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IDENTITIES FOR THE NUMBER OF STANDARD YOUNG TABLEAUX IN SOME (k,ℓ)-HOOKS

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Abstract. Closed formulas are known for $S(k,0;n)$, the number of standard Young tableaux of size n and with at most k parts, where $1 \le k \le 5$. Here we study the analogous problem for $S(k,\ell;n)$, the number of standard Young tableaux of size n which are contained in the (k,ℓ)-hook. We deduce some formulas for the cases $k+\ell \le 4$.

1. Introduction

Given a partition λ of n, which we denote as usual by $\lambda \vdash n$, let χ^λ denote the corresponding irreducible S_n character. Its degree is denoted by $\deg \chi^\lambda = f^\lambda$ and is equal to the number of standard Young tableaux (SYT) of shape λ. (The reader is referred to [8, 9, 13, 15] for introductions into character theory of the symmetric group and symmetric functions.) The number f^λ can be calculated for example by the hook formula (see [8, Theorem 2.3.21], [13, Section 3.10], [15, Corollary 7.21.6]. We consider the number of SYT in the (k,ℓ)-hook. More precisely, given integers $k,\ell,n \ge 0$, we write

$$H(k,\ell;n) = \{\lambda = (\lambda_1,\lambda_2,\dots) \mid \lambda \vdash n \text{ and } \lambda_{k+1} \le \ell\} \quad \text{and} \quad S(k,\ell;n) = \sum_{\lambda \in H(k,\ell;n)} f^\lambda.$$

1.1. **The cases where $S(k,\ell;n)$ are known.** For the "strip" sums $S(k,0;n)$ it is known (see [11] and [15, Ex. 7.16.b]) that

$$S(2,0;n) = \binom{n}{\lfloor \frac{n}{2} \rfloor} \quad \text{and} \quad S(3,0;n) = \sum_{j \ge 0} \frac{1}{j+1}\binom{n}{2j}\binom{2j}{j}.$$

Let $C_j = \frac{1}{j+1}\binom{2j}{j}$ be the Catalan numbers. Gouyou-Beauchamps [7] (see also [15, Ex. 7.16.b]) proved that

$$S(4,0;n) = C_{\lfloor \frac{n+1}{2} \rfloor} \cdot C_{\lceil \frac{n+1}{2} \rceil} \quad \text{and} \quad S(5,0;n) = 6\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} \cdot C_j \cdot \frac{(2j+2)!}{(j+2)!(j+3)!}.$$

As for the "hook" sums, until recently only $S(1,1;n)$ and $S(2,1;n) = S(1,2;n)$ have been calculated:

1. It easily follows that $S(1,1;n) = 2^{n-1}$.

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2. The following identity was proved in [12, Theorem 8.1]:

$$S(2,1;n) = \frac{1}{4}\left(\sum_{r=0}^{n-1}\binom{n-r}{\lfloor\frac{n-r}{2}\rfloor}\binom{n}{r} + \sum_{k=1}^{\lfloor\frac{n}{2}\rfloor-1}\frac{n!}{k!\cdot(k+1)!\cdot(n-2k-2)!\cdot(n-k-1)\cdot(n-k)}\right)+1. \quad (1)$$

1.2. **The main results.** In Section 2 we prove Equation (10), which gives (sort of) a closed formula for $S(3,1;n)$ in terms of the Motzkin-sums function. For the Motzkin-sums function see [14, sequence A005043]. Equation (10) is in fact a "degree" consequence of a formula for S_n characters, of interest in its own right, see Equation (9).

In Section 3 we find some intriguing relations between the sums $S(4,0;n)$ and the "rectangular" sub-sums $S^*(2,2,;n)$ (see Section 3 for their definition), see identities (12) and (13) below.

Finally, in Section 4 we review some cases where the hook sums $S(k,\ell;n)$ are related, in some rather mysterious ways, to hump enumerations on Dyck and on Motzkin paths, see (14), (16), and Theorem 4.1.

As usual, in some of the above identities it is of interest to find bijective proofs, which might explain these identities.

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2. The sums $S(3,1;n)$ and the characters $\chi(3,1;n)$

Define the S_n character

$$\chi(k,\ell;n) = \sum_{\lambda\in H(k,\ell;n)}\chi^\lambda, \qquad \text{so that} \qquad \deg(\chi(k,\ell;n)) = S(k,\ell;n). \quad (2)$$

2.1. **The Motzkin-sums function.** Define the S_n character

$$\Psi(n) = \sum_{k=0}^{\lfloor n/2\rfloor}\chi^{(k,k,1^{n-2k})}, \qquad \text{and denote} \qquad \deg\Psi(n) = a(n). \quad (3)$$

We call $\Psi(n)$ *the Motzkin-sums* character. Note that

$$\deg\chi^{(k,k,1^{n-2k})} = f^{(k,k,1^{n-2k})} = \frac{n!}{(k-1)!\cdot k!\cdot(n-2k)!\cdot(n-k)\cdot(n-k+1)},$$

hence

$$a(n) = \sum_{k=1}^{\lfloor n/2\rfloor}\frac{n!}{(k-1)!\cdot k!\cdot(n-2k)!\cdot(n-k)\cdot(n-k+1)}. \quad (4)$$

By [14, sequence A005043], it follows that $a(n)$ is the Motzkin-sums function. The reader is referred to [14] for various properties of $a(n)$. For example, $a(n)+a(n+1)=$

M_n, where M_n are the Motzkin numbers. Also $a(1) = 0$, $a(2) = 1$, and $a(n)$ satisfies the recurrence

$$a(n) = \frac{n-1}{n+1} \cdot (2 \cdot a(n-1) + 3 \cdot a(n-2)), \qquad \text{for } n \geq 3. \tag{5}$$

Note also that for $n \geq 2$ Equation (1) can be written as

$$S(2,1;n) = \frac{1}{4}\left(\sum_{r=0}^{n-1} \binom{n-r}{\lfloor \frac{n-r}{2} \rfloor}\binom{n}{r} + a(n) - 1\right) + 1. \tag{6}$$

The asymptotic behavior of $a(n)$ can be deduced from that of M_n. We deduce it here, even though it is not needed in the sequel.

Remark 2.1. As n tends to infinity,

$$a(n) \simeq \frac{\sqrt{3}}{8 \cdot \sqrt{2\pi}} \cdot \frac{1}{n\sqrt{n}} \cdot 3^n \qquad \text{and} \qquad a(n) \simeq \frac{1}{4} \cdot M_n.$$

Proof. By standard techniques it can be shown that $a(n)$ has asymptotic behavior

$$a(n) \simeq c \cdot \left(\frac{1}{n}\right)^g \cdot \alpha^n$$

for some constants c, g and α — which we now determine. By [11], we have

$$M_n \simeq \frac{\sqrt{3}}{2\sqrt{2\pi}} \cdot \left(\frac{1}{n}\right)^{3/2} \cdot 3^n.$$

Together with

$$M_n = a(n) + a(n+1) \simeq c \cdot (1+\alpha) \cdot \left(\frac{1}{n}\right)^g \cdot \alpha^n,$$

this implies that $\alpha = 3$, that $g = 3/2$, and that $c = \frac{\sqrt{3}}{8 \cdot \sqrt{2\pi}}$.

□

2.2. **The outer product of S_m and S_n characters.** Given an S_m character χ_m and an S_n character χ_n, we can form their *outer* product $\chi_n \hat{\otimes} \chi_n$. The exact decomposition of $\chi_m \hat{\otimes} \chi_n$ is given by the Littlewood–Richardson rule, see [8, 9, 13, 15]. In the special case that $\chi_n = \chi^{(n)}$, this decomposition is given, below, by Young's rule. Furthermore, we have

$$\deg(\chi_n \hat{\otimes} \chi^{(n)}) = \deg(\chi_n) \cdot \binom{n+m}{n}. \tag{7}$$

Young's Rule (see [9, Ch. I, Sec. 7 and (5.16)]): Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash m$ and denote by λ^{+n} the following set of partitions of $m+n$:

$$\lambda^{+n} = \{\mu \vdash n+m \mid \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots\}.$$

Then

$$\chi^\lambda \hat{\otimes} \chi^{(n)} = \sum_{\mu \in \lambda^{+n}} \chi^\mu.$$

Example 2.2. Given n, it follows that (see [11], [15, Ex. 7.16.b])

$$\chi^{(\lfloor n/2\rfloor)}\hat{\otimes}\chi^{(\lceil n/2\rceil)} = \chi(2,0;n), \quad \text{and by taking degrees,} \quad S(2,0;n) = \binom{n}{\lfloor n/2\rfloor}. \tag{8}$$

2.3. **A character formula for $\chi(3,1;n)$.**

Proposition 2.3. *With the notations of* (2) *and* (3)*,*

$$\chi(3,1;n) = \frac{1}{2}\cdot\left[\chi(2,0,n) + \sum_{j=0}^{n}\Psi(j)\hat{\otimes}\chi^{(n-j)}\right]. \tag{9}$$

By taking degrees, Example 2.2 together with (3) *and* (7) *imply that*

$$S(3,1;n) = \frac{1}{2}\cdot\left[\binom{n}{\lfloor\frac{n}{2}\rfloor} + \sum_{j=0}^{n} a(j)\cdot\binom{n}{j}\right]. \tag{10}$$

Proof. Denote

$$\Omega(n) = \sum_{j=0}^{n}\Psi(j)\hat{\otimes}\chi^{(n-j)},$$

and analyze this S_n character. Young's rule implies the following:

If $\mu\vdash n$, then, by Young's rule, χ^{μ} has a positive coefficient in $\Omega(n)$ if and only if $\mu\in H(3,1;n)$. Moreover, all these coefficients are either 1 or 2, and such a coefficient equals 1 if and only if μ is a partition with at most two rows, say $\mu = (\mu_1,\mu_2)$. It follows that

$$\chi(2,0;n) + \Omega(n) = 2\cdot\sum_{\lambda\in H(3,1;n)}\chi^{\lambda}. \tag{11}$$

This implies (9) and completes the proof of Proposition 2.3. □

3. The sums $S(4,0;n)$ and $S^*(2,2;n)$

Definition 3.1. (1) Let $n = 2m$, $m\ge 2$, and let $H^*(2,2;2m)\subset H(2,2;2m)$ denote the set of partitions $H^*(2,2;2m) = \{(k{+}2, k{+}2, 2^{m-2-k})\vdash 2m \mid k = 0,\dots m{-}2\}$ (the partitions in the $(2,2)$-hook with both arm and leg being rectangular). Furthermore, write

$$S^*(2,2;2m) = \sum_{\lambda\in H^*(2,2;2m)} f^{\lambda}.$$

(2) Let $n = 2m+1$, $m\ge 2$, and let $H^*(2,2;2m+1)\subset H(2,2;2m+1)$ denote the set of partitions $H^*(2,2;2m+1) = \{(k+3, k+2, 2^{m-2-k})\vdash 2m+1 \mid k = 0,\dots m-2\}$ (the partitions in the $(2,2)$-hook with arm nearly rectangular and leg rectangular). Furthermore, let

$$S^*(2,2;2m+1) = \sum_{\lambda\in H^*(2,2;2m+1)} f^{\lambda}.$$

Recall from Section 1.1 that $S(4,0;2m-1) = C_m^2$ and $S(4,0;2m) = C_m \cdot C_{m+1}$. We have the following intriguing identities.

Proposition 3.2. (1) *Let $n = 2m$. Then*

$$S(4,0;2m-2) = C_{m-1} \cdot C_m = S^*(2,2;2m).$$

Explicitly, we have the following identity:

$$\begin{aligned} C_{m-1} \cdot C_m &= \frac{1}{m \cdot (m+1)} \cdot \binom{2m-2}{m-1} \cdot \binom{2m}{m} \\ &= \sum_{k=0}^{m-2} \frac{(2m)!}{k! \cdot (k+1)! \cdot (m-k-2)! \cdot (m-k-1)! \cdot (m-1) \cdot m^2 \cdot (m+1)}. \end{aligned} \tag{12}$$

(2) *Let $n = 2m+1$. Then*

$$\frac{2m+1}{m+2} \cdot S(4,0;2m-1) = \frac{2m+1}{m+2} \cdot C_m^2 = S^*(2,2;2m+1).$$

Explicitly, we have the following identity:

$$\begin{aligned} \frac{2m+1}{m+2} \cdot C_m^2 &= \frac{1}{(m+1) \cdot (m+2)} \cdot \binom{2m}{m} \binom{2m+1}{m} \\ = \sum_{k=0}^{m-2} & \frac{(2m+1)! \cdot 2}{k! \cdot (k+2)! \cdot (m-k-2)! \cdot (m-k-1)! \cdot (m-1) \cdot m \cdot (m+1) \cdot (m+2)}. \end{aligned} \tag{13}$$

Proof. Equation (12) is the specialization of Gauß's ${}_2F_1(a,b;c;1)$ with $a = 2-m, b = 1-m, c = 2$ (cf. [1]), and (13) is similar. Alternatively, the identities (12) and (13) can be verified by the WZ method (cf. [10, 16]). □

4. Hook sums and humps for paths

A Dyck path of length $2n$ is a lattice path, in $\mathbb{Z} \times \mathbb{Z}$, from $(0,0)$ to $(2n,0)$, using up-steps $(1,1)$ and down-steps $(1,-1)$ and never going below the x-axis. A *hump* in a Dyck path is an up-step followed by a down-step.[1]

A Motzkin path of length n is a lattice path from $(0,0)$ to $(n,0)$, using flat-steps $(1,0)$, up-steps $(1,1)$ and down-steps $(1,-1)$, and never going below the x-axis. A hump in a Motzkin path is an up-step followed by zero or more flat-steps followed by a down-step.

We now count *humps* for Dyck and for Motzkin paths and observe the following intriguing phenomena: The hump enumeration in the Dyck case associates the $2 \times n$ rectangular shape $\lambda = (n,n)$ to the $(1,1)$-hook shape $\mu = (n, 1^n)$. Moreover, in the Motzkin case we show below that it associates the $(3,0)$ strip shape partitions $H(3,0;n)$ to the $(2,1)$-hook shape partitions $H(2,1;n)$.

[1] In the Dyck path context, humps are usually called *peaks*. However, we prefer the term "hump" because, in the context of Motzkin paths, this term will indeed differ from "peak."

4.1. **The Dyck case.** The Catalan number

$$C_n = \frac{(2n)!}{n!(n+1)!}$$

is the cardinality of a variety of sets (see [15, Ex. 6.19]); here we are interested in two such sets. First, $C_n = f^{(n,n)}$, the number of SYT of shape (n,n). Second, C_n is the number of Dyck paths of length $2n$. Let $\mathcal{H}D_n$ denote the total number of humps in all Dyck paths of length $2n$. Then

$$\mathcal{H}D_n = \binom{2n-1}{n},$$

see [3, 4, 6]. Since $\binom{2n-1}{n} = f^{(n,1^n)}$, we have

$$C_n = f^{(n,n)} \qquad \text{and} \qquad \mathcal{H}D_n = f^{(n,1^n)}.$$

We denote this association by

$$\mathcal{H} : (n,n) \longrightarrow (n,1^n). \tag{14}$$

4.2. **The Motzkin case.** Like the Catalan numbers, also the Motzkin numbers M_n are the cardinality of a variety of sets (cf. [15, Ex. 6.38], [14, sequence A001006]). The result from [11] that $M_n = S(3,0;n)$ gives the Motzkin numbers a SYT interpretation. Moreover, M_n is the number of Motzkin paths of length n. Let $\mathcal{H}M_n$ denote the total number of humps in all Motzkin paths of length n. Then, according to [14, sequence A097861],

$$\mathcal{H}M_n = \frac{1}{2}\sum_{j\ge 1}\binom{n}{j}\binom{n-j}{j}. \tag{15}$$

We show below that this implies the intriguing identity $\mathcal{H}M_n = S(2,1;n) - 1$, which gives a SYT-interpretation of the numbers $\mathcal{H}M_n$. Thus, the hump enumeration in the Motzkin case associates the $(3,0)$ strip shape partitions $H(3,0;n)$ to the $(2,1)$-hook shape partitions $H(2,1;n)$. We denote this by

$$\mathcal{H} : H(3,0;n) \longrightarrow H(2,1;n). \tag{16}$$

Theorem 4.1. *The number of humps of all Motzkin paths of length n satisfies*

$$\mathcal{H}M_n = S(2,1;n) - 1.$$

Combining Equations (1) and (15), the proof of Theorem 4.1 will follow once the following binomial identity — of interest in its own right — is proved.

Lemma 4.2. *For $n \ge 2$, we have*

$$2\sum_{j=1}^{\lfloor n/2\rfloor}\binom{n}{j}\binom{n-j}{j} = \sum_{r=0}^{n-1}\binom{n-r}{\lfloor\frac{n-r}{2}\rfloor}\binom{n}{r} + a(n) - 1$$
$$= \sum_{r=0}^{n-1}\binom{n-r}{\lfloor\frac{n-r}{2}\rfloor}\binom{n}{r} + \sum_{k=1}^{\lfloor\frac{n}{2}\rfloor-1}\frac{n!}{k!\cdot(k+1)!\cdot(n-2k-2)!\cdot(n-k-1)\cdot(n-k)}. \tag{17}$$

Equation (17) was first verified by the WZ method. About this method, see [10, 16]. Here is an elementary proof which is due to Ira Gessel [5].

Proof. Note first that $a(n)$ is the nth Riordan number, [14, sequence A005043], defined (for example) by

$$\sum_{n=0}^{\infty} a(n)x^n = \frac{2}{1+x+\sqrt{1-2x-3x^2}}.$$

Adding 2 to both sides of (17) gives the equivalent identity

$$2\sum_{j=0}^{\lfloor n/2\rfloor} \binom{n}{j}\binom{n-j}{j} = \sum_{r=0}^{n} \binom{n-r}{\lfloor \frac{n-r}{2}\rfloor}\binom{n}{r} + a(n). \tag{18}$$

Now let us replace r by $n-r$ in the sum on the right-hand side of (18), thereby getting

$$\sum_{r=0}^{n} \binom{r}{\lfloor \frac{r}{2}\rfloor}\binom{n}{r},$$

and then separate the even and odd values of r so that this sum is equal to $u(n)+v(n)$ where

$$u(n) = \sum_j \binom{2j}{j}\binom{n}{2j}$$

and

$$v(n) = \sum_j \binom{2j+1}{j}\binom{n}{2j+1}.$$

Noting that the left-hand side of (18) is $2u(n)$, we see that the identity to be proved is equivalent to

$$u(n) = v(n) + a(n). \tag{19}$$

It is straightforward to show that $u(n)$ is the coefficient of x^n in $(1+x+x^2)^n$ [14, sequence A002426, central trinomial coefficients] and that $v(n)$ is the coefficient of x^{n-1} (or of x^{n+1}) in $(1+x+x^2)^n$ [14, sequence A005717]. With these interpretations for $u(n)$ and $v(n)$, a combinatorial proof of the identity $u(n) - v(n) = a(n)$ has been given by David Callan [Riordan numbers are differences of trinomial coefficients, 2006, `http://www.stat.wisc.edu/~callan/notes/riordan/riordan.pdf`]. Alternatively, Equation (19) follows easily from the known generating functions for $u(n)$, $v(n)$, and $a(n)$, which can all be found in [14] (or derived directly). □

This completes the proof of Theorem 4.1.

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