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A RIDICULOUSLY SIMPLE AND EXPLICIT IMPLICIT FUNCTION THEOREM

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Dedicated to the memory of Pierre Leroux

Abstract. I show that the general implicit-function problem (or parametrized fixed-point problem) in one complex variable has an explicit series solution given by a trivial generalization of the Lagrange inversion formula. I give versions of this formula for both analytic functions and formal power series.

1. Introduction

It is well known to both complex analysts and combinatorialists that the problem of reverting a power series, i.e., solving $f(z) = w$, has an explicit solution known as the Lagrange (or Lagrange–Bürmann) inversion formula [3, 7, 16, 18, 34, 42, 47, 48]. What seems to be less well known is that the more general implicit-function problem $F(z, w) = 0$ also has a simple explicit solution, given by Yuzhakov [49] in 1975 (see also [2, 4, 50]). My purpose here is to give a slightly more general and flexible version of Yuzhakov's formula, and to show that its proof is an utterly trivial generalization of the standard proof of the Lagrange inversion formula. A formal-power-series version of the formula presented here appears also in [42, Exercise 5.59, pp. 99 and 148], but its importance for the implicit-function problem does not seem to be sufficiently stressed.

Let us begin by recalling the Lagrange inversion formula: if $f(z) = \sum_{n=1}^{\infty} a_n z^n$ with $a_1 \neq 0$ (interpreted either as an analytic function or as a formal power series), then

$$f^{-1}(w) \;=\; \sum_{m=1}^{\infty} \frac{w^m}{m} \, [\zeta^{m-1}] \left(\frac{\zeta}{f(\zeta)} \right)^{m} , \tag{1.1}$$

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where $[\zeta^n]g(\zeta) = g^{(n)}(0)/n!$ denotes the coefficient of ζ^n in the power series $g(\zeta)$. More generally, if $h(z) = \sum_{n=0}^{\infty} b_n z^n$, we have

$$h(f^{-1}(w)) \;=\; h(0) + \sum_{m=1}^{\infty} \frac{w^m}{m}\, [\zeta^{m-1}]\, h'(\zeta) \biggl(\frac{\zeta}{f(\zeta)} \biggr)^{m} . \tag{1.2}$$

Proofs of these formulae can be found in many books on analytic function theory [7, 18, 34, 47] and enumerative combinatorics [3, 16, 42, 48].

It is convenient to introduce the function (or formal power series) $g(z) = z/f(z)$; then the equation $f(z) = w$ can be rewritten as $z = g(z)w$, and its solution $z = \varphi(w) = f^{-1}(w)$ is given by the power series

$$\varphi(w) \;=\; \sum_{m=1}^{\infty} \frac{w^m}{m}\, [\zeta^{m-1}] g(\zeta)^m \tag{1.3}$$

and

$$h(\varphi(w)) \;=\; h(0) + \sum_{m=1}^{\infty} \frac{w^m}{m}\, [\zeta^{m-1}] h'(\zeta) g(\zeta)^m . \tag{1.4}$$

There is also an alternate form

$$h(\varphi(w)) \;=\; h(0) + \sum_{m=1}^{\infty} w^m\, [\zeta^m] h(\zeta) \bigl[g(\zeta)^m - \zeta g'(\zeta) g(\zeta)^{m-1} \bigr] . \tag{1.5}$$

Consider now the more general problem of solving $z = G(z,w)$, where $G(0,0) = 0$ and $|(\partial G/\partial z)(0,0)| < 1$. I shall prove here that its solution $z = \varphi(w)$ is given by the function series

$$\varphi(w) \;=\; \sum_{m=1}^{\infty} \frac{1}{m}\, [\zeta^{m-1}] G(\zeta,w)^m . \tag{1.6}$$

More generally, for any analytic function (or formal power series) $H(z,w)$, we have

$$H(\varphi(w),w) \;=\; H(0,w) + \sum_{m=1}^{\infty} \frac{1}{m}\, [\zeta^{m-1}] \frac{\partial H(\zeta,w)}{\partial \zeta}\, G(\zeta,w)^m \tag{1.7}$$

$$\;=\; H(0,w) + \sum_{m=1}^{\infty} [\zeta^m] H(\zeta,w) \Bigl[G(\zeta,w)^m - \zeta \, \frac{\partial G(\zeta,w)}{\partial \zeta}\, G(\zeta,w)^{m-1} \Bigr] . \tag{1.8}$$

The formulae (1.6)–(1.8) are manifestly generalizations of the Lagrange inversion formulae (1.3)–(1.5), to which they reduce when $G(z,w) = g(z)w$ and $H(z,w) = h(z)$. It turns out that the proof of (1.6)–(1.8) is, likewise, virtually a verbatim copy of the standard proof of (1.3)–(1.5): the variables w simply "go for the ride".

The problem of solving $z = G(z,w)$ can alternatively be interpreted as a fixed-point problem for the family of maps $z \mapsto G(z,w)$ parametrized by w. From this point of view, (1.6)–(1.8) are simply a function series giving the unique solution of this fixed-point problem under a suitable "Rouché-contraction" hypothesis [see hypothesis (c) of Theorem 2.3 below]. Once again, the variables w simply "go for the ride".

Before proving (1.6)–(1.8), let us observe how these formulae solve the implicit-function problem $F(z,w) = 0$, where $F(0,0) = 0$ and $(\partial F/\partial z)(0,0) \equiv a_{10} \neq 0$. It suffices to choose *any* analytic function $\gamma(z,w)$ satisfying $\gamma(0,0) \neq 0$, and then define

$$G(z,w) \;=\; z \,-\, \gamma(z,w)F(z,w)\;. \tag{1.9}$$

Clearly $F(z,w) = 0$ is equivalent to $z = G(z,w)$, at least locally in a neighborhood of $(z,w) = (0,0)$. Then $(\partial G/\partial z)(0,0) = 1 - \gamma(0,0)a_{10}$; so for (1.6)–(1.8) to be applicable, it suffices to arrange that $|1-\gamma(0,0)a_{10}| < 1$, which can easily be done by a suitable choice of $\gamma(0,0)$ [namely, by choosing $\gamma(0,0)$ to lie in the open disc of radius $1/|a_{10}|$ centered at $1/a_{10}$]. I wish to stress that each such choice of a function γ gives rise to a valid but different expansion (1.6)–(1.8) for the solution $z = \varphi(w)$. Even in the special case of the inverse-function problem $f(z) = w$, this flexibility exists and is useful (see Example 4.2 below). One important class of choices has $\gamma(0,0) = 1/a_{10}$, so that $(\partial G/\partial z)(0,0) = 0$; this latter condition leads to a slight simplification in the formulae (see Remark 2.4 below). This special class in turn contains two important subclasses:

- Yuzhakov [49] takes γ to be the constant function $1/a_{10}$.[1]
- Alternatively, we can choose γ so that $\gamma(z,0) = z/F(z,0)$ (this definition can still be extended to $w \neq 0$ in many different ways). Then $G(z,0) \equiv 0$, so that $G(z,w)$ has an overall factor w. This leads to a further slight simplification (see again Remark 2.4).

Conversely, the problem of solving $z = G(z,w)$ is of course equivalent to the problem of solving $\widetilde{F}(z,w) = 0$ if we set

$$\widetilde{F}(z,w) \;=\; z \,-\, G(z,w)\;; \tag{1.10}$$

and the condition $(\partial\widetilde{F}/\partial z)(0,0) \neq 0$ is satisfied whenever $(\partial G/\partial z)(0,0) \neq 1$. So our parametrized fixed-point problem has exactly the same level of generality as the implicit-function problem.

The plan of this paper is as follows: First I shall state and prove four versions of the formulae (1.6)–(1.8): one in terms of analytic functions (Theorem 2.3), and three in terms of formal power series (Theorems 3.5, 3.6 and 3.8). Then I shall give some examples and make some final remarks.

Since this paper is aimed at a diverse audience of analysts and combinatorialists, I have endeavored to give more detailed proofs than would otherwise be customary. I apologize in advance to experts for occasionally boring them with elementary observations.

2. Implicit function formula: Analytic version

In the analytic version of (1.6)–(1.8), the variable w simply “goes for the ride”; consequently, w need not be assumed to lie in $\mathbb{C}$, but can lie in the multidimensional complex space $\mathbb{C}^M$ or even in a general topological space W. I shall begin with a simple auxiliary result (Proposition 2.1) that clarifies the meaning of the hypotheses

[1] His subsequent generalization [50] [2, Theorem 20.2 and Proposition 20.4] in effect allows a preliminary change of variables $z' = z - \psi(w)$ for arbitrary ψ satisfying $\psi(0) = 0$.

of Theorem 2.3. I denote open and closed discs in $\mathbb{C}$ by the notations $\mathbb{D}_R = \{z \in \mathbb{C}\colon\ |z| < R\}$ and $\overline{\mathbb{D}}_R = \{z \in \mathbb{C}\colon\ |z| \le R\}$.

Proposition 2.1. *Let W be a topological space, let $R > 0$, and let $G\colon \mathbb{D}_R \times W \to \mathbb{C}$ be a jointly continuous function with the property that $G(\,\cdot\,, w)$ is analytic on $\mathbb{D}_R$ for each fixed $w \in W$. Suppose further that for some $w_0 \in W$ we have $G(0, w_0) = 0$ and $|(\partial G/\partial z)(0, w_0)| < 1$. Then for all sufficiently small $\rho > 0$ and $\epsilon > 0$ there exists an open neighborhood $V_\rho \ni w_0$ such that $|G(z, w)| \le (1-\epsilon)|z|$ whenever $|z| = \rho$ and $w \in V_\rho$.*

Proof. The function

$$f(z) \;=\; \begin{cases} G(z, w_0)/z & \text{if } 0 < |z| < R \\ (\partial G/\partial z)(0, w_0) & \text{if } z = 0 \end{cases} \tag{2.1}$$

is analytic on $\mathbb{D}_R$, hence continuous on $\mathbb{D}_R$; and by hypothesis $|f(0)| < 1$. It follows that for all sufficiently small $\rho > 0$ and $\epsilon > 0$, we have $|f(z)| \le 1 - 2\epsilon$ whenever $|z| \le \rho$. We now use the following simple topological fact:

Lemma 2.2. *If $F\colon X \times Y \to \mathbb{R}$ is continuous and X is compact, then the function $g\colon Y \to \mathbb{R}$ defined by $g(y) = \sup\limits_{x \in X} F(x, y)$ is continuous.*

Applying this with $X = \{z \in \mathbb{C}\colon\ |z| = \rho\}$ and $Y = W$, we conclude that there exists an open neighborhood $V_\rho \ni w_0$ such that $|G(z, w)/z| \le 1 - \epsilon$ whenever $|z| = \rho$ and $w \in V_\rho$. □

For completeness, let us give a proof of the lemma:

Proof of Lemma 2.2. First of all, the compactness of X guarantees that g is everywhere finite. Now, since the supremum of any family of continuous functions is lower semicontinuous, it suffices to prove that g is upper semicontinuous, i.e., that for any $y_0 \in Y$ and any $\epsilon > 0$ there exists an open neighborhood $V \ni y_0$ such that $g(y) < g(y_0) + \epsilon$ for all $y \in V$. To see this, first choose, for each $x \in X$, open neighborhoods $U_x \ni x$ and $V_x \ni y_0$ such that $F(x', y) < F(x,y_0) + \epsilon \le g(y_0) + \epsilon$ whenever $x' \in U_x$ and $y \in V_x$. By compactness, there exists a finite set $\{x_1, \ldots, x_n\}$ such that $\{U_{x_i}\}_{i=1}^n$ covers X. Setting $V = \bigcap\limits_{i=1}^{n} V_{x_i}$ gives the required neighborhood. □

It is easy to see that the conclusion of Lemma 2.2 need not hold if X is noncompact. For instance, with $X = \mathbb{R}$ and $Y = [0, 1]$, take $F(x, y) = xy$ or $F(x, y) = \tanh(xy)$.

We can now state the principal result of this section:

Theorem 2.3 (Implicit function formula — analytic version)**.**
Let V be a topological space, let $\rho > 0$, and let $G, H\colon\ \overline{\mathbb{D}}_\rho \times V \to \mathbb{C}$ be functions satisfying

(a) *G, $\partial G/\partial z$ and H are jointly continuous on $\overline{\mathbb{D}}_\rho \times V$;*
(b) *$G(\,\cdot\,, w)$ and $H(\,\cdot\,, w)$ are analytic on $\mathbb{D}_\rho$ for each fixed $w \in V$; and*
(c) *$|G(z, w)| < |z|$ whenever $|z| = \rho$ and $w \in V$.*

Then for each $w \in V$, there exists a unique $z \in \mathbb{D}_\rho$ satisfying $z = G(z,w)$. Furthermore, this $z = \varphi(w)$ depends continuously on w and is given explicitly by the function series

$$\begin{aligned} H(\varphi(w),w) &= H(0,w) + \sum_{m=1}^{\infty} \frac{1}{m} [\zeta^{m-1}] \frac{\partial H(\zeta,w)}{\partial \zeta} G(\zeta,w)^m & (2.2) \\ &= H(0,w) + \sum_{m=1}^{\infty} [\zeta^m] H(\zeta,w) \Big[G(\zeta,w)^m - \zeta \frac{\partial G(\zeta,w)}{\partial \zeta} G(\zeta,w)^{m-1} \Big] & (2.3) \end{aligned}$$

which are absolutely convergent on V, locally uniformly on V.[2]

If, in addition, V is an open subset of $\mathbb{C}^M$ and G is analytic on $\mathbb{D}_\rho \times V$, then φ is analytic on V.

Finally, if V is a polydisc (or more generally a complete Reinhardt domain) centered at $0 \in \mathbb{C}^M$, and G and H are analytic on $\mathbb{D}_\rho \times V$ and satisfy $G(0,0) = 0$, then $\varphi(w)$ is also given by the Taylor series

$$H(\varphi(w),w) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^M} c_{\boldsymbol{\alpha}} w^{\boldsymbol{\alpha}} \tag{2.4}$$

which is absolutely convergent on V, uniformly on compact subsets of V; here the coefficients $c_{\boldsymbol{\alpha}}$ are given by the absolutely convergent sums

$$\begin{aligned} c_{\boldsymbol{\alpha}} &= [w^{\boldsymbol{\alpha}}] H(0,w) + \sum_{m=1}^{\infty} \frac{1}{m} [\zeta^{m-1} w^{\boldsymbol{\alpha}}] \frac{\partial H(\zeta,w)}{\partial \zeta} G(\zeta,w)^m & (2.5) \\ &= [w^{\boldsymbol{\alpha}}] H(0,w) + \sum_{m=1}^{\infty} [\zeta^m w^{\boldsymbol{\alpha}}] H(\zeta,w) \Big[G(\zeta,w)^m - \zeta \frac{\partial G(\zeta,w)}{\partial \zeta} G(\zeta,w)^{m-1} \Big] . & (2.6) \end{aligned}$$

Remark 2.4. In many cases (2.5) is actually a finite sum. For instance, if $\frac{\partial G}{\partial z}(0,0) = 0$ — as occurs in particular in Yuzhakov's [2, 49, 50] approach — then each factor of $G(\zeta,w)$ brings either ζ^2 or w (at least); simple algebra then shows that the summand in (2.5) is nonvanishing only for $m \le 2|\boldsymbol{\alpha}| - 1$, where $|\boldsymbol{\alpha}| = \sum_{i=1}^{M} \alpha_i$.[3] Under the stronger hypothesis $G(z,0) \equiv 0$, each factor of $G(\zeta,w)$ brings at least one w, so the summand is nonvanishing only for $m \le |\boldsymbol{\alpha}|$.

Analogous comments hold for (2.6), where the conditions are $m \le 2|\boldsymbol{\alpha}|$ and $m \le |\boldsymbol{\alpha}|$, respectively. □

As previously stated, the proof of Theorem 2.3 is a trivial modification of the standard textbook proof of the Lagrange inversion formula [7,34,47], but for completeness let us give it in detail.

[2] "Locally uniformly" means that for each $w \in V$ there exists an open neighborhood $U \ni w$ on which the convergence is uniform. This implies, in particular, that the convergence is uniform on compact subsets of V (and is equivalent to it if V is locally compact).

[3] Yuzhakov [49, Proposition 1] [2, Proposition 20.4] writes $m \le 2|\boldsymbol{\alpha}|$, which is correct but not as strong as it should be.

Proof. Hypotheses (a)–(c) combined with Rouché's theorem imply that for each $w \in V$, the number of roots (including multiplicity) of $z - G(z, w) = 0$ in the disc $|z| < \rho$ is the same as the number of roots of $z = 0$ in this disc, namely one; so let us call this unique (and simple) root $z = \varphi(w)$.[4] It follows that for each $w \in V$, the function

$$H(\zeta, w) \, \frac{1 - \frac{\partial G}{\partial \zeta}(\zeta, w)}{\zeta - G(\zeta, w)} \tag{2.7}$$

is continuous on $|\zeta| \le \rho$ and analytic in $|\zeta| < \rho$ except for a simple pole at $\zeta = \varphi(w)$ with residue $H(\varphi(w), w)$.[5] Cauchy's integral formula therefore gives

$$H(\varphi(w), w) \;=\; \frac{1}{2\pi i} \oint\limits_{|\zeta| = \rho} H(\zeta, w) \, \frac{1 - \frac{\partial G}{\partial \zeta}(\zeta, w)}{\zeta - G(\zeta, w)} \, d\zeta \;. \tag{2.8}$$

Let us now expand

$$\frac{1}{\zeta - G(\zeta, w)} \;=\; \sum_{m=0}^{\infty} \frac{G(\zeta, w)^m}{\zeta^{m+1}} \tag{2.9}$$

and observe that this sum is absolutely convergent, uniformly for ζ on the circle $|\zeta| = \rho$ of integration and locally uniformly for $w \in V$ [this follows from hypothesis (c) and Lemma 2.2]. We therefore have

$$H(\varphi(w), w) \;=\; \sum_{m=0}^{\infty} \frac{1}{2\pi i} \oint\limits_{|\zeta| = \rho} \frac{H(\zeta, w)}{\zeta^{m+1}} \left[1 - \frac{\partial G}{\partial \zeta}(\zeta, w) \right] G(\zeta, w)^m \, d\zeta \;. \tag{2.10}$$

By the Cauchy integral formula, this gives

$$\begin{aligned} H(\varphi(w), w) \;&=\; \sum_{m=0}^{\infty} [\zeta^m] H(\zeta, w) G(\zeta, w)^m \;-\; \sum_{m=0}^{\infty} [\zeta^m] H(\zeta, w) \, \frac{\partial G}{\partial \zeta} \, G(\zeta, w)^m \\ &=\; H(0, w) \;+\; \sum_{m=1}^{\infty} [\zeta^m] H(\zeta, w) G(\zeta, w)^m \\ &\qquad\qquad -\; \sum_{m=1}^{\infty} [\zeta^{m-1}] H(\zeta, w) \, \frac{\partial G}{\partial \zeta} \, G(\zeta, w)^{m-1} \;. \end{aligned} \tag{2.11}$$

This proves the alternate formula (1.8)/(2.3), in which the sum is absolutely convergent on V, locally uniformly on V.

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[4]It is well known (and is easily proved using Rouché's theorem and Lemma 2.2) that this root depends continuously on w. This will also follow from the explicit formula (2.2)/(2.3).

[5]It is in this step that we use the continuity of $\partial G / \partial z$ (as well as that of G and H) on the closed disc $\overline{\mathbb{D}}_\rho$.

To prove (1.7)/(2.2), we start from (2.10) and prepare an integration by parts:

$$\begin{aligned}
\frac{H(\zeta,w)}{\zeta^{m+1}}\frac{\partial G}{\partial \zeta}G^m &= \frac{1}{m+1}\frac{H(\zeta,w)}{\zeta^{m+1}}\frac{\partial}{\partial\zeta}(G^{m+1}) \\
&= \frac{\partial}{\partial\zeta}\biggl(\frac{1}{m+1}\frac{H(\zeta,w)}{\zeta^{m+1}}G^{m+1}\biggr) - \frac{G^{m+1}}{m+1}\frac{\partial}{\partial\zeta}\biggl(\frac{H(\zeta,w)}{\zeta^{m+1}}\biggr) \\
&= \frac{\partial}{\partial\zeta}\biggl(\frac{1}{m+1}\frac{H(\zeta,w)}{\zeta^{m+1}}G^{m+1}\biggr) \\
&\qquad\qquad - G^{m+1}\biggl(\frac{H'(\zeta,w)}{(m+1)\zeta^{m+1}} - \frac{H(\zeta,w)}{\zeta^{m+2}}\biggr)
\end{aligned} \tag{2.12}$$

where the prime denotes $\partial/\partial\zeta$. Since the total derivative gives zero when integrated around a closed contour, we have

$$\begin{aligned}
&H(\varphi(w),w) \\
&\quad = \sum_{m=0}^{\infty}\frac{1}{2\pi i}\oint\limits_{|\zeta|=\rho}\biggl[\frac{H(\zeta,w)}{\zeta^{m+1}}G^m + \frac{H'(\zeta,w)}{(m+1)\zeta^{m+1}}G^{m+1} - \frac{H(\zeta,w)}{\zeta^{m+2}}G^{m+1}\biggr]\,d\zeta\;.
\end{aligned} \tag{2.13}$$

Now the first and third terms in brackets cancel when summed over m, except for the first term at $m = 0$, which gives simply $(1/2\pi i)\oint[H(\zeta,w)/\zeta]\,d\zeta = H(0,w)$. Hence

$$\begin{aligned}
H(\varphi(w),w) &= H(0,w) + \sum_{m=0}^{\infty}\frac{1}{2\pi i}\oint\frac{H'(\zeta,w)}{(m+1)\zeta^{m+1}}G(\zeta,w)^{m+1}\,d\zeta \\
&= H(0,w) + \sum_{m=1}^{\infty}\frac{1}{2\pi i}\oint\frac{H'(\zeta,w)}{m\zeta^{m}}G(\zeta,w)^{m}\,d\zeta \\
&= H(0,w) + \sum_{m=1}^{\infty}\frac{1}{m}\,[\zeta^{m-1}]H'(\zeta,w)G(\zeta,w)^m\;.
\end{aligned} \tag{2.14}$$

This proves the fundamental formula (1.7)/(2.2), in which the sum is absolutely convergent on V, locally uniformly on V.

It follows from this formula [taking $H(z,w) = z$] that if $V \subset \mathbb{C}^M$ and G is analytic, then φ is analytic as well; and if also H is analytic, then so is $w \mapsto H(\varphi(w),w)$.

Finally, if V is a polydisc (or more generally a complete Reinhardt domain) centered at $0 \in \mathbb{C}^M$, and G and H are analytic, then the analytic function $H(\varphi(w),w)$ is given in V by a convergent Taylor series. The coefficients of this Taylor series are given by (2.5)/(2.6) because the absolutely convergent function series (2.2)/(2.3) can be differentiated term-by-term. □

Remark 2.5. The following alternative calculation provides a slightly slicker proof of Theorem 2.3: start from (2.8) and write

$$H(\zeta,w)\,\frac{1-\frac{\partial G}{\partial\zeta}(\zeta,w)}{\zeta - G(\zeta,w)} = H(\zeta,w)\biggl(\frac{1}{\zeta} + \frac{\partial}{\partial\zeta}\log\biggl[1 - \frac{G(\zeta,w)}{\zeta}\biggr]\biggr)\;, \tag{2.15}$$

where hypothesis (c) and Taylor expansion guarantee that the function

$$\log[1 - G(\zeta, w)/\zeta]$$

is well-defined and single-valued on the circle $|\zeta| = \rho$; furthermore, a simple compactness argument extends this to some annulus $\rho_1 < |\zeta| \le \rho$ (locally uniformly in w). Now, the first term in (2.15), when integrated, yields $H(0, w)$. To handle the second term, let us integrate by parts, Taylor-expand the log, and then extract the residue from the resulting Laurent series: we get

$$\begin{aligned} -\frac{1}{2\pi i} \oint H'(\zeta, w) \log\left[1 - \frac{G(\zeta, w)}{\zeta}\right] d\zeta \;&=\; \sum_{m=1}^{\infty} \frac{1}{m} [\zeta^{-1}] H'(\zeta, w) \left(\frac{G(\zeta, w)}{\zeta}\right)^m \\ &=\; \sum_{m=1}^{\infty} \frac{1}{m} [\zeta^{m-1}] H'(\zeta, w) G(\zeta, w)^m \;, \qquad (2.16) \end{aligned}$$

which is (1.7)/(2.2). A similar argument without integration by parts yields the alternate formula (1.8)/(2.3). I thank Alex Eremenko for helpful comments concerning this proof. □

Remark 2.6. Formula (1.7)/(2.2) can alternatively be deduced from the standard Lagrange inversion formula (1.4) by an argument due to Ira Gessel [42, p. 148]: Introduce a new parameter $t \in \mathbb{C}$, and study the equation $z = tG(z, w)$ with solution $z = \Phi(w, t)$. Applying (1.4) with w fixed and t as the variable, we obtain

$$H(\Phi(w, t), w) \;=\; H(0, w) + \sum_{m=1}^{\infty} \frac{t^m}{m} [\zeta^{m-1}] \frac{\partial H(\zeta, w)}{\partial \zeta} G(\zeta, w)^m \;. \qquad (2.17)$$

Setting $t = 1$ yields (1.7)/(2.2). [Hypothesis (c) guarantees that $\sup_{|\zeta|=\rho} |G(\zeta, w)/\zeta| \equiv C < 1$, so that the Lagrange series is convergent for $|t| < 1/C$.] An analogous argument starting from (1.5) yields the alternate formula (1.8)/(2.3). □

Remark 2.7. For some purposes the disc $\mathbb{D}_\rho$ can be replaced an arbitrary domain (connected open set) $D \subset \mathbb{C}$. Hypothesis (c) is then replaced by the assumption that the image $G(D, w)$ is relatively compact in D for all $w \in V$. Under this hypothesis, the map $z \mapsto G(z, w)$ is a strict contraction in the Poincaré metric [26] on $G(D, w)$, i.e., satisfies $d_{\mathrm{Poin}}(G(z_1, w), G(z_2, w)) \le \kappa d_{\mathrm{Poin}}(z_1, z_2)$ for some $\kappa < 1$ (locally uniformly in w). It then follows from the contraction-mapping fixed-point theorem that there is a unique fixed point $\varphi(w)$, which moreover can be obtained by iteration starting at any point of D. That is, if we define

$$G_0(\zeta, w) \;=\; \zeta \qquad (2.18a)$$

$$G_{n+1}(\zeta, w) \;=\; G(G_n(\zeta, w), w) \qquad (2.18b)$$

then $G_n(\zeta, w) \to \varphi(w)$, uniformly for $\zeta \in D$ and locally uniformly for $w \in V$. It would be interesting to know whether this can be used to provide a function series analogous to (1.6)–(1.8) based on the Taylor coefficients of G at an arbitrary point $\zeta_0 \in D$. An analogous argument works for domains $D \subset \mathbb{C}^N$, using the Kobayashi metric [22, 24–26, 30], provided that some iterate $G^n(D, w)$ is Kobayashi-hyperbolic. I thank Alex Eremenko for suggesting the use of the Poincaré and Kobayashi metrics. □

Question 2.8. Can Theorem 2.3 be generalized to allow $H(z,w)$ to have a pole at $z = 0$? Please note that by linearity it suffices to consider $H(z,w) = h(w)z^{-k}$ $(k \ge 1)$; and since $h(w)$ just acts as an overall prefactor, it suffices to consider simply $H(z,w) = z^{-k}$. Of course we will somehow have to restrict attention to the subset of V where $\varphi(w) \neq 0$; and some hypothesis will be needed to guarantee that this subset is nonempty, i.e., that φ is not identically zero. □

3. Implicit function formula: Formal-power-series version

In this section we shall consider $G(z,w)$ to be a formal power series in indeterminates z and $w = (w_i)_{i\in I}$, where I is an arbitrary finite or infinite index set. The coefficients in this formal power series may belong to an arbitrary commutative ring-with-identity-element R. (For some purposes we will want to assume further that the coefficient ring R contains the rationals as a subring. In applications, R will usually be a field of characteristic 0 — e.g. the rationals $\mathbb{Q}$, the reals $\mathbb{R}$, or the complex numbers $\mathbb{C}$ — or a ring of polynomials or formal power series over such a field.) See [8, 16, 18, 21, 36] for basic facts about formal power series. We recall that $R[[w]]$ denotes the ring of formal power series in the indeterminates $w = (w_i)_{i\in I}$ with coefficients in R.

We begin with a well-known proposition asserting the existence and uniqueness of the formal power series $\varphi(w)$ solving the equation $z = G(z,w)$, or equivalently $F(z,w) = 0$. Here R can be an arbitrary commutative ring with identity element; it need not contain the rationals or even be of characteristic 0. (For instance, the ring $\mathbb{Z}_n$ of integers modulo n is allowed.)

Proposition 3.1 (Implicit function theorem for formal power series)**.**

(a) *Let R be a commutative ring with identity element. Let $F(z,w)$ be a formal power series in indeterminates z and $w = (w_i)_{i\in I}$, with coefficients in R; suppose further that $F(0,0) = 0$ and that $(\partial F/\partial z)(0,0)$ is invertible in the ring R. Then there exists a unique formal power series $\varphi(w)$ with zero constant term satisfying $F(\varphi(w),w) = 0$.*

(b) *Let R be a commutative ring with identity element. Let $G(z,w)$ be a formal power series in indeterminates z and $w = (w_i)_{i\in I}$, with coefficients in R; suppose further that $G(0,0) = 0$ and that $1 - (\partial G/\partial z)(0,0)$ is invertible in the ring R. Then there exists a unique formal power series $\varphi(w)$ with zero constant term satisfying $\varphi(w) = G(\varphi(w),w)$.*

If the ring R is a field, then the hypothesis that $(\partial F/\partial z)(0,0)$ be invertible in R means simply that $(\partial F/\partial z)(0,0) \neq 0$. If R is a ring of formal power series over a field, then this hypothesis means that the constant term of $(\partial F/\partial z)(0,0)$ is $\neq 0$. Analogous statements apply to part (b), with $(\partial G/\partial z)(0,0) \neq 1$.

Proof of Proposition 3.1. It suffices to prove either (a) or (b), since they are equivalent under the substitution $F(z,w) = z - G(z,w)$. We shall prove (b). Let us write

$$G(z,w) \;=\; \sum_{k=0}^{\infty} g_k(w)\, z^k \tag{3.1}$$

with each $g_k \in R[[w]]$; by hypothesis $g_0(0) = 0$, and $1 - g_1(0)$ is invertible in the ring R. The equation $\varphi(w) = G(\varphi(w), w)$ can now be written as

$$\varphi(w) \;=\; \sum_{k=0}^{\infty} g_k(w)\, \varphi(w)^k \tag{3.2}$$

or equivalently

$$\varphi(w) \;=\; [1 - g_1(0)]^{-1} \left[g_0(w) \;+\; [g_1(w) - g_1(0)]\, \varphi(w) \;+\; \sum_{k=2}^{\infty} g_k(w)\, \varphi(w)^k \right] . \tag{3.3}$$

Since the series $g_0(w)$, $g_1(w) - g_1(0)$ and $\varphi(w)$ all have zero constant term, we see from (3.3) that the coefficients in $\varphi(w) = \sum_{|\boldsymbol{\alpha}| \ge 1} b_{\boldsymbol{\alpha}} w^{\boldsymbol{\alpha}}$ can be uniquely determined by induction on $|\boldsymbol{\alpha}|$. Conversely, the unique solution of this system of equations necessarily solves $\varphi(w) = G(\varphi(w), w)$. □

A multidimensional generalization of Proposition 3.1, in which z is replaced by a vector of indeterminates $(z_1, \ldots, z_N)$, can be found in Bourbaki [5, p. A.IV.37]. As one might expect, the hypothesis is that the Jacobian determinant $\det(\partial F/\partial z)(0, 0)$ is invertible in R.

We can carry this argument further and provide an explicit formula for $\varphi(w)$. Let us begin with what appears to be a special case, but in fact contains the general result: namely, let us take $G(z, \boldsymbol{g}) = \sum_{n=0}^{\infty} g_n z^n$ where $\boldsymbol{g} = (g_n)_{n=0}^{\infty}$ are indeterminates. We then have the following "universal" version of the Lagrange inversion formula [13, Theorem 6.2]:

Proposition 3.2 (Universal Lagrange inversion formula). *Let $\boldsymbol{g} = (g_n)_{n=0}^{\infty}$ be indeterminates. There is a unique formal power series $\varphi \in \mathbb{Z}[[\boldsymbol{g}]]$ with zero constant term satisfying $\varphi(\boldsymbol{g}) = \sum\limits_{n=0}^{\infty} g_n \varphi(\boldsymbol{g})^n$, and its coefficients are given explicitly by*

$$[g_0^{k_0} g_1^{k_1} g_2^{k_2} \cdots]\, \varphi(\boldsymbol{g})^\ell \;=\; \begin{cases} \ell \, \dfrac{\Big(\sum\limits_{n=0}^{\infty} k_n \,-\, 1\Big)!}{\prod\limits_{n=0}^{\infty} k_n!} & \textit{if } \sum\limits_{n=0}^{\infty} (n-1) k_n = -\ell \\ 0 & \textit{otherwise} \end{cases} \tag{3.4}$$

for all integers $\ell \ge 1$. In particular, for each pair (k_0, k_1) there are finitely many *$(k_2, k_3, \ldots)$ for which (3.4) is nonzero, so that $\varphi(\boldsymbol{g})^\ell$ is a formal power series in g_0, g_1 whose coefficients are* polynomials *in $g_2, g_3, \ldots$: that is, $\varphi \in \mathbb{Z}[g_2, g_3, \ldots][[g_0, g_1]]$.*

We can also write the formula

$$\varphi(\boldsymbol{g})^\ell \;=\; \sum_{m=1}^{\infty} [\zeta^{m-\ell}] \left(\sum_{n=0}^{\infty} g_n \zeta^n \right)^{m-1} \left(\sum_{n=0}^{\infty} (1-n) g_n \zeta^n \right) , \tag{3.5}$$

and in the ring $\mathbb{Q}[[\boldsymbol{g}]]$ we can write

$$\varphi(\boldsymbol{g})^\ell \;=\; \sum_{m=1}^{\infty} \frac{\ell}{m}\, [\zeta^{m-\ell}] \left(\sum_{n=0}^{\infty} g_n \zeta^n \right)^{m} . \tag{3.6}$$

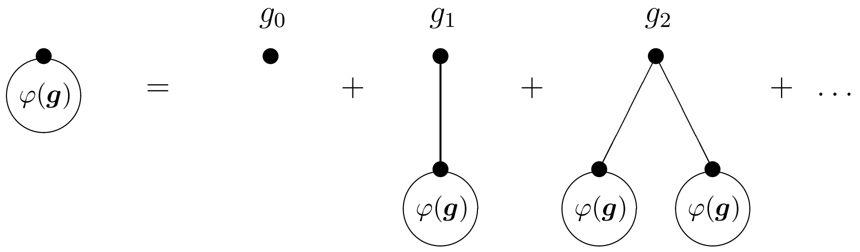

FIGURE 1. The functional equation $\varphi(\boldsymbol{g}) = \sum_{n=0}^{\infty} g_n \varphi(\boldsymbol{g})^n$ for the generating function of unlabeled plane trees.

Proof. The functional equation $\varphi(\boldsymbol{g}) = \sum_{n=0}^{\infty} g_n \varphi(\boldsymbol{g})^n$ is the same equation as is satisfied by the (ordinary) generating function for *unlabeled plane trees* (i.e., rooted trees in which the vertices are unlabeled but the subtrees at each vertex are linearly ordered), in which a vertex having n children gets a weight g_n, and the weight of a tree is the product of its vertex weights (see Figure 1).[6] Since the solution of this equation is unique by Proposition 3.1(b), it follows that $\varphi(\boldsymbol{g})$ is the generating function for unlabeled plane trees with this weighting. More generally, $\varphi(\boldsymbol{g})^\ell$ is the generating function for *unlabeled plane forests* with ℓ components, with the same weighting.[7] It is a well-known (though nontrivial) combinatorial fact [42, Theorem 5.3.10] that the number of unlabeled plane forests with ℓ components having type sequence $(k_0, k_1, \ldots)$ [i.e., in which there are k_n vertices having n children, for each $n \ge 0$] is given precisely by (3.4).

Since the constraint in (3.4) can be written as $k_0 = \ell + \sum_{n=2}^{\infty} (n-1) k_n$, one sees immediately that for each pair (k_0, k_1) there are finitely many $(k_2, k_3, \ldots)$ for which (3.4) is nonzero.

To prove (3.5), let us expand out the summand on the right-hand side, choosing $n = N$ in the last factor: we get

$$[\zeta^{m-\ell}] \left(\sum_{n=0}^{\infty} g_n \zeta^n \right)^{m-1} \left(\sum_{n=0}^{\infty} (1-n) g_n \zeta^n \right)$$
$$= \sum_{N=0}^{\infty} [\zeta^{m-\ell}] \sum_{\substack{k_0, k_1, k_2, \ldots \ge 0 \\ \sum k_n = m}} \binom{m-1}{k_0, \ldots, k_{N-1}, k_N - 1, k_{N+1}, \ldots} (1-N) \left(\prod_{n=0}^{\infty} g_n^{k_n} \right) \zeta^{\sum n k_n},$$

[6]The key fact here is that if Σ is any sum of terms, then a term in the expansion of $\Sigma^n = \Sigma\Sigma \cdots \Sigma$ is obtained by choosing, *in order*, a term of Σ for the first factor, a term of Σ for the second factor, etc. This is why one obtains trees in which the subtrees at each vertex are linearly ordered.

[7]An *unlabeled plane forest* is a forest of rooted trees with unlabeled vertices in which the subtrees at each vertex are linearly ordered and the components of the forest (or equivalently their roots) are also linearly ordered. The reasoning in the preceding footnote explains why $\varphi(\boldsymbol{g})^\ell$ gives rise to forests in which the components are linearly ordered.

where
$$\binom{m-1}{k_0,\ldots,k_{N-1},k_N-1,k_{N+1},\ldots} = \frac{(m-1)!\,k_N}{\prod\limits_{n=0}^{\infty} k_n!}$$
is a multinomial coefficient. The constraints tell us that $\sum_{n=0}^{\infty} k_n = m$ and $\sum_{n=0}^{\infty} nk_n = m-\ell$, so that $\sum_{N=0}^{\infty}(1-N)k_N = \ell$. This is precisely (3.4).

If work over the rationals rather than the integers, things become slightly easier: expanding out the summand in (3.6), we obtain
$$\frac{\ell}{m}\,[\zeta^{m-\ell}]\left(\sum_{n=0}^{\infty} g_n\zeta^n\right)^{m} = \frac{\ell}{m}\,[\zeta^{m-\ell}] \sum_{\substack{k_0,k_1,k_2,\ldots\ge 0\\ \sum k_n=m}} \binom{m}{k_0,k_1,k_2,\ldots}\left(\prod_{n=0}^{\infty} g_n^{k_n}\right)\zeta^{\sum nk_n}\,, \tag{3.7}$$
which again matches (3.4). □

Remark 3.3. To see easily (i.e., without the full combinatorial interpretation) that the numbers (3.4) are indeed integers, it suffices to note that
$$k_i\,\frac{\left(\sum k_n\,-\,1\right)!}{\prod k_n!} \;=\; \binom{\sum k_n\,-\,1}{k_1,\ldots,k_{i-1},k_i-1,k_{i+1},\ldots} \tag{3.8}$$
is a multinomial coefficient and hence an integer, and that $\ell = \sum_{i=0}^{\infty}(1-i)k_i$ by virtue of the constraint. I thank Richard Stanley and Ira Gessel (independently) for this observation. □

Remark 3.4. Let us stress that the proof of the counting result given in [42, Theorem 5.3.10] is purely combinatorial; it is based on a bijection between unlabeled plane forests and a certain class of words on a finite alphabet. We shall use it to deduce the implicit function formula (and in particular the Lagrange inversion formula) as a simple corollary, following the second proof of [42, Theorem 5.4.2]. This approach to the Lagrange inversion formula goes back to Raney [37] and was later simplified by other authors [33,38]. On the other hand, a much easier (though perhaps less enlightening) way of obtaining this counting result is to first prove the Lagrange inversion formula (e.g. by the algebraic argument given in the first proof of [42, Theorem 5.4.2]) and then use it to obtain the enumeration of plane trees or forests as a straightforward application [45] [16, Section 2.7.7]. □

We can now deduce the general implicit function formula as an easy corollary of Proposition 3.2. The key point is that $\varphi(\boldsymbol{g})^\ell$ is a formal power series in g_0, g_1 whose coefficients are *polynomials* in $g_2, g_3, \ldots$: therefore, in the identity $\varphi(\boldsymbol{g}) = \sum_{n=0}^{\infty} g_n\varphi(\boldsymbol{g})^n$ we can make the substitutions $g_n \leftarrow g_n(w)$, where the $g_n(w)$ are formal power series in an arbitrary collection $w = (w_i)_{i\in I}$ of indeterminates, provided that $g_0(w)$ and $g_1(w)$ have zero constant term; the series $g_n(w)$ for $n \ge 2$ are unrestricted. This yields a formal-power-series version of (1.8):

Theorem 3.5 (Implicit function formula — formal-power-series version #1)**.** *Let R be a commutative ring with identity element. Let $G(z,w) = \sum_{n=0}^{\infty} g_n(w)\,z^n$ be a formal power series in indeterminates z and $w = (w_i)_{i\in I}$, with coefficients in R, satisfying $G(0,0) = 0$ and $(\partial G/\partial z)(0,0) = 0$ [i.e., $g_0(0) = 0$ and*

$g_1(0) = 0$]. Then the unique formal power series $\varphi(w)$ with zero constant term satisfying $\varphi(w) = G(\varphi(w), w)$ is given explicitly by

$$\varphi(w)^\ell \;=\; \sum_{k_0,k_1,k_2,\ldots \ge 0} c_\ell(k_0,k_1,k_2,\ldots) \prod_{n=0}^{\infty} g_n(w)^{k_n} \tag{3.9}$$

for all integers $\ell \ge 1$, where $c_\ell(k_0,k_1,k_2,\ldots)$ is given by (3.4). Its coefficients are given by the finite sums

$$[w^{\boldsymbol{\alpha}}]\varphi(w) \;=\; \sum_{m=1}^{2|\boldsymbol{\alpha}|} [\zeta^m w^{\boldsymbol{\alpha}}] \Big[G(\zeta,w)^m \;-\; \zeta \, \frac{\partial G(\zeta,w)}{\partial \zeta} \, G(\zeta,w)^{m-1} \Big] \,. \tag{3.10}$$

More generally, if $H(z,w)$ is any formal power series, we have

$$\begin{aligned} &[w^{\boldsymbol{\alpha}}]H(\varphi(w),w) \\ &\quad= \; [w^{\boldsymbol{\alpha}}]H(0,w) \,+\, \sum_{m=1}^{2|\boldsymbol{\alpha}|} [\zeta^m w^{\boldsymbol{\alpha}}] H(\zeta,w) \Big[G(\zeta,w)^m \;-\; \zeta \, \frac{\partial G(\zeta,w)}{\partial \zeta} \, G(\zeta,w)^{m-1} \Big] \,. \end{aligned} \tag{3.11}$$

Proof. As just observed, substituting $g_n \leftarrow g_n(w)$ in (3.4) proves (3.9).

To prove (3.11), suppose first that $H(z,w) = z^\ell$. If $\ell = 0$, then obviously $\varphi(w)^\ell = 1$. If $\ell \ge 1$, we can substitute $g_n \leftarrow g_n(w)$ in (3.5) to obtain

$$\varphi(w)^\ell \;=\; \sum_{m=1}^{\infty} [\zeta^m] \, \zeta^\ell \Big[G(\zeta,w)^m \;-\; \zeta \, \frac{\partial G(\zeta,w)}{\partial \zeta} \, G(\zeta,w)^{m-1} \Big] \,. \tag{3.12}$$

The general case of $H(z,w) = \sum_{\ell=0}^{\infty} h_\ell(w) z^\ell$ is obtained from these special cases by linearity: we have

$$H(\varphi(w),w) \;=\; H(0,w) \,+\, \sum_{m=1}^{\infty} [\zeta^m] H(\zeta,w) \Big[G(\zeta,w)^m \;-\; \zeta \, \frac{\partial G(\zeta,w)}{\partial \zeta} \, G(\zeta,w)^{m-1} \Big] \,. \tag{3.13}$$

Now take the coefficient of $w^{\boldsymbol{\alpha}}$ on both sides, and observe that contributions come only from $m \le 2|\boldsymbol{\alpha}|$ (see Remark 2.4). □

The implicit function formulae given in Theorem 3.5 — in particular, the variant Lagrange formula (3.10)/(3.11) — are valid in an arbitrary commutative ring-with-identity-element R, even if R is not of characteristic 0, because the numerical coefficients arising in (3.9)–(3.11) are all integers. On the other hand, if we are willing to assume that the ring R contains the rationals as a subring, then we can deduce the slightly more convenient explicit formulae (1.6)/(1.7) for $\varphi(w)$. An argument completely analogous to that leading to (3.11), but based on (3.6) instead of (3.5), proves:

Theorem 3.6 (Implicit function formula — formal-power-series version #2)**.** *Let R be a commutative ring containing the rationals as a subring. Let $G(z,w)$ be a formal power series in indeterminates z and $w = (w_i)_{i \in I}$, with coefficients in R, satisfying $G(0,0) = 0$ and $(\partial G/\partial z)(0,0) = 0$. Then there exists a unique formal* *power series* $\varphi(w)$ *with zero constant term satisfying* $\varphi(w) = G(\varphi(w), w)$*, and its coefficients are given by the finite sums*

$$[w^{\boldsymbol{\alpha}}]\varphi(w) \;=\; \sum_{m=1}^{2|\boldsymbol{\alpha}|-1} \frac{1}{m}\,[\zeta^{m-1}w^{\boldsymbol{\alpha}}]G(\zeta,w)^m \;. \tag{3.14}$$

More generally, if $H(z,w)$ *is any formal power series, we have*

$$[w^{\boldsymbol{\alpha}}]H(\varphi(w),w) \;=\; [w^{\boldsymbol{\alpha}}]H(0,w) \,+\, \sum_{m=1}^{2|\boldsymbol{\alpha}|-1} \frac{1}{m}\,[\zeta^{m-1}w^{\boldsymbol{\alpha}}]\frac{\partial H(\zeta,w)}{\partial\zeta}\,G(\zeta,w)^m \;. \tag{3.15}$$

It is instructive to give two alternate proofs of Theorem 3.6: one deducing the result as a corollary of the usual Lagrange inversion formula for formal power series, and one deducing it from our analytic version of the implicit function formula (Theorem 2.3).

Second Proof of Theorem 3.6. Proposition 3.1(b) gives the existence and uniqueness of $\varphi(w)$. [The proof of (3.14)/(3.15) to be given next will provide an alternate proof of uniqueness.] Now we deduce (3.15) by using the formal-power-series version of Gessel's argument [42, p. 148] mentioned in Remark 2.6: Introduce a new indeterminate t, and study the equation $z = tG(z,w)$ with solution $z = \Phi(w,t)$. Using the Lagrange inversion formula (1.4) for formal power series in the indeterminate t and with coefficients in the ring $R[[w]]$, we obtain[8]

$$H(\Phi(w,t),w) \;=\; H(0,w) \,+\, \sum_{m=1}^{\infty} \frac{t^m}{m}\,[\zeta^{m-1}]\frac{\partial H(\zeta,w)}{\partial\zeta}\,G(\zeta,w)^m \tag{3.16}$$

and hence

$$[w^{\boldsymbol{\alpha}}]H(\Phi(w,t),w) \;=\; [w^{\boldsymbol{\alpha}}]H(0,w) \,+\, \sum_{m=1}^{\infty} \frac{t^m}{m}\,[\zeta^{m-1}w^{\boldsymbol{\alpha}}]\frac{\partial H(\zeta,w)}{\partial\zeta}\,G(\zeta,w)^m \tag{3.17}$$

as equalities between formal power series in t. But by the hypothesis on G, the only nonzero contributions to the sum on the right-hand side of (3.17) come from $m \le 2|\boldsymbol{\alpha}| - 1$ (see Remark 2.4), so each side of (3.17) is in fact a *polynomial* in t (of degree at most $2|\boldsymbol{\alpha}| - 1$). So we can evaluate it at any chosen $t \in R$, in particular at $t = 1$. This proves (3.15). □

Third Proof of Theorem 3.6. We see from the proof of Proposition 3.1 that $\varphi(w)^\ell$ will be given by a universal formula of the form (3.9) with nonnegative integer coefficients $c_\ell(k_0, k_1, k_2, \ldots)$; it remains only to find these coefficients. To do this, it suffices to consider the case $R = \mathbb{C}$ with a single indeterminate w.[9] It furthermore suffices to consider the cases in which $G(z,w)$ is a *polynomial* in z and w (of arbitrarily high degree), since $[w^n]\varphi(w)$ depends only on the $[z^j w^k]G(z,w)$ with $j,k \le n$. But in this

[8]The Lagrange inversion formula for formal power series is most commonly stated for series with coefficients in a field of characteristic 0 [42, Theorem 5.4.2], but in the form (1.3)/(1.4) it also holds for series with coefficients in an arbitrary commutative ring containing the rationals. See e.g. [16, Theorem 1.2.4].

[9]Here it is important that $\mathbb{C}$ is of characteristic 0, in order to avoid losing information about the integers $c_\ell(k_0, k_1, k_2, \ldots)$.

case we can apply the analytic version of the implicit function formula (Theorem 2.3). $\square$

Question 3.7. Can Theorems 3.5 and 3.6 be generalized to allow $H(z,w)$ to be a *Laurent* series in z, at least when w is a single indeterminate and $b_{01} \equiv (\partial G/\partial w)(0,0)$ is invertible in the ring R? (Or slightly more restrictively, when R is a field of characteristic zero and $b_{01} \neq 0$?) Please note that by linearity it suffices to consider $H(z,w) = h(w)z^{-k}$ ($k \geq 1$); and since $h(w)$ just acts as an overall prefactor, it suffices to consider simply $H(z,w) = z^{-k}$. One might try imitating [42, first proof of Theorem 5.4.2], possibly combined with the Gessel idea $z = tG(z,w)$. $\square$

In Theorems 3.5 and 3.6, we have for simplicity assumed that $b_{10} \equiv (\partial G/\partial z)(0,0) = g_1(0)$ is *zero*. This seems more restrictive than Proposition 3.1, where it was assumed only that $1-b_{10}$ is invertible in the ring R, but there is in fact no real loss of generality here. For if $1-b_{10}$ is invertible, then the equation $z = G(z,w)$ is equivalent to $z = \widetilde{G}(z,w)$, where

$$\widetilde{G}(z,w) \;=\; (1-b_{10})^{-1}\,[G(z,w) - b_{10}z] \tag{3.18}$$

satisfies $(\partial\widetilde{G}/\partial z)(0,0) = 0$. We can therefore apply Theorems 3.5 and 3.6 with

$$\widetilde{g}_n(w) \;=\; [1-g_1(0)]^{-1} \times \left\{ \begin{array}{ll} g_n(w) & \text{if } n \neq 1 \\ g_1(w) - g_1(0) & \text{if } n = 1 \end{array} \right\} . \tag{3.19}$$

On the other hand, if $R = \mathbb{R}$ or $\mathbb{C}$ and $|b_{10}| < 1$, we can avoid this preliminary transformation if we prefer: the power-series coefficients will then be given by absolutely convergent infinite sums, which are nothing other than the finite sums based on $\widetilde{G}$ in which each factor $(1-b_{10})^{-m}$ has been expanded out as $\sum\limits_{k=0}^{\infty} \binom{m+k-1}{k} b_{10}^k$. We therefore have:

Theorem 3.8 (Implicit function formula — formal-power-series version #3)**.** *Let R be either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Let $G(z,w)$ be a formal power series in indeterminates z and $w = (w_i)_{i\in I}$, with coefficients in R, satisfying $G(0,0) = 0$ and $|(\partial G/\partial z)(0,0)| < 1$. Then there exists a unique formal power series $\varphi(w)$ with zero constant term satisfying $\varphi(w) = G(\varphi(w),w)$, and its coefficients are given by the absolutely convergent sums*

$$[w^{\boldsymbol{\alpha}}]\varphi(w) \;=\; \sum_{m=1}^{\infty} \frac{1}{m}\,[\zeta^{m-1}w^{\boldsymbol{\alpha}}]G(\zeta,w)^m \,. \tag{3.20}$$

More generally, if $H(z,w)$ is any formal power series, we have

$$[w^{\boldsymbol{\alpha}}]H(\varphi(w),w) \;=\; [w^{\boldsymbol{\alpha}}]H(0,w) + \sum_{m=1}^{\infty} \frac{1}{m}\,[\zeta^{m-1}w^{\boldsymbol{\alpha}}]\frac{\partial H(\zeta,w)}{\partial\zeta}\,G(\zeta,w)^m \,. \tag{3.21}$$

4. Some examples

Example 4.1. Let $G(z,w) = \alpha(w)z + \beta(w)$ with $\beta(0) = 0$. Then (1.6) tells us that

$$\varphi(w) \;=\; \sum_{m=1}^{\infty} \frac{1}{m}\,[\zeta^{m-1}]\,[\alpha(w)\zeta + \beta(w)]^m \;=\; \sum_{m=1}^{\infty} \alpha(w)^{m-1}\beta(w) \,, \tag{4.1}$$

which sums to the correct answer $\beta(w)/[1-\alpha(w)]$ provided that $|\alpha(w)| < 1$. So some condition like $|(\partial G/\partial z)(0,0)| < 1$ is needed in order to ensure convergence of the series (1.6). □

Example 4.2. The inverse-function problem $f(z) = w$ with $f(z) = \sum_{n=1}^{\infty} a_n z^n$ ($a_1 \neq 0$) can be written in the form $z = G(z,w)$ in a variety of different ways. The most obvious choice is

$$G(z,w) \;=\; \frac{z}{f(z)}\, w \;, \tag{4.2}$$

which leads to the usual form of the Lagrange inversion formula:

$$f^{-1}(w) \;=\; \sum_{m=1}^{\infty} \frac{w^m}{m}\, a_1^{-m}\, [\zeta^{m-1}] \biggl(1 + \sum_{n=2}^{\infty} \frac{a_n}{a_1} \zeta^{n-1}\biggr)^{-m} \;. \tag{4.3}$$

An alternative choice, proposed by Yuzhakov [49], is

$$G(z,w) \;=\; \frac{w}{a_1} \,-\, \frac{f(z) - a_1 z}{a_1} \;, \tag{4.4}$$

which leads to

$$f^{-1}(w) \;=\; \sum_{m=1}^{\infty} \frac{a_1^{-m}}{m} \sum_{\ell=0}^{m} \binom{m}{\ell} w^{m-\ell} (-1)^{\ell}\, [\zeta^{m-1}] \biggl(\sum_{n=2}^{\infty} a_n \zeta^n\biggr)^{\ell} \;. \tag{4.5}$$

After some straightforward algebra involving binomial coefficients, both forms can be shown to yield the same result:

$$f^{-1}(w) \;=\; \sum_{m=1}^{\infty} \frac{w^m}{m!} \sum_{\substack{k_2,k_3,\ldots \geq 0 \\ \sum (n-1)k_n = m-1}} (-1)^{\sum k_n}\, a_1^{-(1+\sum n k_n)}\, \frac{(\sum n k_n)!}{k_2!\, k_3!\, \cdots} \prod_{n=2}^{\infty} a_n^{k_n} \tag{4.6}$$

(see also [11]). However, (4.3)/(4.6) expands $f^{-1}(w)$ as a power series in w, while (4.5) expands $f^{-1}(w)$ as a series in a different set of polynomials in w.

Consider, for instance, $f(z) = ze^{-z}$. Then the usual Lagrange inversion formula (1.3) gives

$$f^{-1}(w) \;=\; \sum_{m=1}^{\infty} m^{m-1}\, \frac{w^m}{m!} \;, \tag{4.7}$$

which is well known [42, Propositions 5.3.1 and 5.3.2] to be the exponential generating function for rooted trees (i.e., there are m^{m-1} distinct rooted trees on m labeled vertices). On the other hand, Yuzhakov's version (4.4)/(4.5) gives, after a short calculation, the alternate representation

$$f^{-1}(w) \;=\; \sum_{m=1}^{\infty} P_m(w) \tag{4.8}$$

where

$$P_m(w) \;=\; (-1)^{m-1} \sum_{k=\lceil (m+1)/2 \rceil}^{m} \frac{(m-1)!}{k!\,(k-1)!} \left\{ {k-1 \atop m-k} \right\} w^k \;; \tag{4.9}$$

here $\left\{ {n \atop k} \right\}$ denotes the Stirling subset numbers (also known as Stirling numbers of the second kind), i.e., the number of partitions of an n-element set into k nonempty blocks [17].[10] I wonder whether the coefficients in $P_m(w)$ have any combinatorial meaning. $\square$

Example 4.3. Here is an application from my own current research [41]. The function

$$F(x,w) \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}\, w^{n(n-1)/2} \tag{4.10}$$

arises in enumerative combinatorics in the generating function for the Tutte polynomials of the complete graphs K_n [40, 46] and in statistical mechanics as the grand partition function of a single-site lattice gas with fugacity x and two-particle Boltzmann weight w [39]. Let us consider F as a function of complex variables x and w satisfying $|w| \le 1$: it is jointly analytic on $\mathbb{C} \times \mathbb{D}$ and jointly continuous on $\mathbb{C} \times \overline{\mathbb{D}}$ (here $\mathbb{D}$ and $\overline{\mathbb{D}}$ denote the open and closed unit discs in $\mathbb{C}$, respectively), and it is an entire function of x for every $w \in \overline{\mathbb{D}}$.

When $w = 0$, we have $F(x,0) = 1 + x$, which has a simple zero at $x = -1$. One therefore expects — and can easily prove using Rouché's theorem — that for small $|w|$ there is a unique root of $F(x,w)$ near $x = -1$, which can be expanded in a convergent power series

$$x_0(w) \;=\; -1 \,-\, \sum_{n=1}^{\infty} a_n w^n \,. \tag{4.11}$$

The coefficients $\{a_n\}$ can of course be computed by substituting the series (4.11) into (4.10) and equating term-by-term to zero; but a more efficient method is to use the implicit function formula (1.6). It suffices to set $x = -1 - z$ and define

$$G(z,w) \;=\; \sum_{n=2}^{\infty} \frac{(-1-z)^n}{n!}\, w^{n(n-1)/2} \,. \tag{4.12}$$

We then obtain

$$\begin{aligned} -x_0(w) \;=\; & 1 + \tfrac{1}{2}w + \tfrac{1}{2}w^2 + \tfrac{11}{24}w^3 + \tfrac{11}{24}w^4 + \tfrac{7}{16}w^5 + \tfrac{7}{16}w^6 + \tfrac{493}{1152}w^7 + \tfrac{163}{384}w^8 \\ & + \tfrac{323}{768}w^9 + \tfrac{1603}{3840}w^{10} + \tfrac{57283}{138240}w^{11} + \tfrac{170921}{414720}w^{12} + \ldots \,. \end{aligned} \tag{4.13}$$

I conjecture — but have thus far been unable to prove — that all the coefficients in this power series are nonnegative. Since it is known [31, 32, 35] that $x_0(w)$ is analytic in a complex neighborhood of the real interval $0 < w < 1$, this conjecture would imply, by the Vivanti–Pringsheim theorem [20, Theorem 5.7.1], that $x_0(w)$ is in fact analytic in the whole disc $|w| < 1$, i.e., that the series (4.13) has radius of convergence exactly 1. (It is not hard to show that $x_0(w) \to -\infty$ as $w \uparrow 1$, so that the radius of convergence cannot be bigger than 1.)

[10] The key step in the derivation of (4.8)/(4.9) is the well-known identity [17, eq. (7.49)]

$$(e^{-z} - 1)^k \;=\; k! \sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} \frac{(-z)^n}{n!} \,.$$

Using (1.7) we can also compute power series for functions of $x_0(w)$. For instance, we have

$$\begin{aligned} \log[-x_0(w)] \;=\;& \tfrac{1}{2}w + \tfrac{3}{8}w^2 + \tfrac{1}{4}w^3 + \tfrac{41}{192}w^4 + \tfrac{13}{80}w^5 + \tfrac{85}{576}w^6 + \tfrac{83}{672}w^7 + \tfrac{227}{2048}w^8 \\ & + \tfrac{2065}{20736}w^9 + \tfrac{4157}{46080}w^{10} + \tfrac{6953}{84480}w^{11} + \tfrac{252449}{3317760}w^{12} + \ldots \,. \end{aligned} \tag{4.14}$$

I conjecture that all the coefficients in (4.14) are nonnegative. By exponentiation this implies the preceding conjecture, but is stronger. We also have

$$\begin{aligned} -\frac{1}{x_0(w)} \;=\;& 1 - \tfrac{1}{2}w - \tfrac{1}{4}w^2 - \tfrac{1}{12}w^3 - \tfrac{1}{16}w^4 - \tfrac{1}{48}w^5 - \tfrac{7}{288}w^6 - \tfrac{1}{96}w^7 - \tfrac{7}{768}w^8 \\ & - \tfrac{49}{6912}w^9 - \tfrac{113}{23040}w^{10} - \tfrac{17}{4608}w^{11} - \tfrac{293}{92160}w^{12} - \ldots \,. \end{aligned} \tag{4.15}$$

I conjecture that all the coefficients in (4.15) after the constant term are nonpositive. This implies the preceding two conjectures, but is even stronger. Using MATHEMATICA I have verified all three conjectures through order w^{60}. Indeed, by exploiting the connection between $F(x,w)$ and the generating polynomials $C_n(v)$ of connected graphs on n labeled vertices [40, 41, 46] — or equivalently the inversion enumerator for trees, $I_n(w)$ [42, Exercise 5.48, pp. 93–94 and 139–140] — I have computed the series $x_0(w)$ and verified these conjectures through order w^{775}.

The relative simplicity of the coefficients in (4.15) compared to (4.13)/(4.14) suggests that $-1/x_0(w)$ may have a simpler combinatorial interpretation than $-x_0(w)$ or $\log[-x_0(w)]$. Please note also that since $x_0(w) \to -\infty$ as $w \uparrow 1$, the coefficients $\{b_n\}$ in $-1/x_0(w) = 1 - \sum_{n=1}^{\infty} b_n w^n$ — if indeed they are nonnegative — add up to 1, so they are the probabilities for a positive-integer-valued random variable. What might such a random variable be? Could this approach be used to prove the nonnegativity of $\{b_n\}$?

Please note also that the coefficients $\{a_n\}$ and $\{b_n\}$ satisfy the discrete-time renewal equation

$$a_n \;=\; \sum_{k=1}^{n-1} b_k a_{n-k} \,. \tag{4.16}$$

Therefore, if the $\{b_n\}$ are nonnegative, they can be interpreted [9, Chapter XIII] as the probability distribution for first occurrences (or equivalently for waiting times between successive occurrences) of a recurrent event $\mathcal{E}$, while the $\{a_n\}$ are the probabilities of occurrence *tout court*:

$$a_n \;=\; \mathbb{P}(\mathcal{E} \text{ occurs at the } n\text{th trial}) \tag{4.17a}$$

$$b_n \;=\; \mathbb{P}(\mathcal{E} \text{ occurs for the first time at the } n\text{th trial}) \tag{4.17b}$$

What might such a family of recurrent events be?

For what it's worth, if we define $c_N = 1 - \sum_{n=1}^{N} b_n$, we find empirically (at least up to $N = 775$) that that $c'_N = 2\,N!\,c_N$ is an integer, with

$$\begin{aligned} c'_1, \ldots, c'_{20} \;=\;& 1,\ 1,\ 2,\ 5,\ 20,\ 85,\ 490,\ 3185,\ 23520,\ 199605,\ 1901130, \\ & 19767825,\ 223783560,\ 2806408605,\ 37447860450,\ 540137222625, \\ & 8284392916800,\ 135996789453525,\ 2363554355812650,\ 43437044503677825 \end{aligned} \tag{4.18}$$

Can anyone figure out a combinatorial interpretation of these numbers?

It is also known [41] that $-x_0(w)$ diverges as $w \uparrow 1$ with leading term $e^{-1}(1-w)^{-1}$, which suggests that we have $\lim_{n\to\infty} a_n = e^{-1}$ and $\sum_{n=1}^{\infty} nb_n = e$. If we define $d_N = \sum_{n=1}^{N} nb_n$ and $e_N = \sum_{n=1}^{N} [1/(n-1)! - nb_n]$, we find empirically (at least up to $N = 775$) that $d'_N = 2\,(N-1)!\, d_N$ and $e'_N = 2\,(N-1)!\, e_N$ are integers, with

$$\begin{aligned} d'_1,\ldots,d'_{20} \;=\;& 1,\ 2,\ 5,\ 18,\ 77,\ 420,\ 2625,\ 19110,\ 158025,\ 1457820,\ 14872725,\\ & 166645710,\ 2032946685,\ 26754868140,\ 379216422585,\ 5747274883350,\\ & 92854338001425,\ 1591646029073100,\ 28870013167120125,\ 552364292787857550 \end{aligned} \tag{4.19}$$

$$\begin{aligned} e'_1,\ldots,e'_{20} \;=\;& 1,\ 2,\ 5,\ 14,\ 53,\ 232,\ 1289,\ 8290,\ 61177,\ 515000,\ 4855477,\\ & 50364514,\ 571176005,\ 7098726832,\ 94733907025,\ 1361980060802,\\ & 20893741105009, 342071315736280, 5936899039448717, 108967039136950450 \end{aligned} \tag{4.20}$$

Might these numbers have some combinatorial interpretation? □

5. Possible multidimensional extensions

In this paper I have restricted attention to the implicit-function problem in one complex variable (i.e., $z \in \mathbb{C}$ though w lies in an arbitrary space W). Yuzhakov [2, 49, 50] goes much farther: he gives a beautiful explicit formula for the solution of the multidimensional implicit-function problem $F(z,w) = 0$ with $z \in \mathbb{C}^N$, $w \in \mathbb{C}^M$ and $F\colon \mathbb{C}^N \times \mathbb{C}^M \to \mathbb{C}^N$ under the usual hypothesis that the linear operator $(\partial F/\partial z)(0,0)$ is nonsingular. I suspect that the approach of the present paper can likewise be extended to the corresponding multidimensional situation — that is, $z = G(z,w)$ with $z \in \mathbb{C}^N$, $w \in W$ and $G\colon \mathbb{C}^N \times W \to \mathbb{C}^N$ — under the hypothesis that the linear operator $(\partial G/\partial z)(0,w_0)$ has spectral radius < 1. Indeed, such a result presumably holds when $\mathbb{C}^N$ is replaced by a complex Banach space. As we have seen, the proof of Theorem 2.3 given here applies verbatim when w lies in an arbitrary space, since the variables w simply "go for the ride". But multidimensional z is a genuine generalization; and for lack of time and competence, I have not attempted to pursue it. See [1, 3, 6, 10, 14, 15, 19, 28, 29] for information on multidimensional Lagrange inversion formulae, and [27] for a survey of implicit function theorems.

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