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ON THE COMBINATORICS OF YOUNG–CAPELLI SYMMETRIZERS

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Abstract. We deal with the characteristic zero theory of supersymmetric algebras, regarded as bimodules under the action of a pair of general linear Lie superalgebras, as developed by Brini et al. (see [*Proc. Natl. Acad. Sci. USA* **85** (1988), 1330–1333; *Proc. Natl. Acad. Sci. USA* **86** (1989), 775–778] for the first version, [*Séminaire Lotharingien Combin.* **55** (2007), Article B55g, 117 pp.] for the last version; see also Sergeev [*Mat. Sb. (N.S.)* **123(165)** (1984), 422–430; *Michigan Math. J.* **49** (2001), 113–146] and Cheng and Wang [*Compositio Math.* **128** (2001), 55–94]). The theory had its roots in the pioneering work of Grosshans, Rota and Stein [*Invariant theory and Superalgebras,* Amer. Math. Soc., Providence, RI, 1987], Berele and Regev [*Bull. Am. Math. Soc.* **8** (1983), 337–339; *Adv. Math.* **64** (1987), 118–175] and Sergeev [*Mat. Sb. (N.S.)* **123(165)** (1984), 422–430]. The basic objects of the theory, i.e., symmetrized bitableaux and Young–Capelli symmetrizers, are defined by means of a superalgebraic extension of Capelli's method of virtual variables, and the relations between them are proved in the virtual setting, by means of a Triangularity Lemma, a Nondegeneracy Lemma, and the Superstraightening Law. We give a detailed exposition of the foundations of this theory. In doing this, we establish three new propositions on virtual expressions, and give new, elementary combinatorial proofs of the Triangularity Lemma and of the Nondegeneracy Lemma. With these proofs, we complete the process of giving the theory an elementary combinatorial foundation.

Introduction

We deal with the characteristic zero theory of supersymmetric algebras, regarded as bimodules under the action of a pair of general linear Lie superalgebras, as developed by Brini et al. (see [8, 9] for the first version, [6] for the last version; see [10, 11, 7] for ramifications in other directions; see also [24]). For each module, the theory yields a basis, the basis of standard symmetrized bitableaux, for the space, and a basis, the basis of standard Young–Capelli symmetrizers, for the operator algebra induced by the action, which acts on the former basis in a nondegenerate triangular way. These bases are used to get explicit complete decompositions of the space as a semisimple module, and of the operator algebra as a semisimple algebra. The basic objects are defined by means of a superalgebraic version of Capelli's method of virtual variables, and the relations between them are proved in the virtual setting. Roughly speaking, the foundations of this theory in the first version relied on Grosshans, Rota and Stein's Superstraightening Law [26], and on Berele and Regev's supertensor representation theory of general linear Lie superalgebras and symmetric groups ([1, 2]; see also [35]).

In the last version, Berele and Regev's theory is replaced by the ordinary representation theory of the symmetric group. A first use of Capelli's method of virtual variables in a superalgebraic context can be traced back to the pioneering work of Koszul [28]. The idea of virtual variables is the basic tool introduced by Capelli in order to prove his famous identities [15].

The theory we deal with yields, by specialization, several classical theories. We mention: the ordinary representation theory of the symmetric group, in which symmetrized bitableaux and Young–Capelli symmetrizers turn out to be the classical Young symmetrizers (see, e.g., [33]); the theory of tensor spaces under the action of general linear groups and symmetric groups, in which standard symmetrized bitableaux give bases of the symmetry classes (see, e.g., [40]); the theory of spaces of polynomial functions of several vector variables under the action of general linear groups, in which the basis of standard symmetrized bitableaux becomes the Gordan–Capelli series (in the sense of [39]), and Young–Capelli symmetrizers lead to an explicit form of the Capelli-Deruyts expansion (in the sense of [32]); the Berele–Regev theory of super tensor spaces under the action of general linear Lie superalgebras and symmetric groups [1, 2].

For a detailed description of the relations of the theory with classical invariant theory, as well as of the relations with the representation theory of general linear Lie superalgebras, we refer to [6].

Let $\mathbb{K}$ a be field of characteristic zero, $\mathcal{L}$ and $\mathcal{P}$ two finite $\mathbb{Z}_2$-graded sets, $\mathrm{pl}(\mathcal{L})$ and $\mathrm{pl}(\mathcal{P})$ the corresponding general linear Lie superalgebras, $\mathrm{Super}[\mathcal{L}|\mathcal{P}]$ the supersymmetric algebra over $\mathcal{L} \times \mathcal{P}$, and $\mathrm{Super}_n[\mathcal{L}|\mathcal{P}]$ the $\mathbb{Z}-$homogeneous $n-$th component of this algebra. In [8], Brini, Palareti and Teolis defined symmetrized bitableaux in the space $\mathrm{Super}_n[\mathcal{L}|\mathcal{P}]$, proved that the superstandard symmetrized bitableaux form a linear basis of this space, and got from this basis an explicit complete decomposition of $\mathrm{Super}_n[\mathcal{L}|\mathcal{P}]$ as a semisimple $\mathrm{pl}(\mathcal{L})-$module. Roughly speaking, the proof of the spanning property relies on the Grosshans–Rota–Stein Superstraightening Law [26] and the proof of the linear independence relies on the Berele–Regev theory [1, 2]. In [9], Brini and Teolis constructed a complete decomposition of the operator algebra induced by the action of $\mathrm{pl}(\mathcal{L})$ as a semisimple algebra, by defining the Young–Capelli symmetrizers and proving a Triangularity Theorem for the action of the superstandard Young–Capelli symmetrizers on the superstandard symmetrized bitableaux. The proof of the Triangularity Theorem relies on the Berele–Regev theory [1, 2] and on the properties of Gale–Ryser interpolating matrices [26].

In the last years, the theory has been given a new architecture. In [12], it is outlined how the complete decompositions of the semisimple $\mathrm{pl}(\mathcal{L})$-module $\mathrm{Super}_n[\mathcal{L}|\mathcal{P}]$ and of the semisimple operator algebra induced by the $pl(\mathcal{L})$-action can be simultaneously derived by the Triangularity Theorem for the action of the superstandard Young–Capelli symmetrizers on the superstandard symmetrized bitableaux. In [13] it was shown that the Triangularity Theorem can be directly derived from a Triangularity Lemma and a Nondegeneracy Lemma, and it was announced that these lemmas have elementary combinatorial proofs that will appear elsewhere. We remark that, while the Triangularity Lemma is a refinement of a result in [9] which in [9] had already an elementary combinatorial proof, the Nondegeneracy Lemma, implicitly stated in [8, 9], is exactly the point in which the theory was dependent on the Berele–Regev theory.

In [6], it is given a strong form of the Nondegeneracy Lemma, together with a rather indirect proof based on the ordinary representation theory of the symmetric group.

In this paper, we give a detailed exposition of the foundations of the theory, as well as an outline of the main results. To wit: we give, through Capelli's method of virtual variables, the definitions of the basic objects of the theory; we establish three new propositions on virtual expressions; we give the statements of the Triangularity Lemma, of the Nondegeneracy Lemma and of the Superstraightening Law; we give an outline of the theory till the theorems on complete decompositions; we prove from scratch the three propositions and the two lemmas. All these proofs are new; the proof of the Triangularity Lemma is based on the easy part of the Gale–Ryser Theorem; the proof of the weak form of the Nondegeneracy Lemma is obtained by a direct combinatorial argument; the proof of the strong form of the Nondegeneracy Lemma is based on the Superstraightening Law. With these proofs, we complete the process of giving the theory an elementary combinatorial foundation.

1. Words and Tableaux

In this section we recall some notions pertaining to words and tableaux on a $\mathbb{Z}_2$-graded set. Here, our aim is just to fix terms and notations that will be used throughout.

Let $\mathcal{A}$ be a set endowed with a disjoint union decomposition $\mathcal{A} = \mathcal{A}^+ \sqcup \mathcal{A}^-$ into a subset $\mathcal{A}^+$ of *positive* elements and a subset $\mathcal{A}^-$ of *negative* elements; such a decomposition may be encoded by the mapping $|\ | : \mathcal{A} \to \mathbb{Z}_2$ with values in the two-element field $\mathbb{Z}_2$ which sends the positive and negative elements to 0 and 1, respectively. A set with this structure is called a *signed set*, or a $\mathbb{Z}_2$-*graded set*.

Let $w = x_1 x_2 \cdots x_n$ be a word of length $l(w) = n$ on $\mathcal{A}$. The *support* of w is the set $\mathrm{sp}(w)$ of the elements of $\mathcal{A}$ that occur in w; the *content* of w is the family c_w of the occurrences $c_w(a)$ of the elements a of $\mathcal{A}$ in w.

Let $w = x_1 x_2 \cdots x_n$ and $w' = x'_1 x'_2 \cdots x'_n$ be two words of the same length on $\mathcal{A}$. We say that w and w' are *equivalent*, and we write

$$w \equiv w',$$

if w and w' have the same content. This happens if and only if there is a permutation of the index set $\{1, \ldots, n\}$ which carries w to w'. If there is a $\mathbb{Z}_2$−grade preserving bijection $\mathcal{A} \to \mathcal{A}$ which carries w to w', we say that w and w' are *isomorphic*, and we write

$$w \simeq w'.$$

We say that two sequences $(w, z, \ldots)$ and $(w', z', \ldots)$ of words are isomorphic, and we write $(w, z, \ldots) \simeq (w', z', \ldots)$, if there is a $\mathbb{Z}_2$−grade preserving bijection $\mathcal{A} \to \mathcal{A}$ which carries w to w', z to z', ...

A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of nonnegative integers; the nonnegative integer $\sum \lambda_i$ is called the *weight* of the partition, and is denoted by $|\lambda|$; the sentence "λ has weight n" is rephrased also as "λ is a partition of n", and written $\lambda \vdash n$. A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is *dominated* by a partition $\mu = (\mu_1, \mu_2, \ldots)$ if

$$\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i, \qquad \text{for all } i = 1, 2, \ldots.$$

The relation of dominance is a partial order on partitions, and will be denoted by the symbol $\leq$. We take for granted some basic notions pertaining to partitions, such as the notion of skew partition, horizontal strip, and vertical strip (see, e.g., [31, pp. 1–5]).

A *tableau* over the signed set $\mathcal{A}$ is a sequence

$$S = (w_1, w_2, \ldots, w_p)$$

of words $w_i = w_{i1} w_{i2} \cdots w_{i\lambda_i}$ of weakly decreasing lengths. The partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ is called the *shape* of S, and is denoted by $\mathrm{sh}(S)$. The word w obtained by juxtaposition of the words w_i,

$$w = w_1 w_2 \cdots w_p = w(S)$$

is called the *row word* of S. If n is the length of w, then λ is a partition of n, and we say that the tableau S has *size* n.

The tableau S can be represented by writing the words $w_1, w_2, \ldots, w_p$ with their first entries directly under one another

$$S = \begin{array}{l} w_{11} w_{12} \ldots\ldots\ldots w_{1\lambda_1} \\ w_{21} w_{22} \ldots\ldots w_{2\lambda_2} \\ \vdots \\ w_{p1} w_{p2} \ldots w_{p\lambda_p} \end{array}$$

or, in matrix-like notation, $S = (w_{ij})_{\substack{i=1,\ldots,p \\ j=1,\ldots,\lambda_i}}$.

The *content* of the tableau S is the family c_S of the occurrences $c_S(a)$ of the elements a of $\mathcal{A}$ in S; the *row-content* of S is the family c_S^r of the occurrences $c_S^r(a, i)$ of the elements a of $\mathcal{A}$ in the rows w_i of S; the *column-content* of S is the family c_S^c of the occurrences $c_S^c(a, j)$ of the elements a of $\mathcal{A}$ in the columns z_j of S.

If two tableaux S and S' have the same content, then we say that they are *equivalent*, and we write $S \equiv S'$; if S and S' have the same row-content, we say that they are *row-equivalent*, and we write

$$S \equiv_r S';$$

if S and S' have the same column-content, we say that they are *column-equivalent*, and we write $S \equiv_c S'$.

We denote the sets of all the tableaux over $\mathcal{A}$ of size n, of shape λ, respectively by the symbols

$$\mathfrak{t}_n[\mathcal{A}], \ \mathfrak{t}_\lambda[\mathcal{A}];$$

when no confusion arises, we will suppress the specification $[\mathcal{A}]$.

We assume now that the signed set $\mathcal{A}$ is a linearly ordered set, of the same order type of a subset of the natural numbers. A tableau S over $\mathcal{A}$ is called *standard* if

- each row of S is weakly increasing, with no repetition of negative symbols,
- each column of S is weakly increasing, with no repetition of positive symbols.

It is known that, if $|\mathcal{A}^+| = p$ and $|\mathcal{A}^-| = q$, then there is a standard tableau of shape λ with entries in $\mathcal{A}$ if and only if $\lambda_{p+1} < q + 1$. The set of these partitions is called the *hook set* of the signed set $\mathcal{A}$, and is denoted by $H(\mathcal{A})$.

Let S be a standard tableau over $\mathcal{A}$. For every positive integer p, the object $S^{(p)}$ obtained by considering in S only the first p symbols of $\mathcal{A}$ is a standard tableau; the shapes of these tableaux form a sequence

$$\mathrm{sh}(S^{(1)}) \subseteq \mathrm{sh}(S^{(2)}) \subseteq \ldots = \mathrm{sh}(S) = \mathrm{sh}(S) = \ldots,$$

such that the skew partition $\mathrm{sh}(S^{(p)})/\mathrm{sh}(S^{(p-1)})$ is an horizontal strip or a vertical strip according to the $p-$th symbol of $\mathcal{A}$ is positive or negative. Any sequence of partitions of this type is achieved by exactly one standard tableau.

We use the *partial* order on the set of all standard tableaux over $\mathcal{A}$ defined by setting

$$S \leq T \quad \text{if and only if} \quad S \equiv T, \text{ and } \mathrm{sh}(S^{(p)}) \leq \mathrm{sh}(T^{(p)}), \text{ for all } p = 1, 2, \ldots$$

This partial order on standard tableaux lies between the usual linear order on tableaux [26] and the hyperdominance order on tableaux [20].

We denote the sets of all the standard tableaux over $\mathcal{A}$ of size n, of shape λ, respectively, by the symbols

$$\mathfrak{st}_n[\mathcal{A}], \ \mathfrak{st}_\lambda[\mathcal{A}].$$

When no confusion arises, we will suppress the specification $[\mathcal{A}]$.

2. Setting

In this section, we introduce the supersymmetric algebra $\mathrm{Super}[\mathcal{L}|\mathcal{P}]$ on the direct product of two finite $\mathbb{Z}_2$-graded sets $\mathcal{L}, \mathcal{P}$, left and right superpolarization operators on this algebra, and the bimodules we are interested in. We recall all what we need; we refer to [34, Chapter 0], for the basic superalgebraic notions, and to [6, Sections 3 and 4], for further aspects, namely the representation-theoretical ones.

2.1. The Superalgebra $\mathrm{Super}[\mathcal{A}]$.

Let $\mathbb{K}$ be a field of characteristic 0; by vector space, algebra, linear mapping, etc., we will always mean $\mathbb{K}-$vector space, $\mathbb{K}-$algebra, $\mathbb{K}-$linear mapping, etc..

A $\mathbb{Z}_2$-*grading* on a vector space V is a decomposition $V = V_0 \oplus V_1$ of V as a direct sum of two subspaces V_i, with $i \in \mathbb{Z}_2$. The elements of V_0 and V_1 are said to be *homogeneous*; the grade of an homogeneous element $v \in V$ will be denoted by $|v|$. A $\mathbb{Z}_2$-*grading* on an algebra A is a decomposition $A = A_0 \oplus A_1$ of A as a direct sum of two subspaces A_i, with $i \in \mathbb{Z}_2$, such that $A_i \ A_j \subseteq A_{i+j}$, for all $i, j \in \mathbb{Z}_2$. The superalgebra A is called $\mathbb{Z}_2$-*graded commutative* whenever

$$ab = (-1)^{|a||b|} ba,$$

for every homogeneous elements a, b in A.

A linear mapping $f : V \to W$ from a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ from a $\mathbb{Z}_2$-graded vector space $W = W_0 \oplus W_1$ is said to be homogeneous of degree $i \in \mathbb{Z}_2$ when $f[V_j] \subset W_{j+i}$, for all $j \in \mathbb{Z}_2$. Given a $\mathbb{Z}_2$-graded algebra A, a linear mapping $D : A \to A$, homogeneous of degree $|D|$, is said to be a *left superderivation* whenever

$$D(ab) = D(a)b \ + \ (-1)^{|D||a|} aD(b),$$

for every homogeneous elements a, b in A; a linear mapping $A \leftarrow A : D$, homogeneous of degree $|D|$, is said to be a *right superderivation* whenever

$$(ab)D = (-1)^{|b||D|}(a)Db \ + \ a(b)D,$$

for every homogeneous elements a, b in A.

Let $\mathcal{A}$ be a $\mathbb{Z}_2$-graded set. The free monoid $\mathrm{Mon}[\mathcal{A}]$ on a $\mathcal{A}$ has a natural $\mathbb{Z}_2$-grading given by $|w| = |x_1| + |x_2| + \cdots + |x_n|$, for every word $w = x_1x_2 \cdots x_n$ on $\mathcal{A}$. The *supersymmetric algebra* $\mathrm{Super}[\mathcal{A}]$ generated by the the $\mathbb{Z}_2$-graded set $\mathcal{A}$ is the quotient algebra

$$\mathrm{Super}[\mathcal{A}] = \frac{\mathbb{K}[\mathrm{Mon}[\mathcal{A}]]}{\mathrm{I}[\mathcal{A}]}$$

of the semigroup algebra $\mathbb{K}[\mathrm{Mon}[\mathcal{A}]]$ of the free monoid $\mathrm{Mon}[\mathcal{A}]$ with respect to the two-sided ideal $\mathrm{I}[\mathcal{A}]$ generated by the elements of the form

$$xy - (-1)^{|x||y|}yx, \qquad x, y \in \mathcal{A}.$$

A word w in $\mathrm{Mon}[\mathcal{A}]$ is nonzero in $\mathrm{Super}[\mathcal{A}]$ if and only if each negative symbol occurs in w at most once; two such words give rise, up to a sign, to the same element of $\mathrm{Super}[\mathcal{A}]$ if and only if they have the same content, i.e., they have are equivalent. By choosing one word for each admissible content, one gets a linear basis of this algebra. We define the sign $\epsilon_{w,w'} \in \{-1, 1\}$ of two nonzero equivalent words w, w', by setting

$$w = \epsilon_{w,w'} \ w'.$$

The algebra $\mathrm{Super}[\mathcal{A}]$ has a natural $\mathbb{Z}_2$-grading, given by

$$\begin{aligned}\mathrm{Super}[\mathcal{A}] &= \langle w \in \mathrm{Mon}[\mathcal{A}];\ |w| = 0\rangle \oplus \langle w \in \mathrm{Mon}[\mathcal{A}];\ |w| = 1\rangle \\ &= \mathrm{Super}[\mathcal{A}]_0 \oplus \mathrm{Super}[\mathcal{A}]_1,\end{aligned}$$

and it is $\mathbb{Z}_2$-graded commutative. This algebra has also a natural $\mathbb{Z}$-grading, given by

$$\begin{aligned}\mathrm{Super}[\mathcal{A}] &= \oplus_{n \in \mathbb{N}} \langle w \in \mathrm{Mon}[\mathcal{A}];\ l(w) = n\rangle \\ &= \oplus_{n \in \mathbb{N}} \mathrm{Super}_n[\mathcal{A}],\end{aligned}$$

and the two gradings are *consistent*.

For every x, y in $\mathcal{A}$, there is exactly one left superderivation,

$$\mathcal{D}_{xy} : \mathrm{Super}[\mathcal{A}] \to \mathrm{Super}[\mathcal{A}],$$

homogeneous of degree $|xy| = |x| + |y|$, called the *left superpolarization of* y *to* x, which takes the values $\mathcal{D}_{xy}(u) = \delta_{yu}\ x$ on the generators of $\mathrm{Super}[\mathcal{A}]$. For every x, y in $\mathcal{A}$, there is exactly one right superderivation

$$\mathrm{Super}[\mathcal{A}] \leftarrow \mathrm{Super}[\mathcal{A}] :\ {}_{xy}\mathcal{D},$$

homogeneous of degree $|xy| = |x| + |y|$, called the *right superpolarization of* x *to* y, which takes the values $(u)\ {}_{xy}\mathcal{D} = \delta_{ux}\ y$ on the generators of $\mathrm{Super}[\mathcal{A}]$.

2.2. **The Superalgebra** Super$[\mathcal{L}|\mathcal{P}]$**.** Let $\mathcal{L}, \mathcal{P}$ be two finite $\mathbb{Z}_2$-graded sets, endowed with a linear order. We consider the copy $(\mathcal{L}|\mathcal{P}) = \{(x|y);\ x \in \mathcal{L},\ y \in \mathcal{P}\}$ of the Cartesian product $\mathcal{L} \times \mathcal{P}$, where the $\mathbb{Z}_2-$grade of an ordered pair is defined as the sum of the grades of its components: $|(x|y)| = |xy| = |x| + |y|$. Notice that $(x|y)$ is positive or negative if x and y have equal or different degrees, respectively.

Our discussion will take place in the supersymmetric algebra Super$[\mathcal{L}|\mathcal{P}]$ generated by the $\mathbb{Z}_2$-graded set $(\mathcal{L}|\mathcal{P})$, that is, the quotient algebra

$$\mathrm{Super}[\mathcal{L}|\mathcal{P}] = \frac{\mathbb{K}[\mathrm{Mon}[\mathcal{L}|\mathcal{P}]]}{\mathrm{I}[\mathcal{L}|\mathcal{P}]}$$

of the semigroup algebra $\mathbb{K}[\mathrm{Mon}[\mathcal{L}|\mathcal{P}]]$ of the free monoid $\mathrm{Mon}[\mathcal{L}|\mathcal{P}]$ with respect to the two-sided ideal $\mathrm{I}[\mathcal{L}|\mathcal{P}]$ generated by the elements of the form

$$(x|y)(x'|y') - (-1)^{|xy||x'y'|}(x'|y')(x|y), \qquad (x|y), (x'|y') \in (\mathcal{L}|\mathcal{P}).$$

In particular, this algebra is linearly generated by the "monomials"

$$(x_1|y_1)(x_2|y_2)\cdots(x_n|y_n)$$

in the double symbols $(x_i|y_i)$ in $(\mathcal{L}|\mathcal{P})$, and a monomial is nonzero if and only if it is square-free in the negative double symbols. The algebra Super$[\mathcal{L}|\mathcal{P}]$ has a natural $\mathbb{Z}_2$-grading

$$\mathrm{Super}[\mathcal{L}|\mathcal{P}] = \mathrm{Super}[\mathcal{L}|\mathcal{P}]_0 \oplus \mathrm{Super}[\mathcal{L}|\mathcal{P}]_1,$$

with respect to which it is $\mathbb{Z}_2$-graded commutative, and has also a natural $\mathbb{Z}$-grading

$$\mathrm{Super}[\mathcal{L}|\mathcal{P}] = \oplus_{n\in\mathbb{N}}\mathrm{Super}_n[\mathcal{L}|\mathcal{P}],$$

consistent with the former grading.

For every x, y in $\mathcal{L}$, there is exactly one left superderivation,

$$\mathcal{D}_{xy} : \mathrm{Super}[\mathcal{L}|\mathcal{P}] \to \mathrm{Super}[\mathcal{L}|\mathcal{P}],$$

homogeneous of degree $|xy| = |x| + |y|$, called the *left superpolarization of* y *to* x, which takes the values $\mathcal{D}_{xy}(u|v) = \delta_{yu}\ (x|v)$ on the generators of Super$[\mathcal{L}|\mathcal{P}]$. In particular, the action of a left superpolarization on a monomial is given by

$$\mathcal{D}_{xy}\left((x_1|y_1)(x_2|y_2)\cdots(x_n|y_n)\right)$$
$$= \sum \epsilon_i\ \delta_{yx_i}\ (x_1|y_1)\cdots(x_{i-1}|y_{i-1})(x|y_i)(x_{i+1}|y_{i+1})\cdots(x_n|y_n),$$

where $\epsilon_i = (-1)^{|xy||x_1y_1\cdots x_{i-1}y_{i-1}|}$. Left superpolarizations satisfy the Lie superbracket relations

$$\mathcal{D}_{xy}\mathcal{D}_{zt} - (-1)^{|xy||zt|}\mathcal{D}_{zt}\mathcal{D}_{xy} = \delta_{yz}\ \mathcal{D}_{xt} - (-1)^{|xy||zt|}\delta_{tx}\ \mathcal{D}_{zy}.$$

For every x, y in $\mathcal{P}$, there is exactly one right superderivation

$$\mathrm{Super}[\mathcal{L}|\mathcal{P}] \leftarrow \mathrm{Super}[\mathcal{L}|\mathcal{P}] :\ {}_{xy}\mathcal{D},$$

homogeneous of degree $|xy| = |x|+|y|$, called the *right superpolarization of* x *to* y, which takes the values $(u|v)\ {}_{xy}\mathcal{D} = \delta_{vx}\ (u|y)$ on the generators of Super$[\mathcal{L}|\mathcal{P}]$. In particular, the action of a right superpolarization on a monomial is given by

$$((x_1|y_1)(x_2|y_2)\cdots(x_n|y_n))\ {}_{xy}\mathcal{D}$$
$$= \sum \eta_i\ \delta_{y_ix}\ (x_1|y_1)\cdots(x_{i-1}|y_{i-1})(x_i|y)(x_{i+1}|y_{i+1})\cdots(x_n|y_n),$$

where $\eta_i = (-1)^{|x_{i+1}y_{i+1}\cdots x_n y_n||xy|}$. Right superpolarizations satisfy the Lie superbracket relations

$$ {}_{xy}\mathcal{D} \ {}_{zt}\mathcal{D} - (-1)^{|xy||zt|} \ {}_{zt}\mathcal{D} \ {}_{xy}\mathcal{D} = \delta_{yz} \ {}_{xt}\mathcal{D} - (-1)^{|xy||zt|} \ \delta_{tx} \ {}_{zy}\mathcal{D}. $$

Left superpolarization an right superpolarizations commute:

$$ \left(\mathcal{D}_{xy} \ \underline{\xi}\right) \ {}_{zt}\mathcal{D} = \mathcal{D}_{xy} \ \left(\underline{\xi} \ {}_{zt}\mathcal{D}\right), \qquad \underline{\xi} \in \mathrm{Super}[\mathcal{L}|\mathcal{P}]. $$

Henceforth, n will be a fixed positive integer. The finite-dimensional space $\mathrm{Super}_n[\mathcal{L}|\mathcal{P}]$ is invariant with respect to any left polarization $\mathcal{D}_{xy}$ with $x, y \in \mathcal{L}$, as well as with respect to any right polarization ${}_{zt}\mathcal{D}$ with $z, t \in \mathcal{P}$; the restrictions of these polarizations to the space $\mathrm{Super}_n[\mathcal{L}|\mathcal{P}]$ generate two subalgebras $\mathrm{B}_n[\mathcal{L}|\mathcal{P}]$, $[\mathcal{L}|\mathcal{P}]_n\mathrm{B}$. of the algebra of linear endomorphisms of $\mathrm{Super}_n[\mathcal{L}|\mathcal{P}]$.

We are interested in the bimodule

$$ \mathrm{B}_n[\mathcal{L}|\mathcal{P}] \cdot \mathrm{Super}_n[\mathcal{L}|\mathcal{P}] \cdot [\mathcal{L}|\mathcal{P}]_n\mathrm{B}. $$

3. Basic Objects

The basic objects in the bimodule $\mathrm{B}_n[\mathcal{L}|\mathcal{P}] \cdot \mathrm{Super}_n[\mathcal{L}|\mathcal{P}] \cdot [\mathcal{L}|\mathcal{P}]_n\mathrm{B}$ are defined as the result of expressions containing certain $\mathbb{Z}_2$-graded "virtual variables". The use of $\mathbb{Z}_2$-graded virtual variables allows us to provide simple and transparent definitions of objects with desired symmetry properties; the values of the virtual variables are irrelevant. This leads to work in a wider bimodule and to define basic objects in this wider bimodule.

The formal setting is the following. The finite $\mathbb{Z}_2$-graded sets $\mathcal{L}$ and $\mathcal{P}$ are identified with two disjoint subsets of a $\mathbb{Z}_2$-graded set $\mathcal{X}$ consisting of countably many positive and countably many negative elements. We assume that the linear orders defined on the sets $\mathcal{L}$, $\mathcal{P}$ are extended to a linear order on the set $\mathcal{X}$, which makes $\mathcal{X}$ order-isomorphic to the set $\mathbb{N}$ of natural numbers. Then, we consider the supersymmetric algebra $\mathrm{Super}[\mathcal{X}|\mathcal{X}]$ on the $\mathbb{Z}_2$-graded set $\mathcal{X} \times \mathcal{X}$, left superpolarizations $\mathcal{D}_{xy}$ and right superpolarizations ${}_{xy}\mathcal{D}$, with $x, y \in \mathcal{X}$. These polarizations leave invariant the $\mathbb{N}-$homogeneous component $\mathrm{Super}_n[\mathcal{X}|\mathcal{X}]$, the restrictions of these polarizations to this space generate two subalgebras $\mathrm{B}_n[\mathcal{X}]$ and $[\mathcal{X}]_n\mathrm{B}$ of the algebra of linear endomorphisms of $\mathrm{Super}_n[\mathcal{X}|\mathcal{X}]$, and we consider the bimodule

$$ \mathrm{B}_n[\mathcal{X}] \cdot \mathrm{Super}_n[\mathcal{X}|\mathcal{X}] \cdot [\mathcal{X}]_n\mathrm{B}. $$

For any linear endomorphism φ of the algebra $\mathrm{Super}[\mathcal{X}|\mathcal{X}]$ that stabilizes the subspace $\mathrm{Super}_n[\mathcal{X}|\mathcal{X}]$, the linear endomorphism of the space $\mathrm{Super}_n[\mathcal{X}|\mathcal{X}]$ induced by φ will be denoted by $\varphi|_n$. In this bimodule the expressions which define the basic objects of the theory are given. The consistency of these definitions will follow from the general propositions in the next section.

For any $x, y \in \mathcal{X}$, we will use the following short notations

$$ \underset{\longrightarrow}{xy}, \quad \underline{xy}, \quad \underset{\longleftarrow}{xy}, \qquad \text{for} \qquad \mathcal{D}_{xy}, \quad (x|y), \quad {}_{xy}\mathcal{D}; $$

when no confusion may arise, we will write simply xy for $\underset{\longrightarrow}{xy}$ or $\underset{\longleftarrow}{xy}$.

Let $P = p_1 \cdots p_n$, $Q = q_1 \cdots q_n$, be a pair of words of length n on $\mathcal{X}$.

The *left Capelli row* $\mathrm{H}[P|Q]$, the *biproduct* $(P|Q)$, and the *right Capelli row* $[P|Q]\mathrm{H}$, are defined as follows:

$$\mathrm{H}[P|Q] = \underrightarrow{p_1 a} \ \cdots \underrightarrow{p_n a} \ \underrightarrow{a q_1} \cdots \underrightarrow{a q_n}\Big|_n \ ,$$
$$(P|Q) = p_1 a \ \cdots p_n a \ \left(\underline{a q_1} \cdots \underline{a q_n}\right) = \left(\underline{p_1 a} \cdots \underline{p_n a}\right) a q_1 \ \cdots a q_n,$$
$$[P|Q]\mathrm{H} = \underleftarrow{p_1 a} \ \cdots \underleftarrow{p_n a} \ \underleftarrow{a q_1} \cdots \underleftarrow{a q_n}\Big|_n \ ,$$

where $a \in \mathcal{X}$ is a positive symbol occurring neither in P nor in Q (see [6, Subsections 6.3, 7.2]). Notice that, under transposition of two adjacent entries r and s in P or in Q, each of these objects is transformed into itself, up to the sign $(-1)^{|r||s|}$. Biproducts are far-reaching generalizations of determinants and permanents; they were first defined, as objects satisfying all possible Laplace expansions, in ([26, Chapter 2]). In ([8]), the definition of biproducts in terms of virtual variables was given. Capelli rows are far-reaching generalizations of the classical Capelli operators (see [40, Chapter 2, Section 4]); they were defined, in virtual form, in [9]. The classical Capelli operators were defined in this way in [28].

The *left Capelli* $*-$*row* $\mathrm{H}^*[P|Q]$, the $*-$*biproduct* $(P|Q)^*$, and the *right Capelli* $*-$*row* $[P|Q]^*\mathrm{H}$, are defined in an analogous way, the difference being that now a is a negative symbol.

To any pair (P, Q) of words $P = p_1 \cdots p_m$, $Q = q_1 \cdots q_m$, of the same length on $\mathcal{X}$ we associate a left polarization monomial, a monomial, and a right polarization monomial

$$\underrightarrow{PQ} = \prod_{i=1}^{m} \underrightarrow{p_i q_i}, \qquad \underline{PQ} = \prod_{i=1}^{m} \underline{p_i q_i}, \qquad \underleftarrow{PQ} = \prod_{i=1}^{m} \underleftarrow{p_i q_i};$$

when no confusion may arise, we will write simply simply PQ for $\underrightarrow{PQ}$ or $\underleftarrow{PQ}$. To any pair of tableaux of the same shape on $\mathcal{X}$, we associate their row-words, and then the corresponding monomials. Specifically, to any pair (P, Q) of tableaux

$$P = \begin{array}{l} p_{11} \cdots\cdots\cdots p_{1\lambda_1} \\ p_{21} \cdots\cdots p_{2\lambda_2} \\ \vdots \\ p_{r1} \cdots p_{r\lambda_r} \end{array}, \qquad Q = \begin{array}{l} q_{11} \cdots\cdots\cdots q_{1\lambda_1} \\ q_{21} \cdots\cdots q_{2\lambda_2} \\ \vdots \\ q_{r1} \cdots q_{r\lambda_r} \end{array}$$

of the same shape λ on $\mathcal{X}$ we associate a left polarization monomial, a monomial and a right polarization monomial

$$\underrightarrow{PQ} = \prod_{f=1}^{r} \prod_{g=1}^{\lambda_f} \underrightarrow{p_{fg} q_{fg}}, \qquad \underline{PQ} = \prod_{f=1}^{r} \prod_{g=1}^{\lambda_f} \underline{p_{fg} q_{fg}}, \qquad \underleftarrow{PQ} = \prod_{f=1}^{r} \prod_{g=1}^{\lambda_f} \underleftarrow{p_{fg} q_{fg}};$$

when no confusion may arise, we will write simply PQ for $\underrightarrow{PQ}$ or $\underleftarrow{PQ}$.

In the following, a crucial role will be played by two types of tableaux, which are are used to induce (super)symmetry along rows and (super)skew-symmetry along columns. These tableaux are:

- tableaux filled with positive symbols, such that any two symbols in the same row are equal and any two symbols in the same column are distinct:

$$C = \begin{array}{l} a_1 a_1 \cdots\cdots\cdots a_1 \\ a_2 a_2 \cdots\cdots a_2 \\ \vdots \\ a_r a_r \cdots a_r \end{array} \qquad |a_i| = 0, \qquad a_i \neq a_j \ for \ i \neq j.$$

  These tableaux are said to be of *coDeruyts type*, or coDeruyts for short. *The symbol C and its variations like $C', \dot{C}, C'', \ldots$ will always denote tableaux of coDeruyts type.*
- tableaux filled with negative symbols, such that any two symbols in the same column are equal and any two symbols in the same row are distinct:

$$D = \begin{array}{l} b_1 b_2 \cdots\cdots\cdots b_{\lambda_1} \\ b_1 b_2 \cdots\cdots b_{\lambda_2} \\ \vdots \\ b_1 b_2 \cdots b_{\lambda_r} \end{array} \qquad |b_i| = 1, \qquad b_i \neq b_j \ for \ i \neq j.$$

  These tableaux are said to be of *Deruyts type*, or Deruyts for short. *The symbol D and its variations like $D', \dot{D}, D'', \ldots$ will always denote tableaux of Deruyts type.*

Let (P, Q) be a pair of tableaux of the same shape $\lambda \vdash n$ on $\mathcal{X}$.

The *left Capelli bitableau* $\mathrm{H}[P|Q]$, the *bitableau* $(P|Q)$, and the *right Capelli bitableau* $[P|Q]\mathrm{H}$ are defined as follows:

$$\begin{aligned} \mathrm{H}[P|Q] &= \underset{\longrightarrow}{PC}\ \underset{\longrightarrow}{CQ}\Big|_n, \\ (P|Q) &= PC\ \underline{CQ} = \underline{PC}\ CQ, \\ [P|Q]\mathrm{H} &= \underset{\longleftarrow}{PC}\ \underset{\longleftarrow}{CQ}\Big|_n, \end{aligned}$$

where C is any coDeruyts tableau of shape λ whose support is disjoint from those of P and Q (see [6, Subsections 6.3, 7.4]). Under transposition of two adjacent entries r and s in a row of P or in a row of Q, each of these objects is transformed into itself, up to the sign $(-1)^{|r||s|}$. If P and Q are columns, then these objects are, up to a sign, the left polarization monomial, the monomial, and the right polarization monomial associated to the pair of words (P, Q); if P and Q are rows, then these objects are the left Capelli row, the biproduct, and the right Capelli row associated to the pair of words (P, Q). Bitableaux were first defined, as products of biproducts , in [26]. In [8], the definition of bitableaux in terms of virtual variables was given. Capelli bitableaux were defined, in virtual form, in [9].

The *left Young–Capelli symmetrizer* $\gamma(P, \boxed{Q})$, the *symmetrized bitableau* $(P \| \boxed{Q})$, and the *right Young–Capelli symmetrizer* $(P, \boxed{Q})\gamma$, are defined as follows:

$$\begin{aligned}
\gamma(P, \boxed{Q}) &= \underset{\longrightarrow}{\underline{PC}}\ \underset{\longrightarrow}{\underline{CD}}\ \underset{\longrightarrow}{\underline{DQ}}\ \Big|_n, \\
(P \| \boxed{Q}) &= PC\ CD\ \underline{DQ} = PC\ \underline{CD}\ DQ = \underline{PC}\ CD\ DQ, \\
(P, \boxed{Q})\gamma &= \underset{\longleftarrow}{\underline{PC}}\ \underset{\longleftarrow}{\underline{CD}}\ \underset{\longleftarrow}{\underline{DQ}}\ \Big|_n,
\end{aligned}$$

where C and D are a coDeruyts and a Deruyts tableau of shape λ, C has support disjoint from that of P, D has support disjoint from that of Q (see [6, Subsections 9.1, 10.1]). Notice that, under transposition of two entries in a row of P or in a column of Q, each of these objects is transformed into itself, up to a sign. Symmetrized bitableaux were first defined, in virtual form, in [8]. They are far-reaching generalizations of the members of the classical Gordan–Capelli series (see [39], and [18, 19] for ramifications in other directions). Young–Capelli symmetrizers were defined, in virtual form, in [9]. They are far-reaching generalizations of the classical Young symmetrizers.

The *left Capelli* $*-$*bitableau* $\mathrm{H}^*[P|Q]$, the $*-$*bitableau* $(P|Q)^*$, and the *right Capelli* $*-$*bitableau* $[P|Q]^*\mathrm{H}$ are defined by

$$\begin{aligned}
\mathrm{H}^*[P|Q] &= \underset{\longrightarrow}{\underline{PD}}\ \underset{\longrightarrow}{\underline{DQ}}\ \Big|_n, \\
(P|Q)^* &= PD\ \underline{DQ} = \underline{PD}\ DQ, \\
[P|Q]^*\mathrm{H} &= \underset{\longleftarrow}{\underline{PD}}\ \underset{\longleftarrow}{\underline{DQ}}\ \Big|_n,
\end{aligned}$$

where D is any Deruyts tableau of shape λ whose support is disjoint from those of P and Q [8, 9].

4. Virtual Expressions

The left polarization monomials, the monomials and the right polarization monomials entering into the definitions of the basic objects in the bimodule $\mathrm{B}_n[\mathcal{X}]\cdot\mathrm{Super}_n[\mathcal{X}|\mathcal{X}]\cdot[\mathcal{X}]_n\mathrm{B}$ are always associated to pairs of words of length n over $\mathcal{X}$ with disjoint supports. This leads to consider the class of all the expressions obtained by combining these monomials and polarization monomials. For the sake of brevity, we refer to these expressions as *virtual expressions* of step n.

Any virtual expression of step n

$$PQ\ \underline{RS}$$

yields an element of $\mathrm{Super}_n[\mathcal{X}|\mathcal{X}]$, which can be written in the form

$$PQ\ \underline{RS} = \sum_{P'} \xi_{P'} \underline{P'S},$$

where P' ranges among the words equivalent to P and the coefficients $\xi_{P'}$ are integers. This element depends on the central pair (Q, R) only through its isomorphism type, and is zero unless Q and R have the same content. Analogous remarks can be made for any virtual expression $\underline{PQ}\ RS$.

Proposition 1. *Virtual expressions of step n satisfy the identity*

$$PQ\ \underline{RS} = \underline{PQ}\ RS.$$

This proposition is new. We give a proof in Subsection 9.2.

More generally, the u virtual expressions of step n

$$P_1Q_1\ \cdots\ P_{i-1}Q_{i-1}\ \underline{P_iQ_i}\ P_{i+1}Q_{i+1}\ \cdots\ P_uQ_u, \qquad i = 1, \ldots, u$$

yield the same element in $\mathrm{Super}_n[\mathcal{X}|\mathcal{X}]$, which can be written as

$$\sum_{P', Q'} \zeta_{P'Q'} \underline{P'Q'},$$

where P' ranges among the words equivalent to P_1, Q' ranges among the words equivalent to Q_u, and the coefficients $\zeta_{P'Q'}$ are integers. This element depends on the central pairs $(Q_1, P_2), \ldots, (Q_{u-1}, P_u)$ only trough their isomorphism type, and is zero unless in each of these pairs the words have the same content. This property ensures that all the basic objects for $\mathrm{Super}_n[\mathcal{X}|\mathcal{X}]$ are well-defined, since the row-words of any two coDeruyts tableaux of the same shape are isomorphic, as well as the row-words of any two Deruyts tableaux of the same shape are isomorphic.

Starting from sequences of pairs of words of length n on $\mathcal{X}$ with disjoint supports, one can obtain expressions with values in $\mathrm{Super}_n[\mathcal{X}|\mathcal{X}]$, as well as expressions with values in $\mathrm{B}_n[\mathcal{X}]$ or in $[\mathcal{X}]_n\mathrm{B}$. It turns out that the three types of expressions associated to the same set of sequences satisfy the same linear relations.

Proposition 2. *Let $(P_1^i, Q_1^i), (P_2^i, Q_2^i), \ldots, (P_{m_i}^i, Q_{m_i}^i)$, for $i = 1, \ldots, u$ be a series of sequences of pairs of words of length n on $\mathcal{X}$, such that any two words in the same pair have disjoint supports, let ϑ_i be scalars, and let j_i be indexes with $1 \leq j_i \leq m_i$. Then the following equalities are equivalent*

$$\sum_i \vartheta_i\ \underrightarrow{P_1^iQ_1^i}\ \underrightarrow{P_2^iQ_2^i}\ \cdots\ \underrightarrow{P_{m_i}^iQ_{m_i}^i}\ \Big|_n = \underrightarrow{0},$$

$$\sum_i \vartheta_i\ P_1^iQ_1^i\ \cdots\ P_{j_i-1}^iQ_{j_i-1}^i\ \underline{P_{j_i}^iQ_{j_i}^i}\ P_{j_i+1}^iQ_{j_i+1}^i\ \cdots\ P_{m_i}^iQ_{m_i}^i\ = \underline{0},$$

$$\sum_i \vartheta_i\ \underleftarrow{P_1^iQ_1^i}\ \underleftarrow{P_2^iQ_2^i}\ \cdots\ \underleftarrow{P_{m_i}^iQ_{m_i}^i}\ \Big|_n = \underleftarrow{0}.$$

This proposition is new. We give a proof in Subsection 9.3.

By interchanging left and right, one gets from each of the previous equalities an equivalent equality; for example, from the second equality one gets the equivalent equality

$$\sum_i \vartheta_i\ \overline{Q}_{m_i}^i\overline{P}_{m_i}^i\ \cdots\ \overline{Q}_{j_i+1}^i\overline{P}_{j_i+1}^i\ \underline{\overline{Q}_{j_i}^i\overline{P}_{j_i}^i}\ \overline{Q}_{j_i-1}^i\overline{P}_{j_i-1}^i\ \cdots\ \overline{Q}_1^i\overline{P}_1^i\ =\ \underline{0},$$

where, for each word $R = r_1r_2\cdots r_n$, we have set $\overline{R} = r_n\cdots r_2r_1$.

From Proposition 2 it follows in particular that the value in $\mathrm{B}_n[\mathcal{X}]$ of any virtual expression of step n

$$\underset{\longrightarrow}{P_1Q_1}\ \underset{\longrightarrow}{P_2Q_2}\ \cdots\ \underset{\longrightarrow}{P_uQ_u}\ \Big|_n$$

depends on the pairs $(Q_1, P_2), \ldots, (Q_{u-1}, P_u)$ only trough their isomorphism type, and is zero unless in each of these pairs the words have the same content. This ensures that all the basic objects for $\mathrm{B}_n[\mathcal{X}]$ are well-defined. Analogous remarks can be made for expressions with values in $[\mathcal{X}]_n\mathrm{B}$.

The expansions of virtual expressions with values in $\mathrm{B}_n[\mathcal{X}]$ can be given in the following way.

Proposition 3. *Any virtual expression* $\underset{\longrightarrow}{P_1Q_1}\ \underset{\longrightarrow}{P_2Q_2}\ \cdots\ \underset{\longrightarrow}{P_uQ_u}\ \Big|_n$ *of step n can be written as*

$$\underset{\longrightarrow}{P_1Q_1}\ \underset{\longrightarrow}{P_2Q_2}\ \cdots\ \underset{\longrightarrow}{P_uQ_u}\ \Big|_n = \sum_{P',Q'} \eta_{P',Q'}\underset{\longrightarrow}{P'Q'}\Bigg|_n ,$$

where P', Q' range among the pairs of words of the same length equivalent to subwords of P_1, Q_u, and the coefficients $\eta_{P',Q'}$ are integers.

This proposition is new (compare with [6, Theorems 6.1, 6.2, 6.3]). An analogous result holds for virtual expressions with values in $[\mathcal{X}]_n\mathrm{B}$. We give a proof in Subsection 9.4.

5. Triangularity, Nondegeneracy, and Straightening

5.1. **Triangularity and Nondegeneracy Lemmas.** Our starting point is the following instance of the "easy part" of the Gale–Ryser Theorem.

Scholion 1. *Let (A, B) be a pair of words of the same length on $\mathcal{X}$. If A has the same content of a coDeruyts tableau C, and B has the same content of a Deruyts tableau D, then*

$$\underline{AB} = \begin{cases} \underline{0} & \text{if } \mathrm{sh}(C) \not\leq \mathrm{sh}(D) \\ \eta\ \underline{CD} & \text{if } \mathrm{sh}(C) = \mathrm{sh}(D) \end{cases},$$

with $\eta \in \{0, \pm 1\}$.

For the sake of completeness, we give a proof in Subsection 10.1.

Lemma 1 (Triangularity Lemma I; see [6, Lemma 8.1])**.** *Let (A_1, B_1), (A_2, B_2), $\ldots$, (A_s, B_s) be pairs of words of the same length on $\mathcal{X}$, where any two words in the same pair have disjoint supports, and let $1 \leq i \leq s$. If A_1 has the same content of a coDeruyts tableau C and B_s has the same content of a Deruyts tableau D, then*

$$A_1B_1\ \cdots\ A_{i-1}B_{i-1}\ \underline{A_iB_i}\ A_{i+1}B_{i+1}\ \cdots\ A_sB_s = \begin{cases} \underline{0} & \text{if } \mathrm{sh}(C) \not\leq \mathrm{sh}(D) \\ \vartheta\ \underline{CD} & \text{if } \mathrm{sh}(C) = \mathrm{sh}(D) \end{cases},$$

where ϑ is an integer.

Lemma 2 (Triangularity Lemma II; see [6, Lemma 8.2])**.** *Let (C,S), (T,D) be pairs of tableaux of the same size, where tableaux in the same pair have the same shape and disjoint supports, C is coDeruyts, S and T are standard, and D is Deruyts. Then*

$$CS\ \underline{TD}\ = \begin{cases} \underline{0} & \text{if } S \not\leq T \\ \vartheta\ \underline{CD} & \text{if } S = T \end{cases},$$

where $\vartheta = \pm c_S^r! c_S^c!$ is the product of the factorial of the row-content and the factorial of the column-content of P.

These lemmas are refinements of Propositions 1 and 2 in [9]. They can be regarded as virtual forms of the propositions on Capelli operators in the sense of [22, Theorem 2.1], (see also [21, 20, 27]). In [6], elementary combinatorial proofs of these lemmas are announced. We fulfill this announcement by providing proofs in Subsections 10.2, 10.4.

Let P, Q be two tableaux of the same shape $\lambda \vdash n$. From Lemma 1 it follows that:

- there is exactly one integer ϑ_{PQ}^{+-} such that
$$CP\ \underline{QD} = \vartheta_{PQ}^{+-}\ \underline{CD},$$
for every coDeruyts tableau C of shape λ, with support disjoint from that of P, and for every Deruyts tableau D of shape λ, with support disjoint from that of Q. This integer is called the $(+-)$ *symmetry coefficient* of the two tableaux P and Q;
- for any $\varphi \in \mathrm{B}_n[\mathcal{X}]$, there is exactly one integer $\vartheta_{PQ}^{+-}(\varphi)$ such that
$$CP\ \left(\varphi\ \underline{QD}\right)\ =\ \vartheta_{PQ}^{+-}(\varphi)\ \underline{CD},$$
for every coDeruyts tableau C of shape λ, with support disjoint from that of P, and for every Deruyts tableau D of shape λ, with support disjoint from that of Q.

Example 1. *Given two multilinear tableaux P and Q of the same shape, filled with positive symbols of $\mathcal{X}$, we have: if there are two symbols occurring in the same row of P and in the same column of Q, then $\vartheta_{PQ}^{+-} = 0$; otherwise, there is exactly one multilinear tableau W, of the same shape of P and Q, such that $P \equiv_r W \equiv_c Q$, and $\vartheta_{PQ}^{+-} = \epsilon_{Q,W}$, where $\epsilon_{Q,W}$ is the sign of the permutation that transforms Q into W.*

Given a partition $\lambda \vdash n$, we have the infinite matrices

$$\Theta_{\mathrm{t}_\lambda}^{+-} = \left[\vartheta_{PQ}^{+-}\right]_{P,Q\in\mathrm{t}_\lambda},$$

whose entries are the symmetry coefficients of pairs of tableaux of shape λ over $\mathcal{X}$. Since ϑ_{PQ}^{+-} is zero unless P and Q have the same content, we have that this matrix is a block diagonal matrix, with diagonal blocks parametrized by all the possible multisets of order n over $\mathcal{X}$; furthermore, if two multisets have the same statistic of multiplicities, then the corresponding blocks are equal, up to permutation of rows and columns.

As a consequence of Lemma 2, for every partition $\lambda \vdash n$ the $(+-)$symmetry coefficients of the standard tableaux satisfy the nondegenerate triangularity relations

$$\vartheta^{+-}_{PQ} \begin{cases} = 0 & \text{if } P \not\leq Q \\ \neq 0 & \text{if } P = Q \end{cases} .$$

Therefore, the matrix $\Theta^{+-}_{\mathrm{st}_\lambda} = \left[\vartheta^{+-}_{PQ}\right]_{P,Q,\in \mathrm{st}_\lambda}$ is nonsingular. The inverse matrix is still a block diagonal matrix with the same structure. Its entries are called the *Rutherford coefficients* and are denoted by ϱ^{+-}_{ST}. In the case of Example 1, the Rutherford coefficients are the classical ones [33].

Lemma 3 (Nondegeneracy Lemma; see [6, Lemma 8.3])**.** *Let $C, \dot{D}, \dot{C}, D$ be coDeruyts and Deruyts tableaux of the same shape $\lambda \vdash n$. Then*

$$C\dot{D}\ \underline{\dot{D}\dot{C}}\ \dot{C}D = k_\lambda\ \underline{CD},$$

where $k_\lambda = (-1)^{\binom{n}{2}}\ h_\lambda$, and h_λ is the product of the hook lengths of λ.

In [6], this lemma is derived, in a rather indirect way, by the theorem of essential idempotency of the classical Young Symmetrizers (see, e.g., [4, Chapter IV, Theorem 3.1]). In Subsections 11.3, 11.4, we give a proof based on the Superstraightening Law. We remark that the significance of this lemma for the general theory relies in the statement that $k_\lambda \neq 0$. We give a proof of this weaker statement, based on a direct combinatorial argument, in Subsection 11.2.

We remark that all the statements in this subsection yield relations between virtual expressions. Thus each of these statements yields other equivalent statements, in the sense of Proposition 2.

Other equivalent statements are obtained by interchanging left and right. In particular, given two tableaux P, Q of the same shape $\lambda \vdash n$, there is exactly one integer ϑ^{-+}_{PQ} such that

$$DP\ \underline{QC} = \vartheta^{-+}_{PQ}\ \underline{DC},$$

for every Deruyts tableau D of shape λ, with support disjoint from that of P, and for every coDeruyts tableau C of shape λ, with support disjoint from that of Q. This integer is called the $(-+)$ *symmetry coefficient* of the two tableaux P and Q. The two types of symmetry coefficients satisfy the relation

$$\vartheta^{-+}_{PQ} = \pm\vartheta^{+-}_{QP.}$$

The integers $\vartheta^{-+}_{PQ}(\varphi)$ are defined in an analogous way.

Given a partition $\lambda \vdash n$, we have the infinite matrix

$$\Theta^{-+}_{\mathrm{t}_\lambda} = \left[\vartheta^{-+}_{PQ}\right]_{P,Q\in \mathrm{t}_\lambda} .$$

For every partition $\lambda \vdash n$ the $(-+)$symmetry coefficients of the standard tableaux satisfy the nondegenerate triangularity relations

$$\vartheta^{-+}_{PQ} \begin{cases} = 0 & \text{if } P \not\geq Q \\ \neq 0 & \text{if } P = Q \end{cases} .$$

Therefore, the matrix $\Theta^{-+}_{\mathrm{st}_\lambda}$ is nonsingular. The entries of the inverse matrix are denoted by ϱ^{-+}_{ST}.

5.2. **The Superstraightening Law.** The biproduct has been defined for any pair of words of the same length; for words of different length it is set to be equal to $\underline{0}$. The mapping $(\ |\ ) : \mathrm{Mon}[\mathcal{X}] \times \mathrm{Mon}[\mathcal{X}] \to \mathrm{Super}[\mathcal{X}|\mathcal{X}]$ which assigns to each pair of words their biproduct naturally extends to a bilinear mapping $(\ |\ ) : \mathrm{Super}[\mathcal{X}] \times \mathrm{Super}[\mathcal{X}] \to \mathrm{Super}[\mathcal{X}|\mathcal{X}]$. This mapping is equivariant with respect to the action of left and right superpolarizations:

$$xy\ (A|B) = (xy\ A|B), \qquad (A|B)\ xy = (A|B\ xy).$$

Analogously, the bitableau has been defined for any pair of tableaux of the same shape; for tableaux of different shape it is set to be equal to $\underline{0}$. The mapping $(\ |\ )$: $(\mathrm{Mon}[\mathcal{X}] \times \mathrm{Mon}[\mathcal{X}] \times \cdots) \times (\mathrm{Mon}[\mathcal{X}] \times \mathrm{Mon}[\mathcal{X}] \times \cdots) \to \mathrm{Super}[\mathcal{X}|\mathcal{X}]$ which assigns to each pair of tableaux their bitableau naturally extends to a bilinear mapping $(\ |\ ) : (\mathrm{Super}[\mathcal{X}] \otimes \mathrm{Super}[\mathcal{X}] \otimes \cdots) \times (\mathrm{Super}[\mathcal{X}] \otimes \mathrm{Super}[\mathcal{X}] \otimes \cdots) \to \mathrm{Super}[\mathcal{X}|\mathcal{X}]$. This mapping is equivariant with respect to the action of left (and right) superpolarizations:

$$xy \left(\begin{array}{c|c} u_1 & v_1 \\ \vdots & \vdots \\ u_i & v_i \\ \vdots & \vdots \\ u_p & v_p \end{array} \right) = \sum_{i=1,\dots,p} \xi_i \left(\begin{array}{r|c} u_1 & v_1 \\ \vdots & \vdots \\ xy\ u_i & v_i \\ \vdots & \vdots \\ u_p & v_p \end{array} \right),$$

where $\xi_i = (-1)^{|xy||u_1 \cdots u_{i-1}|}$.

The superalgebra $\mathrm{Super}[\mathcal{X}]$ has a natural structure of bialgebra, in which the coproduct $\Delta(u) = \sum_{(u)} u_{(1)} \otimes u_{(2)}$ of any $u \in \mathrm{Super}[\mathcal{X}]$ is the algebra morphism defined by setting

$$\begin{aligned} &\Delta(1) = 1 \otimes 1 \\ &\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad \text{for all } x \in \mathcal{X}. \end{aligned}$$

Given a positive symbol a in $\mathcal{X}$, and a natural number $n = 0, 1, 2, \dots$, we set $a^{(n)} = \frac{a^n}{n!}$. Notice that $a^{(n)}a^{(m)} = \binom{n+m}{n} a^{(n+m)}$ and $\Delta(a^{(n)}) = \sum_{i=0}^{n} a^{(i)} \otimes a^{(n-i)}$.

Theorem 1 (Superstraightening Law; see[26, Chapter 3, Proposition 10])**.** *Given five words u, v, w, x, y on $\mathcal{X}$, we have*

$$\sum_{(v)} \left(\begin{array}{c|c} uv_{(1)} & x \\ v_{(2)}w & y \end{array} \right) = (-1)^{|u||v|} \sum_{(u),(y)} (-1)^{l(u_{(2)})} \left(\begin{array}{c|c} vu_{(1)} & xy_{(1)} \\ u_{(2)}w & y_{(2)} \end{array} \right).$$

For a "virtual proof", we refer to [6, Subsection 8.1].

Remark 1. *If x, y are words in negative symbols, and y is a subword of x, then we have*

$$\sum_{(v)} \left(\begin{array}{c|c} uv_{(1)} & x \\ v_{(2)}w & y \end{array} \right) = (-1)^{|u||v| + l(u_{(2)})} \sum_{(u)} \left(\begin{array}{c|c} vu_{(1)} & x \\ u_{(2)}w & y \end{array} \right).$$

6. Properties of the Basic Objects

In this section, we state some of the main properties of the basic objects for the bimodule $\mathrm{B}_n[\mathcal{X}]\cdot\mathrm{Super}_n[\mathcal{X}|\mathcal{X}]\cdot[\mathcal{X}]_n\mathrm{B}$; all these properties follow from the Triangularity Lemma, the Nondegeneracy Lemma and the Straightening Law, and we give a sketch of their proofs. We give references to the original papers, as well as to the closest item in [6].

6.1. **Spanning, Linear Independence.** For standard bitableaux, we have the following Spanning Theorem.

Theorem 2 ([26]; [6, Section 8.3]). *Given a pair (P, Q) of tableaux of the same shape $\lambda \vdash n$ on $\mathcal{X}$, the bitableau $(P|Q)$ can be written as a linear combination, with rational coefficients,*

$$(P|Q) = \sum_{S,T} c_{ST}(S|T), \qquad c_{ST} \in \mathbb{Q}$$

of standard bitableaux $(S|T)$, where (S, T) ranges through the pairs of standard tableaux, the first of the same content of P and the second of the same content of Q, with a common shape greater or equal to λ, in the dominance order.

Proof. It follows from a suitable iteration of the Straightening Law. □

Now, each monomial in $\mathrm{Super}_n[\mathcal{X}|\mathcal{X}]$ can be regarded as a bitableau parametrized by a pair of column-tableaux, and thus it can be written as a linear combination, with rational coefficients, of standard bitableaux with shape partition of n. This means that the set of the standard bitableaux of size n on $\mathcal{X}$ spans the space $\mathrm{Super}_n[\mathcal{X}|\mathcal{X}]$. It can be proved, by using the Triangularity Lemma, that standard bitableaux are also linearly independent, and thus form a basis of this space.

For standard symmetrized bitableaux, we have the following Linear Independence Theorem.

Theorem 3 ([6, Subsection 9.4]). *The set of all the standard symmetrized bitableaux of size n on $\mathcal{X}$ is linearly independent.*

Proof. The basic step consists in considering the operators

$$\underline{\xi} \mapsto DP\ \underline{\xi}\ QC, \qquad \underline{\xi} \in \mathrm{Super}_n[\mathcal{X}|\mathcal{X}],$$

for standard tableaux P, Q, Deruyts tableau D with support disjoint from that of P, and coDeruyts tableau C with support disjoint from that of Q, all of the same shape partition of n, and proving that the action of these operators on standard symmetrized bitableaux $(S|\boxed{T})$ of size n satisfies the nondegenerate triangularity relations

$$DP\ (S|\boxed{T})\ QC \ = \begin{cases} \underline{0} & \textit{unless } P \geq S,\ T \geq Q \\ k_\lambda \vartheta^{-+}_{PP}\ \underline{DC}\ \vartheta^{-+}_{QQ}\ \neq \underline{0} & \text{if } P = S,\ T = Q \end{cases},$$

where λ in the second line is the common shape of P, S, T, Q.

This claim can be proved by writing the standard symmetrized bitableau $(S\|\boxed{T})$ in a virtual form, thus obtaining the expression

$$DP\ (S\|\boxed{T})\ QC\ = DP\ S\dot{C}\ \underline{\dot{C}\dot{D}}\ \dot{D}T\ QC,$$

and by applying Lemma 1 (twice), Lemma 2 and Lemma 3. □

6.2. **Triangularity Theorem and Orthonormal Generators.**

Theorem 4 (Triangularity Theorem; see [9]; [6, Subsection 10.2])**.** *The action of Young–Capelli symmetrizers on symmetrized bitableaux satisfies the relations*

$$\begin{aligned}\gamma(P,\boxed{Q})\ (R\|\boxed{S}) &= (P,\boxed{Q})\ (R\|\boxed{S})\gamma \\ &= \begin{cases} k_\lambda\ \vartheta^{-+}_{QR}\ (P\|\boxed{S}) & \textit{if } \mathrm{sh}(Q) = \mathrm{sh}(R)(=\lambda) \\ \underline{0} & \textit{if } \mathrm{sh}(Q) \neq \mathrm{sh}(R) \end{cases},\end{aligned}$$

where the coefficient k_λ is nonzero. For Q, R standard tableaux, the symmetry coefficients ϑ^{-+}_{QR} satisfy the nondegenerate triangularity relations

$$\vartheta^{-+}_{QR} \begin{cases} = 0 & \textit{if } Q \not\geq R \\ \neq 0 & \textit{if } Q = R \end{cases}.$$

Proof. By writing the Young–Capelli symmetrizers and the symmetrized bitableaux in virtual form, the first equality becomes

$$PC\ CD\ DQ\ \underline{R\dot{C}}\ \dot{C}\dot{D}\ \dot{D}S\ =\ PC\ \underline{CD}\ DQ\ R\dot{C}\ \dot{C}\dot{D}\ \dot{D}S,$$

and it is readily seen as a consequence of Proposition 1. The first part of the statement follows by applying in turn Lemma 1 (twice), and Lemma 3. The second part of the statement follows by Lemma 2. □

This theorem leads to the construction of the Orthonormal Generators (see [9]; [6, Subsection 10.3]). Given two standard tableaux P, Q of the same shape λ on $\mathcal{X}$, set

$$Y(P,\boxed{Q}) = \frac{1}{k_\lambda} \sum_{A \in \mathfrak{st}_\lambda[\mathcal{X}]} \varrho^{-+}_{QA}\ \gamma(P,\boxed{A}),$$

$$(R,\boxed{S})Y = \frac{1}{k_\lambda} \sum_{B \in \mathfrak{st}_\lambda[\mathcal{X}]} \varrho^{-+}_{BR}\ (B,\boxed{S})\gamma,$$

where the coefficients ϱ^{-+} are the Rutherford coefficients. Notice that in the first sum the nonzero terms are parametrized by standard tableaux A of shape λ which have the same content of Q. Thus in this sum only a finite number of nonzero terms occurs. An analogous remark holds for the other sum.

The action of the Orthonormal Generators on the standard Symmetrized Bitableaux satisfies the orthogonality relations

$$Y(P,\boxed{Q})\ (R\|\boxed{S})\ =\ (P,\boxed{Q})\ (R\|\boxed{S})Y\ =\ \delta_{QR}\ (P\|\boxed{S}).$$

7. Explicit Complete Decompositions

In this section, we revert to consider the bimodule

$$\mathrm{B}_n[\mathcal{L}|\mathcal{P}] \cdot \mathrm{Super}_n[\mathcal{L}|\mathcal{P}] \cdot [\mathcal{L}|\mathcal{P}]_n\mathrm{B}.$$

We will give some of the main results on this bimodule, and provide a sketch of their proofs. From the remarks made in Section 4 on the expansions of virtual expressions, especially from Proposition 3, we get the following facts.

For any pair (S,T) of tableaux of the same shape partition of n on $\mathcal{L}$ and $\mathcal{P}$, the symmetrized bitableau $(S|\boxed{T})$ associated to (S,T) in the space $\mathrm{Super}_n[\mathcal{X}|\mathcal{X}]$ is a linear combination of monomials associated to pairs of words S', T' equivalent to S and T, and, therefore, it belongs to the space $\mathrm{Super}_n[\mathcal{L}|\mathcal{P}]$.

For any pair (R,S) of tableaux of the same shape partition of n on $\mathcal{L}$, the left Young–Capelli symmetrizer $\gamma(R, \boxed{S})$ associated to (R,S) in the algebra $\mathrm{B}_n[\mathcal{X}]$ is a linear combination of left polarization monomials associated to pairs of words R', S' equivalent to subwords of R and S, and, therefore, it gives rise to an element in the algebra $\mathrm{B}_n[\mathcal{L}|\mathcal{P}]$, that we denote by

$$\gamma_/(R, \boxed{S}).$$

Analogously, for any pair (T,U) of tableaux of the same shape partition of n on $\mathcal{P}$, the right Young–Capelli symmetrizer $(T, \boxed{U})\gamma$ associated to (T,U) in the algebra $[\mathcal{X}]_n\mathrm{B}$ gives rise to an element in the algebra $[\mathcal{L}|\mathcal{P}]_n\mathrm{B}$, that we denote by $(T, \boxed{U})\gamma_/$. Analogous remarks and notations are extended to each of the basic objects, and to the Orthonormal Generators.

The space $\mathrm{Super}_n[\mathcal{L}|\mathcal{P}]$ contains the generating set of standard bitableaux

$$\{(S|T);\ (S,T) \in \mathrm{st}(\mathcal{L}) \times \mathrm{st}(\mathcal{P}):\ sh(S) = sh(T) \vdash n\},$$

as well as the linearly independent set of standard symmetrized bitableaux

$$\left\{(S|\boxed{T});\ (S,T) \in \mathrm{st}(\mathcal{L}) \times \mathrm{st}(\mathcal{P}):\ sh(S) = sh(T) \vdash n\right\}.$$

Since the two sets have the same cardinality, they are both bases of this space: the former is called the *standard basis*, and the latter the *Clebsch–Gordan–Capelli basis* of $\mathrm{Super}_n[\mathcal{L}|\mathcal{P}]$.

Theorem 5 ([8, 9]; [6, Section 12])**.** *The space* $\mathrm{Super}_n[\mathcal{L}|\mathcal{P}]$ *has the complete decomposition*

$$\mathrm{Super}_n[\mathcal{L}|\mathcal{P}] = \bigoplus_{\substack{\lambda \vdash n \\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P})}} \ \bigoplus_{T \in \mathfrak{st}_\lambda(\mathcal{P})} \langle (S|\boxed{T});\ S \in \mathfrak{st}_\lambda(\mathcal{L}) \rangle$$

as a semisimple left module over the algebra $\mathrm{B}_n[\mathcal{L}|\mathcal{P}]$*; the summands of outer direct sum are the isotypic components of the module. The algebra* $\mathrm{B}_n[\mathcal{L}|\mathcal{P}]$ *has the complete decomposition*

$$\mathrm{B}_n[\mathcal{L}|\mathcal{P}] = \bigoplus_{\substack{\lambda \vdash n \\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P})}} \ \bigoplus_{T \in \mathfrak{st}_\lambda(\mathcal{L})} \langle Y_/(S, \boxed{T});\ S \in \mathfrak{st}_\lambda(\mathcal{L}) \rangle$$

as a semisimple algebra; the summands of outer direct sum are the simple subalgebras of the algebra. Analogous results hold for the space $\mathrm{Super}_n[\mathcal{L}|\mathcal{P}]$ *as a right module over the algebra* $[\mathcal{L}|\mathcal{P}]_n\mathrm{B}$*, and for the algebra* $[\mathcal{L}|\mathcal{P}]_n\mathrm{B}$.

Proof. These results follow, through arguments of elementary linear algebra, from the results of the previous section. We only remark that, for any pair (S,T) of standard tableaux of shape $\lambda \in H(\mathcal{L}) \cap H(\mathcal{P})$ on $\mathcal{L}$, the Orthonormal Generator $Y_/(S,\boxed{T})$ is nonzero. Indeed, there is a standard tableau U of shape λ on $\mathcal{P}$, and $Y_/(S,\boxed{T})(T,\boxed{U}) = (T,\boxed{U}) \neq \underline{0}$. □

The irreducible left submodules $\mathcal{S}_T = \langle (S\|\boxed{T});\ S \in \mathfrak{st}_\lambda(\mathcal{L})\rangle$ that appear in the above decomposition are called *Schur modules.* For the irreducible representations of the algebra $\mathrm{B}_n[\mathcal{L}|\mathcal{P}]$ on the Schur modules, we have the following matrix form.

Proposition 4 ([6, Section 13])**.** *For any* $\varphi \in \mathrm{B}_n[\mathcal{L}|\mathcal{P}]$*, the matrix* $\Xi_T(\varphi)$ *that represents the restriction of* φ *to a Schur module* $\mathcal{S}_T$ *of type* λ*, with respect to the basis of standard symmetrized bitableaux is given by*

$$\Xi_T(\varphi) = \left(\Theta^{-+}_{\mathfrak{st}_\lambda(\mathcal{L})}(I)\right)^{-1} \Theta^{-+}_{\mathfrak{st}_\lambda(\mathcal{L})}(\varphi).$$

Proof. For every $V, W \in \mathfrak{st}_\lambda(\mathcal{L})$, we have

$$\begin{aligned} DW\ (\varphi\ VC\ \underline{CD}\ DT) &= DW \sum_{U\in\mathfrak{st}_\lambda(\mathcal{L})} \xi_{T;UV}(\varphi)\ UC\ \underline{CD}\ DT, \\ \vartheta^{-+}_{WV}(\varphi)\ DC\ \underline{CD}\ DT &= \sum_{U\in\mathfrak{st}_\lambda(\mathcal{L})} \vartheta^{-+}_{WU}\ \xi_{T;UV}(\varphi)\ DC\ \underline{CD}\ DT, \\ \vartheta^{-+}_{WV}(\varphi) &= \sum_{U\in\mathfrak{st}_\lambda(\mathcal{L})} \vartheta^{-+}_{WU}\ \xi_{T;UV}(\varphi). \end{aligned}$$

From these equalities, summarized in a matrix equality, we get the formula given in the statement. □

Remark 2. *The space* $\mathrm{Super}_n[\mathcal{L}|\mathcal{P}]$ *has also a basis formed by monomials, e.g. by the monomials* $\underline{PQ}$ *associated to pairs* $(P,Q) = (x_1x_2\cdots x_n, y_1y_2\cdots y_n)$ *of words of length* n *on* $\mathcal{L},\mathcal{P}$*, such that the sequence* $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$ *is weakly increasing in the lexicographic order, and has no repetition of pairs of symbols of different* $\mathbb{Z}_2-$*grade. Thus the set of these pairs of words and the set of the pairs* (S,T) *of standard tableaux of the same shape* $\vdash n$ *on* $\mathcal{L},\mathcal{P}$ *have the same cardinality. A bijection between these sets is provided by the superalgebraic extension of the Robinson–Schensted–Knuth correspondence (see* [5]*, and* [30] *for further developments). This correspondence could be used in place of the straightening laws in the foundations of the theory.*

8. Proofs of the Propositions on Virtual Expressions

8.1. **Basic Remarks.** Let $(P,Q)=(p_1\cdots p_n, q_1\cdots q_n)$ and $(R,S)=(r_1\cdots r_n, s_1\cdots s_n)$ be two pairs of words of length n over $\mathcal{X}$, such that P and Q have disjoint supports, and consider the expression

$$PQ\ \underline{RS}\ = p_1q_1\ \cdots\ p_nq_n\ \left(\underline{r_1s_1}\ \cdots\ \underline{r_ns_n}\right).$$

Any permutation of the elements $\underline{r_i s_i}$ and, by our assumption, any permutation of the polarizations $p_i q_i$ leave the expression unchanged, up to a sign.

By direct computation, we get an expansion of the form

$$p_1 q_1 \ \cdots \ p_n q_n \ \big(\underline{r_1 s_1} \ \cdots \ \underline{r_n s_n}\big) = \sum_{i_1,\ldots,i_n} \epsilon_{i_1,\ldots,i_n} \ \delta_{q_{i_1} r_1} \cdots \delta_{q_{i_n} r_n} \ \underline{p_{i_1} s_1} \ \cdots \ \underline{p_{i_n} s_n},$$

where the sum is taken over all the linear orderings $i_1, \ldots, i_n$ of $1, \ldots, n$ and $\epsilon_{i_1,\ldots,i_n} \in \{-1, +1\}$. This expansion can be written in the form

$$PQ \ \underline{RS} = \sum_{P' \equiv P} \xi_{P'} \ \underline{P'S},$$

where P' ranges among all the words equivalent to P and the coefficients $\xi_{P'}$ are integers. The value of the expression depends on the central pair (Q, R) only through its isomorphism type, and the expression is zero unless Q and R have the same content.

Notice that, for $Q = R = S$,

$$PQ \ \underline{QQ} = c_Q! \ \underline{PQ},$$

where $c_Q!$ is the product of the factorials of the multiplicities of the symbols occurring in Q.

Analogous remarks can be made for the expressions of the form $\underline{PQ} \ RS$, where (P, Q) and (R, S) are pairs of words of length n over $\mathcal{X}$, such that R and S have disjoint supports.

8.2. **Proof of Proposition 1.** Let (P, Q) and (R, S) be two pairs of words of length n over $\mathcal{X}$, where words in the same pair have disjoint supports.

If $c_Q \neq c_R$, then $PQ \ \underline{RS} = \underline{0} = \underline{PQ} \ RS$; if $c_Q = c_R$, then

$$\underline{PQ} \ RS \ = \frac{1}{c_Q!} \ PQ \ \underline{QQ} \ RS \ = \frac{1}{c_R!} \ PQ \ \underline{RR} \ RS \ = PQ \ \underline{RS}.$$

8.3. **Proof of Proposition 2.** Let

$$(P_1^1, Q_1^1), (P_2^1, Q_2^1), \ldots, (P_{m_1}^1, Q_{m_1}^1), \ (P_1^2, Q_1^2), \ldots, (P_{m_2}^2, Q_{m_2}^2), \ \ldots, \ (P_{m_u}^u, Q_{m_u}^u)$$

be a sequence of pairs of words of the same length n on $\mathcal{X}$, such that in each pair the words have disjoint supports. Let $\vartheta_1, \vartheta_2, \ldots, \vartheta_u$ be scalars, and let $j_1, j_2, \ldots, j_u$ be indexes with $1 \leq j_i \leq m_i$. We consider the equalities

$$\sum_i \vartheta_i \ \underrightarrow{P_1^i Q_1^i} \ \underrightarrow{P_2^i Q_2^i} \ \cdots \ \underrightarrow{P_{m_i}^i Q_{m_i}^i} \ \Bigg|_n = \underrightarrow{0}\Big|_n,$$

$$\sum_i \vartheta_i \ P_1^i Q_1^i \ \cdots \ P_{j_i-1}^i Q_{j_i-1}^i \ \underline{P_{j_i}^i Q_{j_i}^i} \ P_{j_i+1}^i Q_{j_i+1}^i \ \cdots \ P_{m_i}^i Q_{m_i}^i \ = \underline{0},$$

$$\sum_i \vartheta_i \ \underleftarrow{P_1^i Q_1^i} \ \underleftarrow{P_2^i Q_2^i} \ \cdots \ \underleftarrow{P_{m_i}^i Q_{m_i}^i} \ \Bigg|_n = \underleftarrow{0}\Big|_n.$$

Any two equalities of the second type are equivalent; indeed, by Proposition 1, their left-hand sides are the same. We can assume that all the rightmost tableaux

$Q^1_{m_1}, Q^2_{m_2}, \ldots, Q^u_{m_u}$ in the various monomials have the same content, let Q be one of them.

We prove the equivalence of the first equality with one equality of the second type.

First we prove that the first equality implies the second. Indeed,

$$\sum \vartheta_i \; P^i_1 Q^i_1 \;\cdots\; P^i_{m_i-1} Q^i_{m_i-1} \; \underline{P^i_{m_i} Q^i_{m_i}} = \frac{1}{c_Q!} \sum \vartheta_i \; P^i_1 Q^i_1 \;\cdots\; P^i_{m_i-1} Q^i_{m_i-1} \; P^i_{m_i} Q^i_{m_i} \; \underline{QQ} \; = \underline{0}.$$

Now we prove that the first equality is implied by the second. Indeed, if (R, S) is any pair of words of length n, and $\dot{R}$ is an isomorphic copy of R with support disjoint from the supports of R and S, then

$$\begin{aligned}\sum \vartheta_i \; P^i_1 Q^i_1 \;\cdots\; P^i_{m_i} Q^i_{m_i} \; \underline{RS} &= \frac{1}{c_R!} \sum \vartheta_i \; P^i_1 Q^i_1 \;\cdots\; P^i_{m_i} Q^i_{m_i} \; \underline{R\dot{R}} \; \dot{R}S \\ &= \frac{1}{c_R!} \sum \vartheta_i \; P^i_1 Q^i_1 \;\cdots\; \underline{P^i_{m_i} Q^i_{m_i}} \; R\dot{R} \; \dot{R}S = \underline{0}.\end{aligned}$$

8.4. **Proof of Proposition 3.** In this subsection, for any word A on $\mathcal{X}$, we denote by $\mathcal{X} - A$ the complement of the support of A in $\mathcal{X}$, and use the symbol $|_{\mathcal{X}-A}$ to mean restriction to the subalgebra Super$[\mathcal{X} - A|\mathcal{X}]$ of the algebra Super$[\mathcal{X}|\mathcal{X}]$.

Step 1. *Let P, Q be two words of the same length n on $\mathcal{X}$. There is a family of integers $(\eta_{P',Q'})$ indexed by the pairs of words P', Q' of the same length equivalent to subwords of P and Q, such that, for every word A of length n with pairwise distinct positive entries occurring neither in P nor in Q,*

$$\underset{\longrightarrow}{PA} \; \underset{\longrightarrow}{AQ}\Big|_{\mathcal{X}-A} = \sum_{P',Q'} \eta_{P',Q'} \; \underset{\longrightarrow}{P'Q'}\Big|_{\mathcal{X}-A}.$$

We use induction on the common length $n \geq 1$ of the words P and Q. For $n = 1$ the statement is true; assume $n > 1$.

Let $P = p_1 p_2 \cdots p_n$, $Q = q_1 q_2 \cdots q_n$, and $A = a_1 a_2 \cdots a_n$, where each symbol a_i is distinct by any other symbol a_j, p_h, q_k. Up to a sign, we have

$$\underset{\longrightarrow}{PA} \; \underset{\longrightarrow}{AQ} = p_n a_n \;\cdots\; p_2 a_2 \; p_1 a_1 \; a_1 q_1 \; a_2 q_2 \;\cdots\; a_n q_n,$$

where, for the sake of brevity, we have denoted left polarizations without under-arrows.

By applying the commutation relation to the central factors $p_1 a_1$ and $a_1 q_1$ we get

$$\begin{aligned}(-1)^{|p_1||q_1|} \; p_n a_n \;\cdots\; p_2 a_2 \; (a_1 q_1 \; p_1 a_1) \; a_2 q_2 \;\cdots\; a_n q_n \\ + p_n a_n \;\cdots\; p_2 a_2 \; (p_1 q_1) \; a_2 q_2 \;\cdots\; a_n q_n \\ + (-1)^{|p_1||q_1|} \; \delta_{p_1 q_1} \; p_n a_n \;\cdots\; p_2 a_2 \; (a_1 a_1) \; a_2 q_2 \;\cdots\; a_n q_n.\end{aligned}$$

The first and the last term of this sum of operators vanish on the subalgebra Super$[\mathcal{X} - A|\mathcal{X}]$. By applying the commutation relations to the second term we get

$$
\begin{aligned}
&(-1)^{|p_1q_1||q_2\cdots q_u|}\ p_na_n\ \cdots\ p_2a_2\ a_2q_2\ \cdots\ a_nq_n\ p_1q_1\\
&+\sum_{i=2}^{n}(-1)^{|p_1q_1||q_2\cdots q_{i-1}|}\ \delta_{p_1q_i}\ p_na_n\ \cdots\ p_2a_2\ a_2q_2\ \cdots a_{i-1}q_{i-1}\ a_iq_1\ a_{i+1}q_{i+1}\ \cdots\ a_nq_n.
\end{aligned}
$$

Now the statement follows by applying the induction hypothesis to the pair $P_1 = p_n \cdots p_2$, $Q_1 = q_2 \cdots q_n$ and to the pairs $P_1 = p_n \cdots p_2$, $Q_i = q_2 \cdots q_{i-1}q_1q_{i+1} \cdots q_n$.

Step 2. *Under the same notation of the previous step, we have*

$$
\underset{\longrightarrow}{PA}\ \underset{\longrightarrow}{AQ}\Big|_n = \sum_{P',Q'} \eta_{P',Q'}\ \underset{\longrightarrow}{P'Q'}\Big|_n .
$$

Let $\underline{RS}$ be a monomial in $\mathrm{Super}_n[\mathcal{X}|\mathcal{X}]$. Then, if B is a word of length n with pairwise distinct positive entries occurring neither in P nor in Q nor in R, we have

$$
PA\ AQ\ \underline{RS} = PB\ BQ\ \underline{RS} = \left[\sum_{P',Q'} \eta_{P',Q'} P'Q'\right] \underline{RS}.
$$

This means that

$$
\underset{\longrightarrow}{PT}\ \underset{\longrightarrow}{TQ}\Big|_n = \sum_{P',Q'} \eta_{P',Q'}\ \underset{\longrightarrow}{P'Q'}\Big|_n .
$$

Step 3. Let (P_1, Q_1), (P_2, Q_2), $\ldots$, (P_u, Q_u) be any sequence of pairs of words of the same length n on $\mathcal{X}$, such that in any pair the words have disjoint supports.

In the space $\mathrm{Super}_n[\mathcal{X}|\mathcal{X}]$ we have an expansion of the form

$$
P_1Q_1\ \cdots\ P_{u-1}Q_{u-1}\ \underline{P_uQ_u} = \sum_{P' \equiv P} \xi_{P'}\ \underline{P'Q_u} = \sum_{P' \equiv P} \zeta_{P'}\ P'T\ \underline{TQ_u},
$$

where P' ranges among the words equivalent to P_1, the coefficients $\xi_{P'}$ are integers, T is a word with pairwise distinct entries which do not occur neither in P_1 nor in Q_u, and each coefficient $\zeta_{P'}$ equals, up to a sign, the corresponding coefficient $\xi_{P'}$. From Proposition 2, this expansion can be lifted to the expansion

$$
\underset{\longrightarrow}{P_1Q_1}\ \cdots\ \underset{\longrightarrow}{P_{u-1}Q_{u-1}}\ \underset{\longrightarrow}{P_uQ_u}\ \Big|_n = \sum_{P' \equiv P} \zeta_{P'}\ \underset{\longrightarrow}{P'T}\ \underset{\longrightarrow}{TQ_u},\Big|_n
$$

in $\mathrm{B}_n[\mathcal{X}]$. By applying the result of previous step to each summand, we get the proof of Proposition 3.

9. Proof of the Triangularity Lemma

9.1. **Proof of the Scholion.** Let (A, B) be a pair of words of the same length on $\mathcal{X}$, such that A has the same content of a coDeruyts tableau C, and B has the same content of a Deruyts tableau D. We set

$$
C = \begin{matrix} a_1^{\lambda_1} \\ a_2^{\lambda_2} \\ \vdots \\ a_p^{\lambda_p} \end{matrix}, \qquad D = \begin{matrix} b_1 \cdots\cdots\cdots b_{\mu_1} \\ b_1 \cdots\cdots b_{\mu_2} \\ \vdots \\ b_1 \cdots b_{\mu_p} \end{matrix},
$$

where the a_i are pairwise distinct positive symbols, the b_j are pairwise distinct negative symbols, and $\sum \lambda_i = \sum \mu_j$. We prove that, under the assumption $\underline{AB} \neq 0$, we have

(1) $\mathrm{sh}(C) \leq \mathrm{sh}(D)$;
(2) $\mathrm{sh}(C) = \mathrm{sh}(D) \Rightarrow \underline{AB} = \eta\ \underline{CD}$, where $\eta \in \{\pm 1\}$.

The variables $\underline{a_i b_j}$ are negative; since $\underline{AB} \neq \underline{0}$, each of these variables appears in this monomial at most once. Let M be the matrix, with rows indexed by $a_1, a_2, \ldots$ and columns indexed by $b_1, b_2, \ldots$, whose $(a_i, b_j)-$th entry is 1 if the variable $\underline{a_i b_j}$ appears in the monomial $\underline{AB}$, and 0 otherwise; the rows of M sum up to $(\lambda_1, \lambda_2, \ldots) = \lambda = \mathrm{sh}(C)$. Let M' be the matrix obtained from M by migrating the $1'$s upward columnwise; the rows of M' sum up to $(\mu_1, \mu_2, \ldots) = \mu = \mathrm{sh}(D)$.

Now, for every $h = 1, 2, \ldots$, we have

$$\lambda_1 + \lambda_2 + \ldots \lambda_h \leq \mu_1 + \mu_2 + \ldots \mu_h.$$

Indeed, the left-hand side counts the $1'$s in the first h rows of M; these $1'$s appear also in the first h rows of M', and the total number of $1'$s in the first h rows of M' is counted by the right-hand side.

If $\lambda = \mu$, then no upward migration takes place, so the $1'$s in the $i-$th row have the column index j ranging from 1 to λ_i, and the monomial is

$$\underline{AB} = \pm \underline{a_1 b_1}\ \cdots \underline{a_1 b_{\lambda_1}}\ \underline{a_2 b_1}\ \cdots \underline{a_2 b_{\lambda_2}}\ \cdots = \pm \underline{CD}.$$

9.2. **Proof of the Triangularity Lemma I.** Let (A_1, B_1), (A_2, B_2), $\ldots$, (A_s, B_s) be pairs of words of the same length on $\mathcal{X}$, where any two words in the same pair have disjoint supports, A_1 has the same content of a coDeruyts tableau C and B_s has the same content of a Deruyts tableau D. Then we have

$$A_1 B_1\ \cdots\ \underline{A_s B_s}\ = \sum_{A \equiv A_1} \xi_A\ \underline{AB_s},$$

where A ranges among the words equivalent to the word A_1, and hence equivalent to C, and the coefficients ξ_A are integers. By the Scholion,

- if $\mathrm{sh}(C) \not\leq \mathrm{sh}(D)$, then for each A we have $\underline{AB_s} = \underline{0}$, thus
$$A_1 B_1\ \cdots\ \underline{A_s B_s}\ = \underline{0};$$
- if $\mathrm{sh}(C) = \mathrm{sh}(D)$, then for each A we have $\underline{AB_s} = \eta_A\ \underline{CD}$, with $\eta_A \in \{0, \pm 1\}$, thus
$$A_1 B_1\ \cdots\ \underline{A_s B_s}\ = \vartheta\ \underline{CD},$$
for some $\vartheta \in \mathbb{Z}$.

9.3. **Expansions of Biproducts.** The expansions of the biproducts

$$(P|Q) = (p_1 \cdots p_n | q_1 \cdots q_n),$$

where all the symbols in P have the same $\mathbb{Z}_2-$degree and all the symbols in Q have the same $\mathbb{Z}_2-$degree, can be arranged in the following table:

$$c_P! \sum_{P' \equiv P} \underline{P'Q} = c_Q! \sum_{Q' \equiv Q} \underline{PQ'} \qquad\qquad c_P! \sum_{P' \equiv P} \underline{P'Q} = \sum_{Q' \equiv Q} \epsilon_{Q,Q'} \ \underline{PQ'}$$

$$\sum_{P' \equiv P} \epsilon_{P,P'} \ \underline{P'Q} = c_Q! \sum_{Q' \equiv Q} \underline{PQ'} \qquad (-1)^{\binom{n}{2}} \sum_{P' \equiv P} \epsilon_{P,P'} \ \underline{P'Q} = (-1)^{\binom{n}{2}} \sum_{Q' \equiv Q} \epsilon_{Q,Q'} \ \underline{PQ'}$$

The rows and the columns of this 2×2 table are indexed by $0, 1$; the $(i, j)-$the entry of the table displays the left and right expansions of the biproduct $(P|Q)$ when all the entries in P have $\mathbb{Z}_2-$degree i and all the entries in Q have $\mathbb{Z}_2-$degree j. When all the symbols in one of the two words, say P, are negative, we assume that P is square-free; if $P' \equiv P$ is a word equivalent to P, then the sign of any permutation which carries P to P' equals the sign $\epsilon_{P,P'}$, defined by the relation $P = \epsilon_{P,P'} \ P'$ in Super$[\mathcal{X}]$.

The table of the expansions of the $*-$biproducts

$$(P|Q)^* = (p_1 \cdots p_n | q_1 \cdots q_n)^*,$$

where all the symbols in P have the same $\mathbb{Z}_2-$degree and all the symbols in Q have the same $\mathbb{Z}_2-$degree, is the transpose of the above table.

9.4. **Proof of the Triangularity Lemma II.** Let (C, S), (T, D) be pairs of tableaux of the same size, where tableaux in the same pair have the same shape and disjoint supports, C is coDeruyts, S and T are standard, and D is Deruyts. We set

$$C = (\alpha_i)_{(i,j) \in \lambda}, \quad S = (s_{ij})_{(i,j) \in \lambda} \quad T = (t_{hk})_{(h,k) \in \mu}, \quad D = (\beta_k)_{(h,k) \in \mu},$$

with $\lambda, \mu \vdash n$, where none of the α's equals any of the s's, and none of the t's equals any of the β's.

Proof of the first part of the statement. Assume that $S \not\leq T$. If $c_S \neq c_T$, then $CS \ \underline{TD} = \underline{0}$. If $c_S = c_T$, then the assumption $S \not\leq T$ means that for some letter $x \in \mathcal{X}$, the shape γ of the subtableau $(s_{ij} : s_{ij} \leq x)$ of the tableau S is not less than or equal, in the dominance order, to the shape δ of the subtableau $(t_{hk} : t_{hk} \leq x)$ of the tableau T. In the following, the symbol of a tableau with a partition as subscript will denote its subtableau of shape that partition. Then we have

$$\begin{aligned}
CS\ \underline{TD} &= \pm \prod_{s_{ij}\le x} \alpha_i s_{ij} \prod_{s_{ij}>x} \alpha_i s_{ij} \left(\prod_{t_{hk}\le x} \underline{t_{hk}\beta_k} \prod_{t_{hk}>x} \underline{t_{hk}\beta_k} \right) \\
&= \pm \left(\prod_{s_{ij}\le x} \alpha_i s_{ij} \prod_{t_{hk}\le x} \underline{t_{hk}\beta_k} \right) \left(\prod_{s_{ij}>x} \alpha_i s_{ij} \prod_{t_{hk}>x} \underline{t_{hk}\beta_k} \right) \\
&= \pm \left(C_\gamma S_\gamma\ \underline{T_\delta D_\delta} \right) \left(\prod_{s_{ij}>x} \alpha_i s_{ij} \prod_{t_{hk}>x} \underline{t_{hk}\beta_k} \right) \\
&= 0,
\end{aligned}$$

by Lemma 1.

Proof of the second part of the statement. Let $S = (s_{ij})_{(i,j)\in\lambda} = T$, with $\lambda \vdash n$.

We use induction on the number of distinct letters which appear in S. For one letter, the statement is true: for $|x| = 0$ we have $\lambda = (n)$, and

$$CS\ \underline{SD} = \alpha_1 \ldots \alpha_1 x \ldots x\ \underline{x \ldots x \beta_1 \ldots \beta_n} = n!\ \underline{\alpha_1 \ldots \alpha_1 \beta_1 \ldots \beta_n} = n!\ \underline{CD};$$

for $|x| = 1$ we have $\lambda = (1, 1, \ldots, 1)$, and

$$CS\ \underline{SD} = \begin{array}{cccc} \alpha_1 & x & x & \beta_1 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_n & x & \underline{x} & \underline{\beta_n} \end{array} = n!\ \underline{CD}.$$

We assume now that S contains more than one letter; we denote by x the greatest letter occurring in S, and denote by γ the shape of the subtableau $(s_{ij} : s_{ij} < x)$ of the tableau S.

We consider the case in which x is positive (the case x negative is analogous). The cells of the shape λ in which x occurs form an horizontal strip; listing these cells starting from the leftmost and moving right and up, we get a sequence

$$(i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m)$$

where the sequence of the row indexes is of the form

$$i_1 = \ldots = i_{m_1} > i_{m_1+1} = \ldots = i_{m_1+m_2} > \ldots > i_{m-m_r} = \ldots = i_m$$

and the sequence of the column indexes is strictly increasing.

We have

$$
\begin{aligned}
CS\ \underline{SD} &= \pm \left(\prod_{(i,j)\in\gamma} \alpha_i s_{ij} \right) \alpha_{i_1} x\ \ldots\ \alpha_{i_m} x \left(\left(\prod_{(i,j)\in\gamma} \underline{s_{ij}\beta_j} \right) \underline{x\beta_{j_1}}\ \cdots\ \underline{x\beta_{j_m}} \right) \\
&= \pm \left(\prod_{(i,j)\in\gamma} \alpha_i s_{ij} \prod_{(i,j)\in\gamma} \underline{s_{ij}\beta_j} \right) \alpha_{i_1} x\ \ldots\ \alpha_{i_m} x\ \underline{x\beta_{j_1}}\ \cdots\ \underline{x\beta_{j_m}} \\
&= \pm \left(\theta^{+-}_{S_\gamma S_\gamma} \prod_{(i,j)\in\gamma} \underline{\alpha_i\beta_j} \right) m_1!m_2!\cdots m_s! \sum \underline{\alpha_{i'_1}\beta_{j_1}}\ \cdots\ \underline{\alpha_{i'_m}\beta_{j_m}},
\end{aligned}
$$

where the sum is extended to the words $i'_1 \cdots i'_m$ equivalent to the word $i_1 \cdots i_m$.

We claim that $i_1 \cdots i_m$ is the only word which really contributes to the whole expression. Indeed, for each cell (i_p, j_p) in the horizontal strip in which occurs x, any cell (i'_p, j_p), with $i'_p < i_p$, belongs to the shape γ, so

$$
\left(\prod_{(i,j)\in\gamma} \underline{\alpha_i\beta_j} \right) \underline{\alpha_{i'_p}\beta_{j_p}} = 0.
$$

Thus the words $i'_1 \cdots i'_m$ whose corresponding summand contributes to the whole expression are subject to the conditions $i'_p \geq i_p$, for all $p = 1, \ldots, m$, and there is only one such word: $i_1 \cdots i_p$.

Now we have

$$
\begin{aligned}
CS\ \underline{SD} &= \pm m_1!m_2!\cdots m_s!\ \theta^{+-}_{S_\gamma S_\gamma} \left(\prod_{(i,j)\in\gamma} \underline{\alpha_i\beta_j} \right) \underline{\alpha_{i_1}\beta_{j_1}} \cdots \underline{\alpha_{i_m}\beta_{j_m}}, \\
&= \pm m_1!m_2!\cdots m_s!\ \theta^{+-}_{S_\gamma S_\gamma} \prod_{(ij)\in\lambda} \underline{\alpha_i\beta_j} \\
&= \pm m_1!m_2!\cdots m_s!\ \theta^{+-}_{S_\gamma S_\gamma}\ \underline{CD}.
\end{aligned}
$$

Finally, by induction, we get

$$
\theta^{+-}_{SS} = \pm m_1!m_2!\cdots m_s!\ \theta^{+-}_{S_\gamma S_\gamma} = \pm m_1!m_2!\cdots m_s!\ c^r_{S_\gamma}!\ c^c_{S_\gamma}! = \pm c^r_S!\ c^c_S!.
$$

10. Proofs of the Nondegeneracy Lemma

10.1. **Expansions of Bitableaux.** Let P, Q be tableaux of the same shape $\lambda \vdash n$ on $\mathcal{X}$, and consider the bitableau

$$
PC\ \underline{CQ}\ =\ \underline{PC}\ CQ = (P|Q),
$$

where C is a coDeruyts tableau of shape λ with no symbol in common with P and Q.

Notice that, upon denoting by $P_1, P_2, \ldots, P_m$ the rows of P, by $A_1, A_2, \ldots, A_m$ the rows of C, by $Q_1, Q_2, \ldots, Q_m$ the rows of Q, we have

$$\begin{aligned}(P|Q) = PC\ \underline{CQ} &= \prod_{i=1}^{m} P_iA_i \prod_{i=1}^{m} \underline{A_iQ_i} \\ &= (-1)^{\sum_{i>j}|P_i||Q_j|} \prod_{i=1}^{m} \left(P_iA_i\ \underline{A_iQ_i}\right) = (-1)^{\sum_{i>j}|P_i||Q_j|} \prod_{i=1}^{m} (P_i|Q_i);\end{aligned}$$

thus, the bitableau of the tableaux P and Q is, up to a sign, the product of the biproducts of the corresponding rows of P and Q.

Among the various left and right expansions of the bitableau $(P|Q)$, we consider only the right expansion in the case in which all the symbols in P and Q are negative; we assume that each row of P and each row of Q is square-free. In this case we have

$$(P|Q) = (-1)^{\binom{n}{2}} \sum_{Q' \equiv_r Q} \epsilon_{Q,Q'}\ \underline{PQ'},$$

where Q' runs through the tableaux row-equivalent to Q, and $\epsilon_{Q,Q'}$ is the sign of the permutations which transform Q to Q'; notice that, if Q' is any tableau of the same shape as Q, then

$$Q \equiv_r Q' \quad \text{if and only if} \quad CQ = \epsilon_{Q,Q'}\ CQ'.$$

Indeed,

$$\begin{aligned}(P|Q) &= (-1)^{\sum_{i>j}\lambda_i\lambda_j} \prod_{i=1}^{m} (P_i|Q_i) \\ &= (-1)^{\sum_{i>j}\lambda_i\lambda_j} \prod_{i=1}^{m} (-1)^{\binom{\lambda_i}{2}} \sum_{Q_i' \equiv Q_i} \epsilon_{Q_i,Q_i'}\ \underline{P_iQ_i'} \\ &= (-1)^{\sum_{i>j}\lambda_i\lambda_j + \sum_i \binom{\lambda_i}{2}} \sum_{Q_1',Q_2',\ldots \equiv Q_1,Q_2,\ldots} \prod_{i=1}^{m} \epsilon_{Q_i,Q_i'}\ \underline{P_iQ_i'} \\ &= (-1)^{\binom{n}{2}} \sum_{Q' \equiv_r Q} \epsilon_{Q,Q'}\ \underline{PQ'}.\end{aligned}$$

The equality $\sum_{i>j}\lambda_i\lambda_j + \sum_i \binom{\lambda_i}{2} = \binom{n}{2}$, comes from the fact that the left hand side counts the number of inversions of the permutation $n(n-1)\ldots 21$ by considering it as split into blocks of order $(\lambda_1, \lambda_2, \ldots, \lambda_m)$.

Let P, Q be tableaux of the same shape $\lambda \vdash n$ on $\mathcal{X}$, and consider the $*-$bitableau

$$PD\ \underline{DQ} \;=\; \underline{PD}\ DQ = (P|Q)^*,$$

where D is a Deruyts tableau of shape λ with no symbol in common with P and Q.

We point out that, among the various expansions, in the case in which all the symbols in Q are negative, we have the right expansion

$$(P|Q)^* = \underline{PD}\ DQ = c_Q^c! \sum_{Q' \equiv_c Q} \underline{PQ'},$$

where Q' runs through the tableaux column-equivalent to Q.

10.2. **Proof of the Nondegeneracy Lemma — Weak Form.** Let $C, \dot{D}, \dot{C}, D$ be coDeruyts and Deruyts tableaux, all of the same shape $\lambda \vdash n$. By Lemma 1, we have

$$C\dot{D}\ \underline{\dot{D}\dot{C}}\ \dot{C}D = k_\lambda\ \underline{CD},$$

for some integer k_λ. We prove that $k_\lambda \neq 0$.

Step 1. We have

$$\underline{\dot{D}\dot{C}}\ \dot{C}D = (-1)^{\binom{n}{2}} \sum_{E \equiv_r D} \epsilon_{E,D}\ \underline{\dot{D}E},$$

where the sum is taken over all the tableaux E which are row-equivalent to the tableau D, and the coefficient $\epsilon_{E,D}$ is defined by the relation

$$\underline{\dot{C}E} = \epsilon_{E,D}\ \underline{\dot{C}D}.$$

Step 2. For every tableau $E \equiv_r D$, we have

$$C\dot{D}\ \underline{\dot{D}E} = \underline{C\dot{D}}\ \dot{D}E = c_E^c! \sum_{F \equiv_c E} \underline{CF},$$

where the sum is taken over all the tableaux F which are column-equivalent to the tableau E.

Step 3. By the previous steps, we have

$$\begin{aligned} C\dot{D}\ \underline{\dot{D}\dot{C}}\ \dot{C}D &= (-1)^{\binom{n}{2}} \sum_{E \equiv_r D} \epsilon_{E,D}\ C\dot{D}\ \underline{\dot{D}E} \\ &= (-1)^{\binom{n}{2}} \sum_{E \equiv_r D} \epsilon_{E,D}\ c_E^c! \sum_{F \equiv_c E} \underline{CF}. \end{aligned}$$

Since F must be equivalent to D, we have that $\underline{CF} = \underline{0}$ unless $F \equiv_r D$; in the case $F \equiv_r D$, we have

$$\underline{CF} = \epsilon_{F,D}\ \underline{CD}.$$

Then we have

$$C\dot{D}\ \underline{\dot{D}\dot{C}}\ \dot{C}D = (-1)^{\binom{n}{2}} \left(\sum_{E \equiv_r D} \epsilon_{E,D}\ c_E^c! \sum_{\substack{F \equiv_c E \\ F \equiv_r D}} \epsilon_{F,D} \right) \underline{CD}.$$

Now, the set

$$\{G \mid\ G \equiv_r D\}$$

of the tableau which are row-equivalent to the tableau D splits, under the equivalence relation $\equiv_c$ of column-equivalence, into some classes $\mathcal{C}_0, \mathcal{C}_1, \ldots$; so, noting that the coefficient $c_E^c!$ is constant over each class and denoting by γ_i its value on $\mathcal{C}_i$, we get

$$\begin{aligned} \sum_{E \equiv_r D} \epsilon_{E,D}\ c_E^c! \sum_{\substack{F \equiv_c E \\ F \equiv_r D}} \epsilon_{F,D} &= \sum_i \gamma_i \sum_{E \in \mathcal{C}_i} \epsilon_{E,D} \sum_{\substack{F \equiv_c E \\ F \equiv_r D}} \epsilon_{F,D} \\ &= \sum_i \gamma_i \sum_{E \in \mathcal{C}_i} \epsilon_{E,D} \sum_{F \in \mathcal{C}_i} \epsilon_{F,D} = \sum_i \gamma_i \left(\sum_{E \in \mathcal{C}_i} \epsilon_{E,D} \right)^2 \geq 0. \end{aligned}$$

As a matter of fact, this integer is strictly positive, since for the equivalence class, say $\mathcal{C}_0$, of the tableau D we have

$$\sum_{E \in \mathcal{C}_0} \epsilon_{E,D} = \epsilon_{D,D} = 1.$$

10.3. **An Application of the Straightening Law.** We will use the following application of the Straightening Law.

Proposition 5. *Let $x, \ldots, y, z$ be j pairwise distinct negative letters, let $l_1 \geq l_2 \geq \cdots \geq l_r$, and let D be any Deruyts tableau of shape $(j + l_1, j + l_2, \ldots, j + l_r)$.*

- *Let $a_1, a_2, \ldots, a_r$ be pairwise distinct positive letters. Then*

$$a_1 z \left(\begin{array}{l|c} x \cdots yza_1^{(l_1)} & \\ x \cdots yza_2^{(l_2)} & D \\ \vdots & \\ x \cdots yza_r^{(l_r)} & \end{array} \right) = (-1)^{j-1}(l_1 + r) \left(\begin{array}{l|c} x \cdots ya_1^{(l_1+1)} & \\ x \cdots yza_2^{(l_2)} & D \\ \vdots & \\ x \cdots yza_r^{(l_r)} & \end{array} \right).$$

- *Let a_1 be a positive symbol and let v_i, for $i = 2, \ldots, r$, be words of length l_i not containing z. Then*

$$a_1 z \left(\begin{array}{l|c} x \cdots yza_1^{(l_1)} & \\ x \cdots yzv_2 & D \\ \vdots & \\ x \cdots yzv_r & \end{array} \right) = (-1)^{j-1}(l_1 + r) \left(\begin{array}{l|c} x \cdots ya_1^{(l_1+1)} & \\ x \cdots yzv_2 & D \\ \vdots & \\ x \cdots yzv_r & \end{array} \right).$$

Notice that $l_1 + r$ is the hook length of the cell $(1, j)$ in the Ferrers diagram of the common shape $(j + l_1, j + l_2, \ldots, j + l_r)$ of the tableaux.

Proof. We prove the first statement of the Proposition; the second follows from the first by acting on both sides with the left polarization monomial

$$v_2 a_2^{l_2} \ \cdots \ v_r a_r^{l_r}.$$

We notice that

$$a_1 z \left(\begin{array}{l|c} x \cdots yza_1^{(l_1)} & \\ x \cdots yza_2^{(l_2)} & D \\ \vdots & \\ x \cdots yza_r^{(l_r)} & \end{array} \right) = \sum_{i=1}^{r} (-1)^{ij-1} \left(\begin{array}{l|c} x \cdots yza_1^{(l_1)} & \\ \vdots & \\ x \cdots ya_1 a_i^{(l_i)} & D \\ x \cdots yza_{i+1}^{(l_{i+1})} & \\ \vdots & \\ x \cdots yza_r^{(l_r)} & \end{array} \right).$$

The first statement of the proposition follows from the following facts.

(1) The first term of the sum is

$$(-1)^{j-1}\left(\begin{array}{c|c} x\cdots ya_1a_1^{(l_1)} & \\ x\cdots yza_2^{(l_2)} & D \\ \vdots & \\ x\cdots yza_r^{(l_r)} & \end{array}\right) = (-1)^{j-1}(l_1+1)\left(\begin{array}{c|c} x\cdots ya_1^{(l_1+1)} & \\ x\cdots yza_2^{(l_2)} & D \\ \vdots & \\ x\cdots yza_r^{(l_r)} & \end{array}\right).$$

(2) The second term equals

$$-\left(\begin{array}{c|c} x\cdots yza_1^{(l_1)} & \\ x\cdots ya_1a_2^{(l_2)} & \\ x\cdots yza_3^{(l_3)} & D \\ \vdots & \\ x\cdots yza_r^{(l_r)} & \end{array}\right) = (-1)^{j-1}\left(\begin{array}{c|c} x\cdots ya_1^{(l_1+1)} & \\ x\cdots yza_2^{(l_2)} & D \\ \vdots & \\ x\cdots yza_r^{(l_r)} & \end{array}\right).$$

Indeed, by the straightening law relative to the diagonalization of $a_1^{(l_1+1)}$, we have

$$\left(\begin{array}{c|c} x\cdots yza_1^{(l_1)} & D' \\ x\cdots ya_1a_2^{(l_2)} & \end{array}\right) = -\left(\begin{array}{c|c} x\cdots ya_1^{(l_1+1)} & D' \\ zx\cdots ya_2^{(l_2)} & \end{array}\right) = (-1)^{j}\left(\begin{array}{c|c} x\cdots ya_1^{(l_1+1)} & D' \\ x\cdots yza_2^{(l_2)} & \end{array}\right).$$

(3) All the terms from the second on are equal:

$$(-1)^{ij-1}\left(\begin{array}{c|c} x\cdots yza_1^{(l_1)} & \\ \vdots & \\ x\cdots ya_1a_i^{(l_i)} & D \\ x\cdots yza_{i+1}^{(l_{i+1})} & \\ \vdots & \\ x\cdots yza_r^{(l_r)} & \end{array}\right) = (-1)^{(i-1)j-1}\left(\begin{array}{c|c} x\cdots yza_1^{(l_1)} & \\ \vdots & \\ x\cdots ya_1a_{i-1}^{(l_{i-1})} & D \\ x\cdots yza_i^{(l_i)} & \\ \vdots & \\ x\cdots yza_r^{(l_r)} & \end{array}\right),$$

for $i=2,\ldots,r$.

Indeed, by the straightening law relative to the diagonalization of a_1, we have

$$\left(\begin{array}{c|c} \vdots & \\ x\cdots yza_{i-1}^{(l_{i-1})} & D'' \\ x\cdots ya_1a_i^{(l_i)} & \end{array}\right) = -\left(\begin{array}{c|c} \vdots & \\ x\cdots ya_1a_{i-1}^{(l_{i-1})} & D'' \\ zx\cdots ya_i^{(l_i)} & \end{array}\right) - \left(\begin{array}{c|c} \vdots & \\ x\cdots yza_1a_{i-1}^{(l_{i-1}-1)} & D'' \\ a_{i-1}x\cdots ya_i^{(l_i)} & \end{array}\right)$$

$$= (-1)^{j}\left(\begin{array}{c|c} \vdots & \\ x\cdots ya_1a_{i-1}^{(l_{i-1})} & D'' \\ x\cdots yza_i^{(l_i)} & \end{array}\right);$$

the second term in the first equality vanishes since it has the factor

$$\left(\begin{array}{c|c} x\cdots yza_1^{(l_1)} & D''' \\ x\cdots yza_1a_{i-1}^{(l_{i-1}-1)} & \end{array}\right),$$

which in turn vanishes, by the straightening law relative to the diagonalization of $a_1^{(l_1+1)}$.

□

10.4. **Proof of the Nondegeneracy Lemma.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \vdash n$ be a partition of n, and let

$$h_{ij} = \lambda_i + \lambda'_j - i - j + 1$$

be the hook length of the cell (i, j) in the shape λ.

Let $a_1, a_2, \ldots, a_p$ be positive pairwise distinct symbols, let $x_1, x_2, ..., x_{\lambda_1}$ be negative pairwise distinct symbols, and let D be any Deruyts tableau of shape λ.

From the second point of the previous proposition we get

$$a_1 x_j \left(\begin{array}{l|c} x_1 \cdots\cdots x_j a_1^{(\lambda_1 - j)} & \\ x_1 \cdots\cdots x_{\lambda_2} & D \\ \vdots & \\ x_1 \cdots x_{\lambda_p} & \end{array} \right) = (-1)^{j-1} h_{1j} \left(\begin{array}{l|c} x_1 \cdots\cdots x_{j-1} a_1^{(\lambda_1 - j + 1)} & \\ x_1 \cdots\cdots x_{\lambda_2} & D \\ \vdots & \\ x_1 \cdots x_{\lambda_p} & \end{array} \right),$$

for every $j = 1, \ldots, \lambda_1$. By iterating this identity, we get

$$\left(\prod_{j=1}^{\lambda_1} a_1 x_j \right) \left(\begin{array}{l|c} x_1 \cdots\cdots\cdots x_{\lambda_1} & \\ x_1 \cdots\cdots x_{\lambda_2} & D \\ \vdots & \\ x_1 \cdots x_{\lambda_p} & \end{array} \right) = (-1)^{\binom{\lambda_1}{2}} \left(\prod_{j=1}^{\lambda_1} h_{1j} \right) \left(\begin{array}{l|c} a_1^{(\lambda_1)} & \\ x_1 \cdots\cdots x_{\lambda_2} & D \\ \vdots & \\ x_1 \cdots x_{\lambda_p} & \end{array} \right).$$

By iterating this identity, we get

$$\left(\prod_{i=1}^{p} \prod_{j=1}^{\lambda_i} a_i x_j \right) \left(\begin{array}{l|c} x_1 \cdots\cdots\cdots x_{\lambda_1} & \\ x_1 \cdots\cdots x_{\lambda_2} & D \\ \vdots & \\ x_1 \cdots x_{\lambda_p} & \end{array} \right) = \epsilon \left(\prod_{i=1}^{p} \prod_{j=1}^{\lambda_i} h_{i,j} \right) \left(\begin{array}{l|c} a_1^{(\lambda_1)} & \\ a_2^{(\lambda_2)} & D \\ \vdots & \\ a_p^{(\lambda_p)} & \end{array} \right),$$

where $\epsilon = (-1)^{\sum_{h>k} \lambda_h \lambda_k + \sum_h \binom{\lambda_h}{2}} = (-1)^{\binom{n}{2}}$.

Now, for

$$C = \begin{array}{l} a_1^{\lambda_1} \\ a_2^{\lambda_2} \\ \vdots \\ a_p^{\lambda_p} \end{array}, \qquad \dot{D} = \begin{array}{l} x_1 \cdots\cdots\cdots x_{\lambda_1} \\ x_1 \cdots\cdots x_{\lambda_2} \\ \vdots \\ x_1 \cdots x_{\lambda_p} \end{array}$$

and $\dot{C}$ any coDeruyts tableau of shape λ, we get

$$\begin{aligned} C\dot{D} \ \left(\underline{\dot{D}\dot{C}} \ \dot{C}D \right) &= \left(\prod_{i=1}^{p} \prod_{j=1}^{\lambda_i} a_i x_j \right) \left(\begin{array}{l|c} x_1 \cdots\cdots\cdots x_{\lambda_1} & \\ x_1 \cdots\cdots x_{\lambda_2} & D \\ \vdots & \\ x_1 \cdots x_{\lambda_p} & \end{array} \right) \\ &= \epsilon \left(\prod_{i=1}^{p} \prod_{j=1}^{\lambda_i} h_{i,j} \right) \left(\begin{array}{l|c} a_1^{(\lambda_1)} & \\ a_2^{(\lambda_2)} & D \\ \vdots & \\ a_p^{(\lambda_p)} & \end{array} \right) = (-1)^{\binom{n}{2}} h_\lambda \underline{CD}, \end{aligned}$$

which is the statement of Proposition 5.

Remark 3. *From the second identity in the proof, we get*

$$[x_{\lambda_1}\cdots x_1|x_1\cdots x_{\lambda_1}]\left(\begin{array}{l|l} x_1\cdots\cdots\cdots x_{\lambda_1} & y_1\cdots\cdots\cdots y_{\lambda_1}\\ x_1\cdots\cdots x_{\lambda_2} & y_1\cdots\cdots y_{\lambda_2}\\ \vdots & \vdots\\ x_1\cdots x_{\lambda_p} & y_1\cdots y_{\lambda_p}\end{array}\right)$$
$$=\left(\prod_{j=1}^{\lambda_1}h_{1j}\right)\left(\begin{array}{l|l} x_1\cdots\cdots\cdots x_{\lambda_1} & y_1\cdots\cdots\cdots y_{\lambda_1}\\ x_1\cdots\cdots x_{\lambda_2} & y_1\cdots\cdots y_{\lambda_2}\\ \vdots & \vdots\\ x_1\cdots x_{\lambda_p} & y_1\cdots y_{\lambda_p}\end{array}\right).$$

This identity in turn, for $\lambda_1=\lambda_2=\ldots=\lambda_p(=m)$, yields

$$[x_m\cdots x_1|x_1\cdots x_m]\ \ (x_1\cdots x_m|x_1\cdots x_m)^p=\langle p\rangle_m\ \ (x_1\cdots x_m|x_1\cdots x_m)^p\,,$$

where $\langle p\rangle_m=p(p+1)\ldots(p+m-1)$. This identity is equivalent to the Cayley identity for the action of the Ω operator on a product of determinants [38].

Appendix: An Algebra of Virtual Expressions

Given a $\mathbb{Z}_2$-graded set $\mathcal{X}$ consisting of countably many positive and countably many negative symbols, and given a positive integer n, we define a new associative algebra over the rational numbers $\mathbb{Q}$.

First, consider the set of all pairs (P,Q) of words, of length n with disjoint supports, over $\mathcal{X}$; for the sake of brevity, we write PQ instead of (P,Q). This set generates a free unitary associative algebra T over $\mathbb{Q}$; we denote the product of two generators PQ and RS by juxtaposition $PQ\ RS$, with an inner blank; the products $A_1B_1\ A_2B_2\ \cdots\ A_mB_m$ form a linear basis of this algebra. Consider the linear morphisms defined by setting

$$\begin{aligned}
\underset{\rightarrow}{\varphi}:\mathrm{T}\to\mathrm{B}_n[\mathcal{X}],\qquad &\underset{\rightarrow}{\varphi}\,(A_1B_1\ \cdots\ A_mB_m)=\underset{\longrightarrow}{A_1B_1}\ \cdots\ \underset{\longrightarrow}{A_mB_m}|_n,\\
\underline{\varphi}:\mathrm{T}\to\mathrm{Super}_n[\mathcal{X}|\mathcal{X}],\qquad &\underline{\varphi}\,(A_1B_1\ \cdots\ A_mB_m)=A_1B_1\ \cdots\underline{A_iB_i}\ \cdots\ A_mB_m,\\
\underset{\leftarrow}{\varphi}:\mathrm{T}\to[\mathcal{X}]_n\mathrm{B},\qquad &\underset{\leftarrow}{\varphi}\,(A_1B_1\ \cdots\ A_mB_m)=\underset{\longleftarrow}{A_1B_1}\ \cdots\ \underset{\longleftarrow}{A_mB_m}|_n;
\end{aligned}$$

we explicitly notice that the second map is well-defined, by Proposition 1. The crucial fact is that all these morphisms have the same kernel

$$\mathrm{Ker}=\mathrm{Ker}\left(\underset{\rightarrow}{\varphi}\right)=\mathrm{Ker}\left(\underline{\varphi}\right)=\mathrm{Ker}\left(\underset{\leftarrow}{\varphi}\right),$$

by Proposition 2; this kernel is a two-sided ideal of T, since $\underset{\rightarrow}{\varphi}$ and $\underset{\leftarrow}{\varphi}$ are algebra morphisms.

We consider the quotient algebra

$$\overline{\mathrm{T}}=\frac{\mathrm{T}}{\mathrm{Ker}}.$$

In this algebra, we have the identity

$$A_1B_1\ A_2B_2\ \cdots\ A_mB_m\ =\ A_1B_1'\ A_2'B_2'\ \cdots\ A_m'B_m,$$

under the assumption $(B_1, A_2) \simeq (B'_1, A'_2)$, ..., $(B_{m-1}, A_m) \simeq (B'_{m-1}, A'_m)$, and the common value of these expressions is zero unless $B_1 \equiv A_2$, ..., $B_{m-1} \equiv A_m$. We have an expansion

$$A_1B_1\ A_2B_2\ \cdots\ A_mB_m\ =\ \sum_{A'\equiv A} \zeta_{A'}\ A'Q\ QB_m,$$

where A' ranges among the words equivalent to A_1, Q is a word with pairwise distinct positive entries which do not occur neither in A_1 nor in B_m, and the coefficients $\zeta_{A'}$ are integers. Therefore, the algebra is linearly generated by the products $AQ\ QB$, with Q a word with pairwise distinct positive entries which do not occur neither in A nor in B.

In the algebra $\overline{\mathrm{T}}$, for any pair (R, S) of tableaux of the same shape $\lambda \vdash n$, we have three basic objects, given by

$$RC\ CS, \qquad RC\ CD\ DS, \qquad RD\ DS,$$

where C and D are any coDeruyts and Deruyts tableaux of shape λ; the images of these objects under the mappings $\underset{\rightarrow}{\varphi}$, $\underline{\varphi}$, $\underset{\leftarrow}{\varphi}$ are the left Capelli rows, the bitableaux, the right Capelli rows, the left Young–Capelli symmetrizers, the symmetrized bitableaux, ...

The basic triangularity and nondegeneracy relations are

$$A_1B_1\ \cdots\ A_sB_s\ = \begin{cases} \qquad 0 & \text{if } \mathrm{sh}(C) \not\leq \mathrm{sh}(D) \\ \\ \vartheta\ CD & \text{if } \mathrm{sh}(C) = \mathrm{sh}(D) \end{cases}, \qquad (A_1 \equiv C, B_s \equiv D)$$

$$CP\ QD\ = \begin{cases} \qquad\quad 0 & \text{if } P \not\leq Q \\ \\ \pm c^r_P!\ c^c_P!\ CD & \text{if } P = Q \end{cases}, \qquad (P, Q\ \in \mathfrak{st})$$

$$C\dot{D}\ \dot{D}\dot{C}\dot{C}D = (-1)^{\binom{n}{2}} h_\lambda\ CD \qquad (\mathrm{sh}(C) = \ldots = \mathrm{sh}(D) = \lambda).$$

There are also identities coming from the Straightening Laws (compare with [28], [10, 11]).

Finally, we notice that the group algebra $\mathbb{Q}[\mathbb{S}_n]$ of the symmetric group $\mathbb{S}_n$ occurs in a natural way in this context. Specifically, let $P = x_1x_2\cdots x_n$ be a word in pairwise distinct positive symbols; then the subspace $\overline{\mathrm{S}}$ linearly generated by the products

$$P_1Q\ QP_2,$$

where P_1 and P_2 range through the words equivalent to P, and Q is a word in pairwise distinct positive symbols not occurring in P, is a subalgebra of the algebra $\overline{\mathrm{T}}$.

The map Φ that sends the product $P'Q\ QP''$, where

$$P' = x_{i_1}x_{i_2}\cdots x_{i_n}, \qquad P'' = x_{j_1}x_{j_2}\cdots x_{j_n},$$

to the the permutation

$$\begin{pmatrix} i_1\ i_2\ \cdots\ i_n \\ j_1\ j_2\ \cdots\ j_n \end{pmatrix}$$

induces an algebra isomorphism from the algebra $\overline{\mathrm{S}}$ to the group algebra $\mathbb{Q}[\mathbb{S}_n]$ of the symmetric group $\mathbb{S}_n$. In this isomorphism, the objects of Young–Capelli type associated to pairs of multilinear tableaux in the symbols $x_1, x_2, \ldots, x_n$ are sent to the classical Young symmetrizers.

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