. However, all the involved functions are homogeneous polynomials, and it is easily determined that $c(n,\ell,\ell')$ has degree 0, thus it is a constant. □

4.2. Determination of the constant $c(n,\ell,\ell')$. We can evaluate directly the constant for $n=1$, which is $c(1,\ell,\ell') = (-1)^{\binom{\ell'+1}{2}}$, and we know that, for $n \ge 2$ and $\gcd(\ell+2, \ell'+1) > 1$, $c(n,\ell,\ell') = 0$. In the rest of this section we will complete the proof of expression (1.8), for the remaining case $n \geq 2$ and $\gcd(\ell+2, \ell'+1) = 1$. This is done by induction in n, i.e., we will prove that, for (n, ℓ, ℓ') as above,

$$\frac{c(n,\ell,\ell')}{c(n-1,\ell,\ell')} = (-1)^{\binom{\ell+1}{2}}.$$

Now that we only have to determine the constant, we have the freedom of choosing simpler values also for the z_k's, in addition to the ones for the x_i's and the y_j's.

First of all, in equation (4.1) let us specialize $x_i = q^i z_1$ for $1 \leq i \leq \ell$. In this way we find that the matrix entries M_{ij} for $1 \leq i \leq \ell$ take the form[2]

$$M_{ij} = z_1^{\ell'} U_{\ell'}(1, q^i) \frac{U_{\ell+1}(y_j, z_1)}{y_j - q^i z_1} \prod_{r=2}^{2n-2} \frac{U_{\ell+1}(z_r, z_1)}{z_r - q^i z_1} s_{\lambda_{n-1,\ell,\ell'}}(\vec{z}_{\setminus 1}, y_j). \tag{4.13}$$

As we have done in the proof of Proposition 7, when we compute the determinant of the matrix M, for $1 \leq i \leq \ell$ we extract the factor

$$z_1^{\ell'} U_{\ell'}(1, q^i) \prod_{r=2}^{2n-2} \frac{U_{\ell+1}(z_r, z_1)}{z_r - q^i z_1} \tag{4.14}$$

from the i-th row, and find that

$$\det M = F(z_1; \vec{z}_{\setminus 1}) \det M', \tag{4.15}$$

where

$$F(z_1; \vec{z}_{\setminus 1}) = \frac{z_1^{\ell'\ell}}{U_{\ell'}(1, q^{\ell+1})} \prod_{r=2}^{2n-2} U_{\ell+1}^{\ell-1}(z_r, z_1)\ (z_r - q^{\ell+1} z_1),$$

and the matrix M' coincides with M in the last $N - \ell$ rows, while each of the first ℓ rows is divisible by the factor in equation (4.14).

We now substitute the expression (4.15) for $\det M$ into the definition of $Q_{n,\ell,\ell'}(\vec{z})$ and then into equation (4.4), taking into account also the substitutions of the variables in the Vandermonde determinant in the denominator

$$\Delta(\vec{x}) \quad \longrightarrow \quad z_1^{\binom{\ell}{2}} \Delta'(z_1, \vec{x}_{\setminus 1,\ldots\ell}), \tag{4.16}$$

$$\Delta'(z_1, \vec{x}_{\setminus 1,\ldots\ell}) := \Delta(q, q^2, \ldots, q^\ell)\ \Delta(\vec{x}_{\setminus 1,\ldots\ell}) \prod_{\substack{1 \leq i \leq \ell \\ \ell+1 \leq k \leq N}} (q^i z_1 - x_k), \tag{4.17}$$

$$\Delta(q, q^2, \ldots, q^\ell)^2 = (-1)^{\binom{\ell+1}{2}} (q^{-1} - q^{-2})^2 (\ell+2)^{\ell-2}. \tag{4.18}$$

We obtain

$$\begin{aligned} Q_{n,\ell,\ell'}(\vec{z}) &= \frac{F(z_1; \vec{z}_{\setminus 1}) \det M'}{z_1^{\binom{\ell}{2}} \Delta'(z_1, \vec{x}_{\setminus 1,\ldots\ell}) \Delta(\vec{y})} \\ &= c(n,\ell,\ell') \prod_{i=1}^{2n-2} z_i^{\ell'(\ell+1)}\ s^{\ell}_{\mu_{2n-2,\ell+1}}(\vec{z})\ s^{\ell(n-2)+\ell'-1}_{\lambda_{n-1,\ell,\ell'}}(\vec{z}). \end{aligned} \tag{4.19}$$

[2]That is, nothing else but $\widetilde{M}_{ij}$ in (4.2), under the full replacement $x_i \to q^i z_1$.

We eliminate the factors appearing on both sides of the previous equation, and we obtain

$$\begin{aligned}(4.20)\quad \frac{\det M'}{U_{\ell'}(1,q^{\ell+1})\Delta'(z_1,\vec{x}_{\setminus 1,\dots\ell})\Delta(\vec{y})} &= c(n,\ell,\ell')z_1^{\binom{\ell}{2}+\ell'} \\ &\times \prod_{i=2}^{2n-2} z_i^{\ell'(\ell+1)} \prod_{r=2}^{2n-2} \frac{U_{\ell+1}(z_r,q^{\ell+1}z_1)}{z_r-z_1} s^{\ell}_{\mu_{2n-3,\ell+1}}(\vec{z}_{\setminus 1})\, s^{\ell(n-2)+\ell'-1}_{\lambda_{n-1,\ell,\ell'}}(\vec{z})\,.\end{aligned}$$

Note that, among other things, we have eliminated all the factors $z_r - q^{\ell+1}z_1$ on both sides. This allows us to set $z_2 = q^{\ell+1}z_1$. Furthermore, we choose to specialize $y_i = q^i z_1$, for $1 \le i \le \ell$ (the Vandermonde factor $\Delta(\vec{y})$ in equation (4.20) is then to be treated similarly to what is done in (4.16) for $\Delta(\vec{x})$).

It is easy to see which simplifications occur on the factorized right-hand side of equation (4.20)

$$(4.21)\qquad \prod_{i=2}^{2n-2} z_i^{\ell'(\ell+1)} \to q^{\ell'} z_1^{\ell'(\ell+1)} \prod_{i=3}^{2n-2} z_i^{\ell'(\ell+1)}$$

$$(4.22)\qquad \prod_{r=2}^{2n-2} \frac{U_{\ell+1}(z_r,q^{\ell+1}z_1)}{z_r-z_1} \to \frac{z_1^{\ell}(\ell+2)}{q^{-2}-q^{-1}} \prod_{r=3}^{2n-2} \frac{U_{\ell+1}(z_r,q^{\ell+1}z_1)}{z_r-z_1}$$

$$(4.23)\qquad s^{\ell}_{\mu_{2n-3,\ell+1}}(\vec{z}_{\setminus 1}) \to \prod_{r=3}^{2n-2} U^{\ell}_{\ell+1}(z_r,q^{\ell+1}z_1)\; s^{\ell}_{\mu_{2n-4,\ell+1}}(\vec{z}_{\setminus 1,2})$$

$$(4.24)\qquad s_{\lambda_{n-1,\ell,\ell'}}(\vec{z}) \to z_1^{\ell'} U_{\ell'}(1,q^{\ell+1}) \prod_{r=3}^{2n-2} \frac{U_{\ell+1}(z_r,z_1)}{z_r-q^{\ell+1}z_1} s_{\lambda_{n-2,\ell,\ell'}}(\vec{z}_{\setminus 1,2})\,.$$

Even more drastic simplifications arise on the left-hand side of equation (4.20). For $i > \ell$ and $j \le \ell$, the entries M'_{ij} consist of the Schur polynomials $s_{\lambda_{n,\ell,\ell'}}$ evaluated at a set of variables including a triple satisfying the wheel condition (namely, z_1, $y_j = q^j z_1$ and $z_2 = q^{\ell+1}z_1$), therefore they vanish because of Proposition 3. Similarly, for $i \le \ell$ and $j \le \ell$, with the only exception of $i = j$, M'_{ij} vanishes because of the factor $\frac{U_{\ell+1}(y_j,z_1)}{y_j-q^i z_1} = \prod_{1\le k\le \ell+1;\, k\ne i}(y_j - q^k z_1)$ (cf. equation (4.13)). As a result,

$$(4.25)\qquad \det M' = \Big(\prod_{i=1}^{\ell} M'_{ii}\Big)\ \det M'_{\{\ell+1,\dots,N\},\{\ell+1,\dots,N\}}.$$

The diagonal factors M'_{ii} read

$$M'_{ii} = \frac{z_1^{\ell+\ell'}(\ell+2)q^{\ell i}U_{\ell'}(q^{\ell+1},q^i)}{1-q^{-i}} \prod_{r=3}^{2n-2} \frac{U_{\ell+1}(z_r,q^{\ell+1}z_1)}{z_r-q^i z_1} s_{\lambda_{n-2,\ell,\ell'}}(\vec{z}_{\setminus 1,2}).$$

Most importantly, the minor of the matrix M' restricted to the last $N-\ell$ rows and columns is easily related to the matrix M for the system of size $n-1$, where the indices of the variables z_k run from 3 to $2n-2$, while the indices of the x_i's and y_j's run from $\ell+1$ to $N = (n-1)\ell+\ell'+1$. More precisely, $M'_{\ell+i,\ell+j}$, at size n and under the specializations above, is proportional to M_{ij} at size $n-1$, the proportionality factor for the pair (i,j) being

$$z_1^{\ell'} U_{\ell'}(1,q^{\ell+1}) \left(\prod_{r=3}^{2n-2} \frac{U_{\ell+1}(z_r,z_1)}{z_r - q^{\ell+1} z_1} \right) \frac{U_{\ell+1}(x_{\ell+i},z_1)}{x_{\ell+i} - q^{\ell+1} z_1} \, \frac{U_{\ell+1}(y_{\ell+j},z_1)}{y_{\ell+j} - q^{\ell+1} z_1}. \tag{4.26}$$

(The relevant fact is that this quantity factorizes into a term depending on x_i only, and a term depending on y_j only, these terms thus factorize in the evaluation of the determinant.) Thus we get

(4.27)

$$\begin{aligned} \det M'_{\{\ell+1,\dots,N\},\{\ell+1,\dots,N\}} = \left[z_1^{\ell'} U_{\ell'}(1,q^{\ell+1}) \left(\prod_{r=3}^{2n-2} \frac{U_{\ell+1}(z_r,z_1)}{z_r - q^{\ell+1} z_1} \right) \right]^{N-\ell} \\ \times \prod_{i=1}^{N-\ell} \frac{U_{\ell+1}(x_{\ell+i},z_1)}{x_{\ell+i} - q^{\ell+1} z_1} \, \prod_{j=1}^{N-\ell} \frac{U_{\ell+1}(y_{\ell+j},z_1)}{y_{\ell+j} - q^{\ell+1} z_1} \\ \times \Delta(x_{\ell+1},\dots,x_N)\, \Delta(y_{\ell+1},\dots,y_N)\, Q_{n-1,\ell,\ell'}(z_3,\dots,z_{2n})\,. \end{aligned}$$

In this equation we can replace $Q_{n-1,\ell,\ell'}(\vec{z}_{\setminus 1,2})$ by its expression given by equation (4.4) — the factor $c(n-1,\ell,\ell')$ emerges at this point — then, we can substitute (4.27) in (4.20), using (4.25). In this way we arrive at a fully factorized form on both sides of equation (4.20). Cancellation of common factors is a matter of simple algebra,[3]which, in the end, leads to the recurrence relation

$$c(n,\ell,\ell') = (-1)^{\binom{\ell+1}{2}} c(n-1,\ell,\ell')\,,$$

as was to be proven. □

Appendix A. Basic facts on symmetric polynomials

A *partition* λ is a (finite or infinite) non-increasing sequence of non-negative integers, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \cdots)$ containing only finitely many non-zero terms. The number of non-zero terms (or *parts*) $\ell(\lambda)$, and the value of the sum $|\lambda| = \sum_{i=1}^{k} \lambda_i$, are called the *length* and the *weight* of the partition, respectively. When there is no confusion, we use the same name λ for the finite partition $(\lambda_1, \lambda_2, \dots, \lambda_k)$ and for the infinite one $(\lambda_1, \lambda_2, \dots, \lambda_k, 0, 0, \dots)$. Seen as an arrangement of cells (as, e.g., in Figure 2), λ is often called a *Young diagram*.

Given an ordered ℓ-tuple of indeterminates $\vec{z} = \{z_i\}_{1\leq i\leq \ell}$, the *Vandermonde determinant* $\Delta(\vec{z})$ is defined as the determinant of the $\ell \times \ell$ matrix $V = (V_{ij})_{1\leq i,j\leq \ell}$ with $V_{ij} = z_i^{\ell-j}$. It is well known that $\Delta(\vec{z}) = \prod_{1\leq i<j\leq \ell}(z_i - z_j)$. For a partition λ of length at most ℓ, one similarly defines the *shifted Vandermonde determinant* $\Delta_\lambda(\vec{z})$ as the determinant of the $\ell\times\ell$ matrix $V = (V_{ij})_{1\leq i,j\leq \ell}$ with $V_{ij} = z_i^{\lambda_j+\ell-j}$. Thus $\Delta(\vec{z}) \equiv \Delta_{(0,0,\dots,0)}(\vec{z})$.

[3]Useful relations at this point are (4.18) and

$$\prod_{i=1}^{\ell} \frac{q^{\ell i}}{1-q^{-i}} = \frac{q^{-2}-q^{-1}}{\ell+2}\,; \qquad \prod_{i=1}^{\ell} U_{\ell'}(q^{\ell+1},q^i) = \frac{q^{\ell'}}{U_{\ell'}(1,q^{\ell+1})}\,.$$

Then the *Schur polynomial* associated to λ is the function in ℓ indeterminates

$$s_\lambda(\vec{z}) = \frac{\Delta_\lambda(\vec{z})}{\Delta(\vec{z})}\,. \tag{A.1}$$

It is indeed a polynomial, it is symmetric in all its variables, and homogeneous of degree $|\lambda|$. The Schur functions are at the heart of algebraic combinatorics [**15**] and enjoy several remarkable properties (see [**10**]). Here we limit ourselves to presenting the few simple results we need in the paper, among which the following "splitting formula".

Proposition 9. *Let $(\lambda_1,\dots,\lambda_k)$ and $(\mu_1,\dots,\mu_h)$ be two partitions such that $\lambda_k \geq \mu_1$. Let ν denote the partition $\nu = (\lambda_1,\dots,\lambda_k,\mu_1,\dots,\mu_h)$. Then we have*

$$\lim_{\epsilon\to 0} \frac{s_\nu(z_1,\dots,z_k,\epsilon y_1,\dots,\epsilon y_h)}{\epsilon^{|\mu|}} = s_\lambda(z_1,\dots,z_k)\, s_\mu(y_1,\dots,y_h)\,. \tag{A.2}$$

This generalizes the simple property that a Schur polynomial $s_\lambda(\vec{z})$ in ℓ variables has maximum degree λ_1, and mimimum degree λ_ℓ, in any of its variables. For the connoisseurs, the proposition can easily be proven in several ways, for example by using the decomposition formula for Schur function $s_\alpha(\vec{x},\vec{y}) = \sum_{\beta\subseteq\alpha} s_\beta(\vec{x}) s_{\alpha/\beta}(\vec{y})$ (see, e.g., [**10**, eq. (5.9)]), and simple properties of skew Schur functions (which we do not introduce). Here we provide a more verbose but completely self-contained proof.

Proof. Using the defining equation (A.1), we are led to study the behaviour of $\Delta_\gamma(\vec{z},\epsilon\vec{y})$ as $\epsilon\to 0$, for the cases $\gamma=\nu$ (in the numerator) and $\gamma = 0$ (in the denominator). More generally, consider $\gamma = (\gamma_1,\dots,\gamma_{k+h}) \equiv (\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_h)$. Recall that $\Delta_\gamma(\vec{z},\epsilon\vec{y})$ is defined as the determinant of the matrix $V = (V_{ij})_{1\leq i,j\leq k+h}$ with $V_{ij} = z_i^{\gamma_j+k+h-j}$ for $i\leq k$ and $V_{ij} = (\epsilon y_{i-k})^{\gamma_j+k+h-j}$ for $i>k$. Consider the Laplace expansion of V along the first k rows,

$$\det V = \sum_{\substack{I\subseteq[k+h]\\ |I|=k}} \epsilon(I,[k]) \det V_{[k],I} \det V_{[k]^c,I^c}\,.$$

As the summand with index I has a factor $\epsilon^{\sum_{j\in I^c}(\gamma_j+k+h-j)}$, the term with $I=[k]$ has a factor $\epsilon^{|\beta|+\binom{h}{2}}$, and all other terms contain a higher power of ϵ. Thus,

$$\begin{aligned}\frac{\Delta_\gamma(\vec{z},\epsilon\vec{y})}{\epsilon^{|\beta|+\binom{h}{2}}} &= \det(z_i^{\alpha_j+k+h-j})_{1\leq i,j\leq k}\ \det(y_i^{\beta_j+k+h-(k+j)})_{1\leq i,j\leq h}\ +\mathcal{O}(\epsilon)\\ &= \Big(\prod_{i=1}^k z_i^h\Big)\,\Delta_\alpha(\vec{z})\ \Delta_\beta(\vec{y})\ +\mathcal{O}(\epsilon)\,.\end{aligned} \tag{A.3}$$

Comparing this equation for $\gamma=\nu$ and $\gamma=0$ allows us to conclude. □

The bivariate homogeneous Chebyshev polynomials of the second kind are defined by

$$U_k(x,y) = \frac{x^{k+1}-y^{k+1}}{x-y} = x^k + x^{k-1}y + \cdots + y^k\,. \tag{A.4}$$

Define the *staircase partition* $\mu_{n,\ell}$ as the partition

$$\mu_{n,\ell} = \big(\ell n-\ell, \ell n - 2\ell, \dots, \ell, 0, 0, \dots\big)\,. \tag{A.5}$$

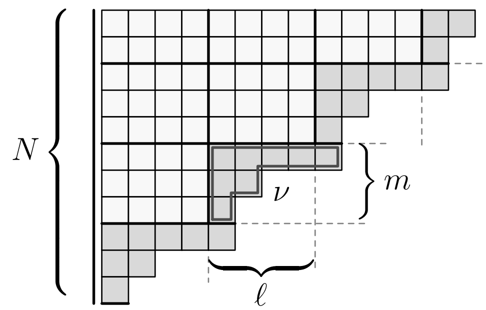

FIGURE 3. An example of partition $\lambda(N,m,\ell,\nu)$ with $N=11$, $m=3$, $\ell=4$ and $\nu=(5,2,1)$.

The associated Schur function in n variables is easily evaluated by (A.1):

$$s_{\mu_{n,\ell}}(\vec{z}) = \frac{\Delta_{\mu_{n,\ell}}(\vec{z})}{\Delta(\vec{z})} = \frac{\Delta(z_1^{\ell+1},\dots,z_n^{\ell+1})}{\Delta(z_1,\dots,z_n)} = \prod_{1\le i<j\le n} \frac{z_i^{\ell+1}-z_j^{\ell+1}}{z_i-z_j} = \prod_{1\le i<j\le n} U_\ell(z_i,z_j)\,. \tag{A.6}$$

Appendix B. Properties of staircase Schur functions

Let us consider three non-negative integers N, m and ℓ, with $m\ge 1$, and a partition ν of length $\ell(\nu)\le m$, with $\nu_1-\nu_m\ge\ell$. We define the partition $\lambda(N,m,\ell,\nu)$ as follows: for $0\le i<N$, consider the unique way of writing $N-i=am+b$, with $a\ge 0$ and $0\le b<m$ (that is, $a=\lfloor (N-i)/m\rfloor$ and $b\equiv N-i \pmod{m}$). Then

$$\lambda(N,m,\ell,\nu)_{N-i} = am+\nu_b\,.$$

(See Figure 3.) We call such diagrams *m-staircase diagrams*, and the Schur functions $s_{N,m,\ell,\nu}(\vec{z})\equiv s_{\lambda(N,m,\ell,\nu)}(\vec{z})$ in N variables *m-staircase Schur functions.* These functions generalize the (1-)staircase and 2-staircase functions defined in (1.6) and (1.3), corresponding to taking $m=1$ and 2, respectively, $\nu_m=0$, and N a multiple of m ($\nu_1\equiv\ell'$ for 2-staircase functions). In this section we set $q=\exp(\frac{2\pi i}{\ell+m})$.

We say that a symmetric function in N variables $f(z_1,\dots,z_N)$ satisfies the (m,ℓ)-*wheel condition* if, for $I=\{i_1,\dots,i_{m+1}\}\subseteq[N]$ and $K=\{k_1,\dots,k_{m+1}\}\subseteq[\ell+m]$,

$$f(z_1,\dots,z_N)|_{z_{i_a}=q^{k_a}w,\,a=1,\dots,m+1}=0\,.$$

Note that, as we deal with symmetric polynomials, it is not necessary to take ordered m-tuples in place of subsets. We call a specialization $z_{i_a}=q^{k_a}w$ of the form above a "wheel hyperplane". Proposition 3 is the 2-staircase function specialization of the following more general proposition. The proof we produce below is a minor variation of the paragraphs concerning Theorem 4 of [**18**].

Proposition 10. *The symmetric function in N variables $s_{N,m,\ell,\nu}(\vec{z})$ satisfies the (m,ℓ)-wheel condition.*

PROOF. Consider the generic wheel hyperplane $z_{i_a}=q^{k_a}w$ for $i_a\in I$ and $k_a\in K$ as above. Write $\lambda=\lambda(N,m,\ell,\nu)$ for brevity. Represent $s_{N,m,\ell,\nu}(\vec{z})$ as a ratio of a shifted Vandermonde determinant by a Vandermonde determinant, Δ_λ/Δ, as in equation (A.1). Since, even under the specialization, the variables z_i are all distinct, the Vandermonde determinant in the denominator is non-singular, so it suffices to prove that the shifted Vandermonde determinant vanishes. The shifted entries of the partition are $\tilde{\lambda}_i = \lambda_i + (N-i)$, and, writing $i = N-am-b$, we have $\tilde{\lambda}_{N-am-b} = (\ell+m)a+b+\nu_{m-b}$. Note in particular that

$$\tilde{\lambda}_{N-am-b} \equiv b + \nu_{m-b} \pmod{\ell+m}, \tag{B.1}$$

independently of a. Consider the matrix $V_{ij} = z_i^{\tilde{\lambda}_j}$, such that $\Delta_\lambda = \det V$. Let V' denote the rectangular minor of V restricted to the $m+1$ rows in I, and write $j = N-am-b$ as above. Then, because of equation (B.1),

$$V'_{ij} = z_i^{\tilde{\lambda}_j} = w^{\tilde{\lambda}_j} q^{k_i(N-(\ell+m)a_j-b_j-\nu_{b_j})} = w^{\tilde{\lambda}_j} q^{Nk_i} q^{-k_i(b_j+\nu_{m-b_j})}\,.$$

As $b+\nu_b$ for $b \in \{0,\dots,m-1\}$ takes m distinct values, V' has rank at most m, while it has $m+1$ rows. This proves that $\det V = 0$. □

Now we present a generalization of Proposition 4.

Proposition 11. *For $I = \{i_1,\dots,i_m\} \subseteq [N]$ and $K = \{k_1,\dots,k_m\} \subseteq [\ell+m]$, $s_{N,m,\ell,\nu}(\vec{z})$ satisfies the recursion*

$$\begin{aligned} s_{N,m,\ell,\nu}(\vec{z})(\vec{z}_{\setminus I}, q^{k_1}w,\dots,q^{k_m}w) \\ = s_\nu(q^{k_1},\dots,q^{k_m}) w^{|\nu|} \Bigg(\prod_{j\in[N]\setminus I} \prod_{h\in[\ell+m]\setminus K} (z_j - q^h w) \Bigg) s_{N-m,m,\ell,\nu}(\vec{z}_{\setminus I})\,. \end{aligned} \tag{B.2}$$

Proof. From Proposition 10 it follows that, for I and K as above, $s_{N,m,\ell,\nu}(\vec{z})$ satisfies the equation

$$s_{N,m,\ell,\nu}(\vec{z}_{\setminus I}, q^{k_1}w,\dots,q^{k_m}w) = \Bigg(\prod_{j\in[N]\setminus I} \prod_{h\in[\ell+m]\setminus K} (z_j - q^h w) \Bigg) F^{(K)}_{N,m,\ell,\nu}(\vec{z}_{\setminus I}, w)\,, \tag{B.3}$$

for some polynomial $F^{(K)}_{N,m,\ell,\nu}(\vec{z}_{\setminus I}, w)$. Rewrite the equation above in the form

$$\Delta_{\lambda(N,m,\ell,\nu)}(\vec{z}, q^{k_i}w) = \Delta(\vec{z}, q^{k_i}w) \prod_{j,h} (z_j - q^h w) F^{(K)}_{N,m,\ell,\nu}.$$

An easy computation of minimal and maximal degree in w for all the factors in this expression other than $F^{(K)}$ shows that $F^{(K)}_{N,m,\ell,\nu}(\vec{z}_{\setminus I}, w)$ is homogeneous of degree $|\nu|$ in w. Thus,

$$F^{(K)}_{N,m,\ell,\nu}(\vec{z}_{\setminus I}, w) \equiv w^{|\nu|} \lim_{v\to 0} \frac{F^{(K)}_{N,m,\ell,\nu}(\vec{z}_{\setminus I}, v)}{v^{|\nu|}},$$

and, in order to determine this quantity, it suffices to divide both sides of equation (B.3) by $w^{|\nu|}$ and take the limit $w \to 0$. Using Lemma A.2, we find for the left-hand side of equation (B.3)

$$\lim_{w\to 0} \frac{s_{N,m,\ell,\nu}(\vec{z}_{\setminus I}, q^{k_1}w,\dots,q^{k_m}w)}{w^{|\nu|}} = s_\nu(q^{k_1},\dots,q^{k_m}) s_{N-m,m,\ell,\nu}(\vec{z}_{\setminus I}) \prod_{j\notin I} z_j^\ell\,.$$

The factor $\prod_{j\notin I} z_j^\ell$ simplifies with the the same term appearing on the right-hand side, from the limit of the product of binomials $z_j - q^h w$. Therefore we end up with

$$F^{(K)}_{N,m,\ell,\nu}(\vec{z}_{\setminus I}, w) = w^{|\nu|} s_\nu(q^{k_1}, \dots, q^{k_m}) s_{N-m,m,\ell,\nu}(\vec{z}_{\setminus I})\,. \tag{B.4}$$

□

For a symmetric polynomial $P(\vec{z})$ in N variables, and $1 \le k \le N$, call $d_k(P)$ the maximum degree of P in (any) k variables simultaneously. In what follows, when the number of variables is clear, we will use the short notations $d \equiv d_1$ and $D \equiv d_N$. Recall that, for a Schur function $s_\lambda(\vec{z})$, $d_k = \lambda_1 + \cdots + \lambda_k$.

Among the staircase Schur functions considered in the propositions above, the subclass $\nu_1 = \cdots = \nu_m = 0$ (i.e., $\nu = \varnothing$) has the further property of being "of minimal degree" among all symmetric functions satisfying the wheel condition, in various senses involving this set of degrees d_k. The following proposition describes some of the possible choices. It is a generalization to the m-staircase case of the $m = 2$ situation analysed in [**18**, Thm. 4], but, in contrast to Proposition 10, the proof technique is substantially different, as the Lagrange interpolation argument used in [**18**] is specific to $m = 2$. (With higher values, some degree counting hypothesis is not met.)

Determining the uniqueness of a function satisfying a precise set of conditions and degree bounds is often a useful tool when one wants to "prove that two (families of) functions are the same". Although this could appear as a rare coincidence, this line of reasoning has already proven to be of value in several enumeration problems related to integrable systems, ranging from the recognition of the Izergin determinant [**6**], and its identification as a Schur function [**16**], up to the "higher-spin" cases in [**18**]. We report the following result, with the hope that it may be useful in generalizations of six-vertex and loop models involving simultaneously both "higher-spin" and "higher rank", i.e., higher values of m (besides $m = 2$) in representations of the quantum affine algebra q-deforming $\mathfrak{sl}(m)$.

Proposition 12. *Let $N = am + b$, with $a \ge 0$ and $1 \le b \le m$. The symmetric polynomial in N variables $s_{N,m,\ell}(\vec{z}) := s_{N,m,\ell,\varnothing}(\vec{z})$ has $(D, d, d_m) = (D^*, d^*, d^*_m)$, with*

$$(D^*, d^*, d^*_m) = \left(a\ell\left(\tfrac{m(a-1)}{2} + b\right), a\ell, (N-m)\ell\right)\,.$$

It is the unique symmetric function (up to multiplication by a scalar) satisfying the (m, ℓ)-wheel condition and any of the following degree conditions:

(*a*) $d \le d^*$ *and* $D \le D^*$*;*
(*b*) $d_m \le d^*_m$*;*
(*b*$'$) $d \le d^*$ *and* m *divides* N*;*
(*c*) $f_{m,\ell}(D, d) \le f_{m,\ell}(D^*, d^*)$*, for* $f_{m,\ell}(D, d) = \frac{\ell}{m} D + \frac{d(d+\ell)}{2}$*.*

Proof. Clearly (b') is implied by (b), as $d^*_m = m\, d^*$ if m divides N, and $d_m \le m\, d$ for any polynomial, so it suffices to concentrate on the three cases (a), (b), and (c). Observe that a degree condition $d \le d^*$ alone would fail to guarantee uniqueness, since, if $b < m$, any $s_{N,m,\ell,\nu}$ such that ν has at most $m - b$ non-zero parts and $\nu_1 \le \ell$ would work.

The fact that the Schur functions above satisfy the claimed wheel condition has been already proven in Proposition 10, and the degrees are easily calculated. So we just have to prove degree minimality, and uniqueness.

If we have $a = 0$ (i.e., $N \leq m$), for arbitrary m and ℓ, the statement is trivial because the wheel condition is empty (there are no wheel hyperplanes), and indeed $s_{N,m,\ell}(\vec{z}) = 1$ in this case.

The case $m = 1$, and arbitrary N and ℓ, is also fairly simple. A polynomial in N variables $P(\vec{z})$ satisfies the $(1,\ell)$-wheel condition if and only if, for all $i < j$ and $1 \leq k \leq \ell$, it is divisible by $z_i - q^k z_j$. Therefore the polynomial of minimal degree satisfying the wheel condition consists of the product of these factors, and indeed coincides with $s_{N,1,\ell}(\vec{z}) \equiv s_{\mu_{N,\ell}}(\vec{z})$.

The proof for generic values of m and N, and any of the degree conditions in the list, is done by a double induction on N and m, using the cases above as a basis. Let us assume the statement to be true up to the value $m-1$, and, for the value m, up to $N-1$ variables. Then suppose that $P(\vec{z})$ is a symmetric polynomial in N variables which satisfies the (m,ℓ)-wheel condition, and with a degree triple (D, d, d_m) satisfying any of the conditions. We want to show that, up to rescaling $P(\vec{z})$ by a constant factor, $P(\vec{z}) = s_{N,m,\ell}(\vec{z})$.

We know from Proposition 11 that, for I and K appropriate sets (i.e., $I \subseteq [N]$ and $K \subseteq [\ell + m]$, $|I| = |K| = m$), $P(\vec{z})$ satisfies the equation

$$P(\vec{z}_{\setminus I}, q^{k_1} w, \dots, q^{k_m} w) = \Bigg(\prod_{\substack{j \in [N] \setminus I \\ h \in [\ell+m] \setminus K}} (z_j - q^h w) \Bigg) F^{(K)}_{N,m,\ell}(\vec{z}_{\setminus I}, w) \,, \tag{B.5}$$

for some polynomial $F^{(K)}_{N,m,\ell}(\vec{z}_{\setminus I}, w)$, symmetric in the $N - m = (a-1)m + b$ variables $\{z_j\}_{j \notin I}$, and satisfying the (m,ℓ)-wheel condition for the remaining variables z_j.

Call $d(F)$ the maximum degree of F in one variable, seen as a polynomial in variables z_j only, $d_w(F)$ the degree as a polynomial in w, and $D_w(F)$ the maximum total degree of F in z_j's and w. From the degree triple of P and the binomial factors of P, it is easy to realize that

$$D_w(F) \leq D(P) - (N-m)\ell \,, \tag{B.6}$$

$$d(F) \leq d(P) - \ell \,, \tag{B.7}$$

$$d_w(F) \leq d_m(P) - (N-m)\ell \leq m\, d(P) - (N-m)\ell \,. \tag{B.8}$$

(The inequalities come from the fact that cancellations may occur in P from the specialization. The equation for $d_w(F)$ is obtained by considering the m-tuple of variables $\{z_i\}_{i \in I}$ in P.) Furthermore, if $N \geq 2m$, $d_m(F)$ is defined, and we can also infer

$$d_m(F) \leq d_m(P) - m\ell \,.$$

(This equation is obtained by considering the m-tuple of variables $\{z_i\}_{i \in J}$, for some J of size m and disjoint from I, in P).

From the above bounds on the degree of F, and the fact that $F^{(K)}_{N,m,\ell}(\vec{z}_{\setminus I}, w)$ must satisfy the (m,ℓ)-wheel condition for m-tuples of the $N - m$ remaining variables z_j, we can prove in the various cases one of the inequalities

$$D_w(F) \leq D^*(N-m, m, \ell) \,; \tag{B.9a}$$

$$d_w(F) \leq 0 \quad \text{and} \quad \big(\; d_m(F) \leq d^*_m(N-m, m, \ell) \quad \text{or} \quad d(F) = 0 \;\big) \,; \tag{B.9b}$$

$$f_{m,\ell}\big(D_w(F), d(F)\big) \leq f_{m,\ell}\big(D^*(N-m, m, \ell), d^*(N-m, m, \ell)\big) \,; \tag{B.9c}$$

and by induction on N we conclude that

$$F^{(K)}_{N,m,\ell}(\vec{z}_{\setminus I}, w) = c_K \; s_{N-m,m,\ell}(\vec{z}_{\setminus I}) \,, \tag{B.10}$$

for some constant c_K. However, c_K cannot depend on K either. This is seen by specializing equations (B.5) and (B.10) to $w = 0$, which gives

$$P(\vec{z}_{\setminus I}, 0, \dots, 0) = c_K \; s_{N-m,m,\ell}(\vec{z}_{\setminus I}) \prod_{j \in [N] \setminus I} z_j^{\ell} \,. \tag{B.11}$$

So, up to a multiplicative factor in P, we know that for any I and K as above, the specializations of $P(\vec{z})$ and of $s_{N,m,\ell}(\vec{z})$ to $z_{i_a} = q^{k_a} w$ are equal. This can be rephrased by saying that the difference $R(\vec{z}) := s_{N,m,\ell}(\vec{z}) - P(\vec{z})$ is a symmetric polynomial satisfying the $(m-1, \ell+1)$-wheel condition, and furthermore this implies easily that $D(R)$, $d(R)$, and $d_m(R)$ are a triple of entries smaller or equal to some triple (D, d, d_m) satisfying (one of) the degree condition(s) under consideration (because P does this by hypothesis, and the Schur function does it explicitly, and in the difference the degree can at most decrease due to cancellations).

As all the degree conditions in our proposition are monotone (in particular, $f_{m,\ell}(D+\alpha, d+\beta) \geq f_{m,\ell}(D, d)$ if $\alpha, \beta \geq 0$), the quantities in the conditions, as functions of $D(R)$, $d(R)$, and $d_m(R)$, are bounded above by the analogous quantities as functions of D^*, d^*, and d^*_m (for parameters (m, ℓ)).

Making an induction hypothesis in m, these degree bounds are to be compared with the bounds for a symmetric function in N variables, satisfying the $(m-1, \ell+1)$-wheel condition stated in the proposition. Therefore, in our range of interest $m \geq 2$, $a \geq 1$, write $N = am + b = \tilde{a}(m-1) + \tilde{b}$, with $1 \leq \tilde{b} \leq m-1$. Clearly $\tilde{a} \geq a$. The triple (D, d, d_m), pertinent to the bounds on the degrees of R for the (m, ℓ) case, reads

$$D = \ell\big(m\tbinom{a}{2} + ab\big) \,; \tag{B.12}$$

$$d = \ell a \,; \tag{B.13}$$

$$d_m = \ell(N - m) \,; \tag{B.14}$$

while the triple pertinent to the bounds for the $(m-1, \ell+1)$ case reads

$$D' = (\ell+1)\big((m-1)\tbinom{\tilde{a}}{2} + \tilde{a}\tilde{b}\big) \,; \tag{B.15}$$

$$d' = (\ell+1)\tilde{a} \,; \tag{B.16}$$

$$d'_{m-1} = (\ell+1)(N - m + 1) \,. \tag{B.17}$$

In particular, $f_{m,\ell}(D, d) = \ell^2 aN/m$ and $f_{m-1,\ell+1}(D', d') = (\ell+1)^2 \tilde{a}N/(m-1)$. As we have

$$d < d' \,; \tag{B.18a}$$

$$d_{m-1} \leq d_m < d'_{m-1} \,; \tag{B.18b}$$

$$f_{m-1,\ell+1}(D, d) < f_{m-1,\ell+1}(D', d') \,; \tag{B.18c}$$

(the last inequality comes with some algebra: the difference is

$$f_{m-1,\ell+1}(D, d) - f_{m-1,\ell+1}(D', d') = -\frac{(\ell+1)N}{m-1}\big((\ell+1)\tilde{a} - \ell a\big) - \frac{\ell(\ell+m)}{m-1}\binom{a+1}{2}$$

and is visibly negative), for any of the conditions in our list we reach the conclusion that $R(\vec{z}) = 0$. $\square$

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References

[1] D.M. Bressoud, *Proofs and confirmations. The story of the alternating sign matrix conjecture.* MAA Spectrum. Mathematical Association of America, Washington, DC; Cambridge University Press, Cambridge, 1999.

[2] F. Colomo and A.G. Pronko, *On two-point boundary correlations in the six-vertex model with domain wall boundary conditions,* J. Stat. Mech. **2005** (2005), P05010; `ar`χ`iv:math-ph/0503049`.

[3] B. Feigin, M. Jimbo, T. Miwa, E. Mukhin and Y. Takeyama, *Symmetric polynomials vanishing on the diagonals shifted by roots of unity,* Int. Math. Res. Not. **18** (2003), 999–1014; `ar`χ`iv:math/0209126`.

[4] B. Feigin, M. Jimbo, T. Miwa and E. Mukhin, *Symmetric polynomials vanishing on the shifted diagonals and Macdonald polynomials,* Int. Math. Res. Not. **18** (2003), 1015–1034; `ar`χ`iv:math/0209042`.

[5] T. Fonseca and P. Zinn-Justin, *On the doubly refined enumeration of alternating sign matrices and totally symmetric self-complementary plane partitions,* Electron. J. Combin. **15** (2008), Art. #R81, 35 pp.; `ar`χ`iv:0803.1595`.

[6] A.G. Izergin, *Partition function of the six-vertex model in a finite volume,* Sov. Phys. Dokl. **32** (1987), 878–879.

[7] V.E. Korepin, *Calculation of norms of Bethe wave functions,* Commun. Math. Phys. **86** (1982), 391–418.

[8] C. Krattenthaler, *Advanced Determinant Calculus,* Sém. Lothar. Combin. **42** ("The Andrews Festschrift") (1999), Art. B42q, 67 pp.

[9] G. Kuperberg, *Another proof of the alternating-sign matrix conjecture,* Int. Math. Res. Not. **3** (1996), 139–150; `ar`χ`iv:math/9712207`.

[10] I.G. Macdonald, *Symmetric Functions and Hall Polynomials.* Oxford Mathematical Monographs. Oxford University Press, 1999.

[11] A. Lascoux, *Pfaffians and representations of the symmetric group,* Acta Math. Sinica (Eng.) **25** (2009), 1929–1950; `ar`χ`iv:math/0610510`.

[12] W.H. Mills, D.P. Robbins and H. Rumsey, *Alternating-sign matrices and descending plane partitions,* J. Combin. Theory Ser. A **34** (1983), 340–359.

[13] T. Muir, *A Treatise on the Theory of Determinants,* revised and enlarged by W.H. Metzler, Courier Dover Publ., reprint of the 1939 edition, 2003.

[14] S. Okada, *Enumeration of symmetry classes of alternating sign matrices and characters of classical groups,* J. Alg. Combin. **23** (2006), 43–69; `ar`χ`iv:math/0408234`.

[15] R.P. Stanley, *Enumerative Combinatorics,* vol. 2, Cambridge University Press, Cambridge, 2001.

[16] Yu. Stroganov, *Izergin–Korepin determinant at a third root of unity,* Theor. Math. Phys. **146** (2004), 65–76 [russian: Teor. Mat. Fiz. **146** (2004), 53–62]; `ar`χ`iv:math-ph/0204042`.

[17] D. Zeilberger, *Proof of the alternating sign matrix conjecture,* Electron. J. Combin. **3**(no. 2; "The Foata Festschrift") (1996), Art. #R13, 84 pp.

[18] P. Zinn-Justin, *Combinatorial point for fused loop models,* Comm. Math. Phys. **272** (2007), 661–682; arχiv:math-ph/0603018.

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