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COMBINATORIAL SPECIES AND FEYNMAN DIAGRAMS

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Dedicated to the memory of Pierre Leroux

Abstract. A Feynman diagram is a graphical construction that describes certain interactions in physics. Most calculations with such diagrams reduce to consideration of connected Feynman diagrams. These in turn may be constructed from line-irreducible Feynman diagrams, those for which removal of a single line does not destroy connectivity. The purpose of this article is to exhibit the combinatorial nature of this construction in the framework of species of structures. The main result is a dissymmetry theorem for connected Feynman diagrams. This purely combinatorial theorem relates the species of connected diagrams to species with less symmetry, such as the species of connected diagrams with a designated line-irreducible subdiagram. There is also a discussion of the relation of this result to the Legendre transform.

1. Introduction

This article is on the relation between constructions in combinatorics and corresponding constructions in physics. The framework is the theory of species of structures in the sense of Joyal [10] (see also the book [2]). The physics topic is Feynman diagrams. Feynman diagrams are combinatorial structures involving graphs constructed in a rather complicated way.

In graph theory there is a fundamental concept of connected graph. There are also stronger notions of connectivity. A connected graph is 2-connected if it does not have a cut-vertex: a vertex whose removal disconnects the graph. A connected graph is 2-edge-connected if it does not have a bridge: an edge whose removal disconnects the graph. In the Feynman diagram context the notion corresponding to 2-edge-connected graph is 1-particle-irreducible diagram. For simplicity we henceforth use the terminology line-irreducible diagram.

There are identities that reduce sums over connected diagrams to sums over line-irreducible diagrams. They may be formulated in terms of the Legendre transform. Such results are standard in the physics literature and may be found in textbooks [13, Chapter 16]. Current discussions may be found, for instance, in [6] or [9]. The lecture notes [7] are a particularly good introduction. The goal of the present work is to formulate and prove these identities in the context of species of structures. The identities should take the form of bijections between sets of combinatorial structures. In short, the numerical identities should be consequences of combinatorial identities.

This species approach depends on a combinatorial fact about trees known as the dissymmetry theorem. This principle was applied to statistical mechanics by Leroux [11]. In his treatment the natural setting is graphs with the concept of 2-connectivity. An article by Brydges and Leroux [3] explored what happens in the graph case with the 2-edge-connectivity concept. For the Feynman diagram case considered here the natural notion is line-irreducibility.

The species identities are relations between finite sets. When infinite series are encountered, they are interpreted in a formal sense. See [8] for a survey of species techniques and of their applications to statistical mechanics. The formal combinatorics of Feynman diagrams (with oriented lines) is systematically explored by Abdesselam in [1]. Another topic of analysis is concerned with convergence of expansions. See, for example, Brydges and Martin [4, Section VIII] for the state of the art.

The Feynman diagrams under consideration here are labeled Feynman diagrams. The theory of combinatorial species works particularly smoothly with labeled structures. In this case the only symmetry considered is permutation of the labels, and the corresponding generating series (exponential generating function) is a relatively uncomplicated construct. That is the reason for concentrating on that case in the present treatment. By contrast, many expositions deal with isomorphism classes of Feynman diagrams. These are abstract graphs with vertices and edges, allowing multiple edges and loops. The reason for using isomorphism classes of diagrams is to compress the sums involved so as to have less terms to compute. This is important in physics for practitioners of perturbation theory who need to evaluate some quantity—say a scattering amplitude—up to some order in the coupling constant. The theory of combinatorial species also describes isomorphism classes, but this requires a deeper understanding of the automorphisms that describe the symmetries. The natural tool is then the cycle index series.

While the isomorphism class description of Feynman diagrams is not needed for the main body of the present article, it is relevant to understanding its relation to other approaches to the combinatorics of

Feynman diagrams. For this reason, an appendix is included that discusses this topic. In particular, this allows a more detailed comparison with the contributions [3] and [1].

Feynman diagrams have an important interpretation in quantum field theory. They describe the perturbation of free quantum particle propagation by interactions. The vertices represent the interactions, and the lines represent the free particle propagation. Feynman diagrams also have an interpretation in the theory of spatial stochastic processes. Here they describe the perturbation of a Gaussian process by nonlinear interactions. The vertices come from the interaction terms, and the lines represent the Gaussian covariances. In the following we shall use the language of perturbations of Gaussian processes. This corresponds to the quantum field case with chargeless bosons. The lack of charge corresponds to the combinatorial fact that the lines are unoriented.

2. Feynman diagrams

2.1. **Exponential generating functions.** For each n consider a finite set U with cardinality n. A point i in U is called a *label.* One possible choice is $U = \{1, \ldots, n\}$. However, it can be useful to employ a subset V of U as another label set. In other circumstances one can take a set partition Γ of U as a label set. So it is best to allow a general finite set at the outset. Combinatorial objects are often built from such a label set; the exact nature of this set is usually not important.

Fix a finite set $\mathcal{X}$ (possibly very large). This is the color palette. Each x in $\mathcal{X}$ is a *color.* Consider a function $a : U \to \mathcal{X}$. This is called a *colored set.* Thus the label i has color a_i. The case where we do not care to use colors is obtained by using a color palette with only one color, for instance, gray. We may refer to this situation as *monochromatic.*

In some interpretations the colors may be thought of as the index set for a stochastic process. Each color is a point in space (or a point in time). A colored set is a way of describing several points. Thus, for instance, for a colored set $a : \{i, j\} \to \mathcal{X}$ this describes points a_i, a_j in space, which may be the same point.

Such an index set may also be thought of as the coordinate index set for a product space of dimension N, where N is the number of colors. An element ϕ of this product space is an N-component vector that assigns to each index x in $\mathcal{X}$ a number $\phi(x)$, the x component of the vector. More generally, a tensor of rank 2 assigns to each ordered pair of indices x, y a number $\phi(x, y)$. A tensor of rank n is defined in a similar way as a number depending on n indices. In the applications considered here the tensors are symmetric tensors. Thus in the rank 2 case we have $\phi(x, y) = \phi(y, x)$.

The notion of colored set gives a more flexible way of describing such tensors. Consider a label set U_n of cardinality n. Say that for each colored set $a : U_n \to \mathcal{X}$ there is a corresponding coefficient $f(a)$. Suppose that $f(a)$ depends only on the number of points in U_n of each color. Then $f(a)$ is a *symmetric tensor* of rank n in a space of dimension N. The colors are the tensor indices. For example, if $n = 2$ the tensor is rank 2. For a colored set $a : \{i, j\} \to \mathcal{X}$ there are two indices a_i, a_j, which may be the same. Permuting the labels gives the same value of $f(a)$.

The *exponential generating function* for this family of tensors is a formal power series in many variables z_x, one for each color x in $\mathcal{X}$. It is given by the formal sum

$$F(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:U_n \to \mathcal{X}} f(a) z^a, \tag{1}$$

where $z^a = \prod_{j \in U_n} z_{a_j}$. In the monochromatic case the tensors are scalar coefficients that depend only on n, and there is only one variable.

Exponential generating functions give a convenient summary of important tensor operations. The most fundamental operation is combinatorial multiplication. Say that $G(z)$ corresponds to the tensors $g(a)$, and $H(z)$ corresponds to the tensors $h(a)$. Then the product $F(z) = G(z)H(z)$ corresponds to the tensors

$$f(a) = \sum_{\langle V, W \rangle} g(a_V) h(a_W), \tag{2}$$

where V, W range over ordered pairs of disjoint sets with union U, and where a_V and a_W are the restrictions of a to V and W. Combinatorial multiplication splits the set in all possible ways, multiplies the weights of the constituents, and adds the results.

Composition is even more important. It is useful to allow two color palettes. Say that $G(w)$ is defined with color palette $\mathcal{Y}$. Furthermore, say that for each y in $\mathcal{Y}$ there is an exponential generating function $H_y(z)$ with color palette $\mathcal{X}$. Suppose for simplicity that each $H_y(0) = 0$. Then the composition $F(z) = G(H(z))$ makes sense as an exponential generating function with color palette $\mathcal{X}$. The corresponding operation on tensors $a : U \to \mathcal{X}$ is

$$f(a) = \sum_{\Gamma} \sum_{b:\Gamma \to \mathcal{Y}} g(b) \prod_{V \in \Gamma} h_{b_V}(a_V), \tag{3}$$

where $a : U \to \mathcal{X}$, and where Γ ranges over set partitions of U. Combinatorial composition partitions the set in all possible ways, with all possible colorings of the blocks, multiplies the weight of the partition with the weights of the blocks of the partition, and adds the results. If it happens that the set U is empty, then the only set partition is the empty partition Γ with no blocks. The product over the blocks V in Γ is the empty product with value one.

An important special case is when $G(w)$ is monochromatic and there is only one $H(z)$. In particular, if we take the monochromatic $G(w) = \exp(w)$, then the corresponding tensors are simply the scalar 1. So in that case $F(z) = \exp(H(z))$ is given by

$$f(a) = \sum_{\Gamma} \prod_{V \in \Gamma} h(a_V), \tag{4}$$

where $a : U \to \mathcal{X}$, and where Γ ranges over set partitions of U. The combinatorial exponential partitions the set in all possible ways, multiplies the weights of the blocks of the partition, and adds the results.

2.2. **Gaussian expectations.** Let $\mathcal{X}$ be a palette of colors, and consider Gaussian random variables ϕ^x indexed by x in $\mathcal{X}$. This means that there is a Gaussian measure on the N-dimensional product space $\mathbf{R}^{\mathcal{X}}$. The ϕ^x are the coordinate variables with induced one-dimensional Gaussian measures. The expectation of a function $h(\phi)$ with respect to the Gaussian measure will be denoted $\langle h(\phi)\rangle_0$. A Gaussian measure is always determined by its mean function and its covariance function. In the following suppose that the Gaussian random variables have mean zero and covariance

$$C^{xy} = \langle \phi^x \phi^y \rangle_0. \tag{5}$$

We could also write the covariance as C^a, where $a : \{i, j\} \to \mathcal{X}$ is a colored set with $a_i = x$ and $a_j = y$. In other words, it is a rank 2 tensor.

An expectation of a monomial in random variables is called a moment. In particular, a covariance of mean zero random variables is a second moment. The higher moments $\langle \phi^a \rangle_0 = \langle \prod_j \phi^{a_j} \rangle_0$ of the family of Gaussian random variables should be determined by the exponential generating function

$$\langle \exp(\sum_x J_x \phi^x)\rangle_0 = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a : U_n \to \mathcal{X}} \langle \phi^a \rangle_0 J_a, \tag{6}$$

where $J_a = \prod_j J_{a_j}$. Here the J_a are arbitrary complex variables. In the Gaussian case this expression is well-defined. Furthermore, since the variables are Gaussian, the moment generating function has an explicit expression in terms of the covariance:

(7)
$$\langle \exp(\sum_x J_x \phi^x) \rangle_0 = \exp(\frac{1}{2} \sum_x \sum_y C^{xy} J_x J_y) = \exp(\frac{1}{2} \sum_{a:U_2 \to \mathcal{X}} C^a J_a).$$

It follows from the combinatorial exponential that the higher moments are given in terms of covariances for $a : U \to \mathcal{X}$ by

$$\langle \phi^a \rangle_0 = \sum_{\sigma \in \mathrm{Mat}[U]} \prod_{W \in \sigma} C^{a_W}, \tag{8}$$

where σ ranges over all set partitions of U into two-point blocks. Such a partition is called a *perfect matching* of U. Write the weight of the perfect matching σ as

$$C(a, \sigma) = \prod_{W \in \sigma} C^{a_W}. \tag{9}$$

Then

$$\langle \phi^a \rangle_0 = \sum_{\sigma \in \mathrm{Mat}[U]} C(a, \sigma) \tag{10}$$

is the sum of weights of perfect matchings.

2.3. **Vacuum Feynman diagrams.** In the theory of random fields non-Gaussian processes are built from Gaussian processes. Start with a set $\mathcal{X}$ indexing a Gaussian process. Consider variables ϕ^x for $x \in \mathcal{X}$. The *action* written in terms of the fields ϕ takes the form

$$S(\phi) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{a:U_n \to \mathcal{X}} S_a \phi^a. \tag{11}$$

The ϕ^a is the product $\prod_j \phi^{a_j}$, where the product is over the label set. The coefficients S_a are symmetric tensors. The action is a given quantity that defines the process of interest. In some applications it is restricted to be a polynomial of low degree, such as three or four. In that case, there are only finitely many parameters that define the action.

The exponential of the action is

$$\exp(S(\phi)) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:U_n \to \mathcal{X}} [\sum_{\Gamma \in \mathrm{Par}[U_n]} \prod_{V \in \Gamma} S_{a_V}] \phi^a. \tag{12}$$

The Γ range over set partitions of the label set. The a_V is the restriction of a to the subset V of the label set. In the future it will be convenient to write

$$S(a,\Gamma) = \prod_{V\in\Gamma} S_{a_V}. \tag{13}$$

The *partition function* is the expectation

(14)

$$Z = \langle \exp(S(\phi))\rangle_0 = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:U_n\to\mathcal{X}} \sum_{\Gamma\in\mathrm{Par}[U_n]} \sum_{\sigma\in\mathrm{Mat}[U_n]} S(a,\Gamma)C(a,\sigma).$$

A *vacuum Feynman diagram* γ for label set U consists of a set partition Γ of U and a perfect matching σ of U. Each block in Γ is a *vertex*, and each pair $\{i,j\}$ in σ is a *line*. The labels i,j are the ends of the lines. The weight of the vacuum Feynman diagram is the product $\mathrm{Feyn}(a,\gamma) = S(a,\Gamma)C(a,\sigma)$. The result above says that the partition function is given by a sum of weights of diagrams.

Each vacuum Feynman diagram defines a graph (with loops and multiple edges). Each block in the set partition Γ is a vertex, and an edge from such a vertex to another corresponds to a line with one end in one vertex and the other end in the other vertex.

The partition function Z may be thought of as a function of the parameters that define the action. The physics is determined by this function. The partition function does not have a direct physical interpretation, but ratios such as the logarithmic differential dZ/Z determine relevant expected values. The quantities that vary in such a differential expression are certain parameters in the action. The particular parameters that are chosen depend on the physical quantity that one is trying to compute. In physics it is usual to focus on the *connected function* or *free energy* F defined so that $Z = \exp(F)$. Then the differential $dF = dZ/Z$ is a quantity of interest. The combinatorial term "connected function" comes from the fact that its computation involves connected Feynman diagrams. In physics "free energy" often refers loosely to a quantity that is a multiple of a thermodynamic free energy. In any case the quantity F, regarded as a function of action parameters, is central to the theory.

2.4. **Feynman diagrams with legs and with amputated lines.** There are more general kinds of Feynman diagrams. A *Feynman diagram with legs* for label set U consists of a set partition Γ of a subset U_0 of U together with a perfect matching σ of U. A line that does not have both ends in U_0 is called a *leg*. A label that is not in the vertex subset U_0 is a *leg label*. One way to get a Feynman diagram with legs is to take a vacuum Feynman diagram, remove all the one-point vertices, and replace them with leg labels.

Another kind of Feynman diagram has vertices with labels that do not belong to lines. An *amputated Feynman diagram* γ on a label set U is a set partition Γ of U together with a perfect matching σ of a subset of U. A label that is not matched is an *amputated line label.* It may be thought of as one end point of a potential or virtual line, and may be pictured as a half-line.

The most general form of Feynman diagram γ on label set U would consist of a set partition Γ of a subset of U and a perfect matching σ of another subset of U. Then there could be ordinary labels that belong both to the partition and the perfect matching, leg labels that belong to the perfect matching but not the partition, amputated line labels that belong to the partition but not the perfect matching, and amputated leg labels that belong to neither one. In each case the weight of the diagram is $\mathrm{Feyn}(a, \gamma) = S(a, \Gamma)C(a, \sigma)$.

3. Species of structures

3.1. **Species equations.** The most basic species notion is that of an assignment to each finite set U of a finite set $F[U]$ of combinatorial objects, with a suitable naturality requirement. Often U is called the label set, and the elements of $F[U]$ are constructed from U. An elementary example is the power set species P that assigns to each set U with n elements the set $P[U]$ of all subsets of U, which is a set with 2^n elements. Many other examples appear in [3, Chapter 1].

There is a more general concept of weighted species. A weighted species is one for which counting of $F[U]$ is replaced by adding weights associated to the elements of this set. In the example of the power set species one could weight each subset of U with k elements by t^k. The total weight of $P[U]$ would then be $\sum_{k=0}^{n} \binom{n}{k} t^k = (1 + t)^n$. The weighted species notion is explained in [2, Section 2.3].

A multisort species is one for which the label set U is categorized into sorts, and such that the combinatorial constructions depend on this categorization. In the present exposition the sorts are called colors. This more advanced notion occurs in [2, Section 2.4]. In the following the species that occur will usually be weighted multisort species. A species built over a set U with no sort distinction might be called monochromatic. The case where the focus is on counting the set $F[U]$ of combinatorial objects is obtained by setting the weight of each object equal to one. In the power set example the value of $(1 + t)^n$ with $t = 1$ is the count 2^n.

We have seen that in species theory it is useful to have a concept of weighted set. The notation here is chosen so operations on weighted sets appear very similar to operations on sets. Fix a commutative ring R (perhaps the real numbers or the complex numbers). A *weighted set* is a finite set together with a weight function A from the set to R. The finite set is the domain $\mathrm{dom}(A)$, and the function A is from $\mathrm{dom}(A)$ to R.

Suppose that A, B are weighted sets. There are natural concepts of disjoint union $A+B$ and cartesian product $A \times B$. The disjoint union is produced in the obvious way. If $A : \mathrm{dom}(A) \to R$ is a weighted set, and $B : \mathrm{dom}(B) \to R$ is a weighted set, then $A+B : \mathrm{dom}(A)+\mathrm{dom}(B) \to R$ is the weighted set defined on the disjoint union $\mathrm{dom}(A) + \mathrm{dom}(B)$ of sets by $(A + B)(\alpha) = A(\alpha)$ for α in $\mathrm{dom}(A)$ and $(A + B)(\beta) = B(\beta)$ for β in $\mathrm{dom}(B)$. Similarly, the cartesian product $A \times B : \mathrm{dom}(A) \times \mathrm{dom}(B) \to R$ is defined on the cartesian product $\mathrm{dom}(A) \times \mathrm{dom}(B)$ of sets by the *multiplicative property* $(A \times B)(\alpha, \beta) = A(\alpha)B(\beta)$. The case of sets without any special weighting is obtained by giving each element weight one. In that case the weighted set may be identified with the set.

If $A : \mathrm{dom}(A) \to R$ is a weighted set, then it has *total weight*

$$|A| = \sum_{\alpha \in \mathrm{dom}(A)} A(\alpha). \tag{15}$$

This is an obvious generalization of cardinality. It is obvious that $|A + B| = |A| + |B|$. When the multiplicative property holds it is also true that $|A \times B| = |A|\,|B|$. This follows from the distributive law. The idea is that the domain of a weighted set is a set of combinatorial objects. One wants to manipulate these sets of combinatorial objects, while at the same time keeping track of their weights. Since the weighted set operations automatically include the corresponding operations on sets of combinatorial objects, one can treat the weighted sets as sets, but with extra structure. It may be convenient to think of the elements α of the domain of A as being elements of the weighted set A. With this convention it is possible to write $\alpha \in A$ for an element of the domain of A and to write $A(\alpha)$ for the weight of this α.

In the following it is convenient to consider disjoint sums and cartesian products of indexed families of weighted sets. Suppose that for each p in some index set there is a weighted set A_p. The weighted set $\sum_p A_p$ has domain set consisting of all pairs (p, α), where α is in the domain set of A_p. The weight of (p, α) is $A_p(\alpha)$. The weighted set $\prod_p A_p$ has domain set consisting of all functions $p \mapsto \tau_p$ such that each τ_p is in the domain set of A_p. The weight of τ is $\prod_p A_p(\tau_p)$.

A *species* F assigns to each colored set $a : U \to \mathcal{X}$ a corresponding set $\mathrm{dom}(F[a])$ of combinatorial objects together with a weight function $F[a] : \mathrm{dom}(F[a]) \to R$. In other words, it takes a colored set a to a weighted set $F[a]$. Furthermore, it is assumed that permutations of the labels that carry along the colors induce corresponding permutations of the weighted sets that carry along the weights. The total weight of $F[a]$ is obtained by summing the weights of the individual combinatorial objects in $\mathrm{dom}(F[a])$. The total weight is denoted

$$f(a) = |F(a)| = \sum_{\alpha \in \mathrm{dom}(F[a])} F[a](\alpha). \tag{16}$$

The end product of species theory is the computation of total weights of combinatorial objects built from colored sets. This is a natural generalization of counting.

For each species F there is a corresponding *exponential generating function* $F(z)$ with tensor coefficients given by the total weights $f(a)$. The operations on species are designed to give corresponding operations on the exponential generating functions.

Given two species G and H, there is another species $G + H$ that is the *combinatorial sum*. It is defined by $(G+H)[a] = G[a]+H[a]$, where the sum on the right is weighted set disjoint sum. The corresponding exponential generating functions satisfy $(G + H)(z) = G(z) + H(z)$.

Given two species G and H, there is a species $G * H$ that is the *combinatorial product*. It is defined by

$$(G * H)[a] = \sum_{\langle V,W \rangle} G[a_V] \times H[a_W]. \tag{17}$$

This is a disjoint sum of cartesian products of weighted sets. The $\langle V, W\rangle$ that index the disjoint sum range over ordered pairs of subsets with $V \cap W = \emptyset$ and $V + W = U$. Also a_V denotes the restriction of a to V. It follows that the exponential generating functions satisfy

$$(G * H)(z) = G(z)H(z). \tag{18}$$

Consider two color palettes $\mathcal{Y}$ and $\mathcal{X}$. Say that G is a species on $\mathcal{Y}$ colored sets, and H_y for y in $\mathcal{Y}$ are species on $\mathcal{X}$ colored sets. Suppose that for each y that $H_y(\emptyset) = \emptyset$. The *combinatorial composition* $G \circ H$ as a species on $\mathcal{X}$ colored sets is defined by

$$(G \circ H)[a] = \sum_{\Gamma} \sum_{b:\Gamma \to \mathcal{Y}} G[b] \times \prod_{V \in \Gamma} H_{b(V)}[a_V]. \tag{19}$$

This is a disjoint sum of cartesian products of weighted sets. The sum is indexed by set partitions Γ of U and colorings $b : \Gamma \to \mathcal{Y}$ of these partitions. Each cartesian product is indexed by the blocks of the partition. The corresponding exponential generating functions satisfy

$$(G \circ H)(z) = G(H(z)). \tag{20}$$

The combinatorial composition is complicated enough to make it worth looking at the construction in more detail. Consider a colored set $a : U \to \mathcal{X}$. An element α of the underlying domain set of the weighted set $(G \circ H)[a]$ is of the form $\alpha = (\Gamma, b, \lambda, \tau)$. Here the element of the index set for the disjoint sum is given by specifying a set partition Γ of U and a coloring $b : \Gamma \to \mathcal{Y}$. The λ is a structure in the underlying domain set of $G[b]$. For each block V in Γ the value τ_V is in the underlying domain set of $H_{b(V)}[a_V]$. The weight of α is the product $G[b](\lambda) \prod_{V \in \Gamma} H_{b(V)}[a_V](\tau_V)$.

An important special case is when G is monochromatic and there is only one H species. In particular, we may take for G the species E that assigns to each set a single point with weight one. Then $E \circ H$ is the combinatorial exponential given by

$$(E \circ H)[a] = \sum_{\Gamma} \prod_{V \in \Gamma} H[a_V]. \tag{21}$$

The corresponding exponential generating functions satisfy

$$(E \circ H)(z) = \exp(H(z)). \tag{22}$$

There are other useful operations on species. If F is a species, then there is a species F'_x given by *combinatorial derivative*. It is defined by

$$F'_x[a] = F[a^x], \tag{23}$$

where $a^x : U + \{*\} \to \mathcal{X}$ is equal to a on U and has value x on $*$. Here $*$ is a label point that is not in U. Then the exponential generating function satisfies

$$F'_x(z) = \frac{\partial F(z)}{\partial z_x}. \tag{24}$$

The combinatorial derivative is perhaps best regarded as a partial derivative, where the coordinate index is the color x. In the following the index may be used either as a subscript or as a superscript, depending on the context. This represents an attempt to follow tensor calculus conventions.

3.2. **Feynman diagram species.** A first notion of Feynman diagram species is the following. Fix a color palette $\mathcal{X}$. For each colored set $a : U \to \mathcal{X}$ there is a set $Z[U]$ of vacuum Feynman diagrams with label set U. Each diagram γ consists of a set partition Γ of U and a perfect matching σ of U. The weight of a diagram is $\mathrm{Feyn}(a, \gamma) = S(a, \Gamma)C(a, \sigma)$. The weighted set of all such diagrams is $Z[a]$.

A vacuum Feynman diagram is *connected* if the label set cannot be partitioned into blocks that are not coupled by the diagram. Let Z be the species of vacuum Feynman diagrams, and let F be the species of connected vacuum Feynman diagrams. By the general theory,

$$Z = E \circ F. \tag{25}$$

This says that every vacuum Feynman diagram gives rise to a set partition on each block of which there is a connected vacuum Feynman diagram. The exponential generating function identity is $Z(z) = \exp(F(z))$.

The simplest connected vacuum Feynman diagrams are for $n = 2$. There is a diagram with one vertex and a single loop; there is also a diagram with two vertices and a line that forms a bridge between them. The next simplest case is with $n = 4$. There is a diagram with one vertex and a double loop; there is a diagram with two vertices and two lines joining them; there is a diagram with two vertices, one loop, and a line that forms a bridge between the vertices; there is a diagram with three vertices and two bridge lines.

The weight of each diagram is given by the Feynman rules. For instance, consider the $n = 4$ vacuum diagram with two vertices and two lines joining them. Say more specifically that the underlying set is $U = \{i, j, k, \ell\}$ and the partition has blocks $\{i, j\}$ and $\{k, \ell\}$. The lines are $\{i, k\}$ and $\{j, \ell\}$. The weight is then $S_{a_i\, a_j} C^{a_i\, a_k} C^{a_j\, a_\ell} S_{a_k\, a_l}$ For another example, consider the $n = 4$ diagram with two vertices, one loop, and a bridge. Say that the partition has blocks $\{i\}$ and $\{j, k, \ell\}$ and the lines are the bridge $\{i, j\}$ and the loop $\{k, \ell\}$. The weight for this diagram is $S_{a_i} C^{a_i\, a_j} S_{a_j\, a_k\, a_\ell} C^{a_k\, a_\ell}$.

3.3. **Indicator species.** Indicator species are combinatorial constructions that return at most one object. They are not particularly interesting in themselves, but they are excellent building blocks for more complicated species. The simplest example is the monochromatic set indicator species E. Given a label set U, the value of $E[U]$ is a single point of weight one. The corresponding exponential generating function is a function of a single variable w; it is just the exponential function $E(w) = \exp(w)$.

There are also indicator species that act on colored sets. We shall need the one point of color x indicator species X_x that returns a single point of weight one, but only provided that the input is a colored set with one element of color x. For every other input the return is the empty set. The exponential generating function of X_x is the coordinate function $X_x(z) = z_x$.

A more interesting example is the *line indicator species* L. For a colored label set $a : \{i, j\} \to \mathcal{X}$ with two points it gives a single point with weight $C^{a_i a_j}$. Otherwise it returns the empty set. The line indicator species has quadratic exponential generating function $L(z) = \frac{1}{2} \sum_x \sum_y C^{xy} z_x z_y$. The perfect matching species Mat satisfies

$$\mathrm{Mat} = E \circ L. \tag{26}$$

This already shows the power of the indicator species concept.

It will be useful to have the combinatorial partial derivative L'^x of the line indicator species. The value of this on a one-point colored set $a : \{j\} \to \mathcal{X}$ is a single point with weight $C^{x a_j}$. In all other cases it returns the empty set. This is a species that recognizes a single point, but regards it as having a line from an external point of color x. The exponential generating function of this species is $L'^x(z) = \sum_y C^{xy} z_y$.

3.4. **Connected Feynman diagrams with a leg or amputated line.** Here are two useful species. The first is the species $F^{\uparrow x}$ of connected Feynman diagrams with an external leg of color x. Given a colored set $a : U \to \mathcal{X}$, this species consists of all connected Feynman diagrams τ on the disjoint union $U + \{*\}$, where $*$ is an external point that is a leg label of color x. Each connected diagram τ has a set partition Γ of U and a perfect matching σ of $U + \{*\}$. The weight of a diagram is $\mathrm{Feyn}(a^x, \tau) = S(a^x, \Gamma) C(a^x, \sigma)$. These weighted diagrams define the weighted set $F^{\uparrow x}[a]$. The total weight of this weighted set contributes to the coefficient of the z^a term in the exponential generating function. The external point $*$ plays a role similar to that of the external point in the definition of the derivative species, except that only the perfect matching part of the diagram extends to this point.

The second is the species of $F^{\odot}_y$ of connected Feynman diagrams with a amputated line of color y. This assigns to each colored set $a : U \to \mathcal{X}$ a set partition Γ of U, an amputated line label j in U with $a_j = y$, and a perfect matching σ_j of the difference set $U - \{j\}$, and this is done in such a way as to produce a connected diagram γ_j. The weight of a diagram is $\mathrm{Feyn}(a, \gamma_j) = S(a, \Gamma) C(a, \sigma_j)$. These weighted diagrams define the weighted set $F^{\odot}_y[a]$. Again the total weight of this weighted set contributes to the coefficient of the z^a term in the exponential generating function. This construction is somewhat like the combinatorial product construction in the special case when the disjoint union $U = \{j\} + (U - \{j\})$ involves one-point subsets. Again only the perfect matching part of the diagram is restricted.

Lemma 3.1. *There is a species isomorphism*

$$F^{\uparrow x} = L'^{x} \circ F^{\odot}. \tag{27}$$

The proof is to note that the weighted set $(L'^{x} \circ F^{\odot})[a]$ is computed with a trivial partition with only one block U. An object in the underlying set is given by U, a color $b(U)$, and a connected Feynman diagram γ_j with some specified amputated line label j in U with $a_j = b(U)$. The weight is $C^{xb(U)}\mathrm{Feyn}(a, \gamma_j) = C^{xa_j}\mathrm{Feyn}(a, \gamma_j)$. Map this to a connected diagram τ by adding a line from j to an external leg label $*$ of color x. This map may be reversed by defining j to be the vertex adjacent to $*$. The weight $\mathrm{Feyn}(a^x, \tau)$ is the same. This gives an object in the weighted set $F^{\uparrow x}[a]$.

The lemma says that a connected Feynman diagram with a leg to an external point is obtained by adjoining a line to a connected Feynman diagram with an amputated line. The exponential generating function identity is $F^{\uparrow x}(z) = \sum_y C^{xy} F^{\odot}_y(z)$.

It is easy to generate examples of connected diagrams with single amputated lines. For $n = 1$ there is a single vertex with an amputated line. For $n = 3$ there is the one vertex diagram with loop and amputated line; also there is the two vertex line with bridge line and one amputated line. For $n = 5$ there is a diagram with one vertex and a double loop and one amputated line, and there is a diagram with two vertices and two lines joining them and one amputated line. However now there are two kinds of diagrams with two vertices, one loop, and a line that forms a bridge between the vertices, since one can place the amputated line in the vertex with the loop or the vertex without the loop. Also, there are two kinds of diagrams with three vertices and two bridge lines, since one can place the amputated line in the center vertex or in an end vertex.

Adding a line and an external point to an amputated diagram gives the corresponding diagram with a leg line to an external point. The simplest such diagram has a one point vertex with a line to an external point. The next simplest such diagram has a loop and a line to an external point; in the physics literature this is sometimes called a "tadpole" diagram.

4. Line-irreducible diagrams

4.1. **Lumps and bridges.** A line in a connected Feynman diagram is a *bridge* if removing it produces a diagram that is not connected. When all bridges are removed the resulting diagram will have amputated lines. The label set may then be partitioned into connected components on each one of which is a connected diagram with amputated lines.

A connected diagram with amputated lines is a *line-irreducible diagram* if it has no bridge. This notion is closely related to the graph theory notion of 2-edge-connected graph. It is also a possible formulation of the physics concept of 1-particle-irreducible diagram.

Some obvious examples of line-irreducible diagrams are the line-irreducible vacuum diagrams. For instance, there is the $n = 2$ vertex diagram with one vertex and a loop. Also, there is the $n = 4$ vacuum diagram with one vertex and a double loop, as well as the $n = 4$ vacuum diagram with two vertices and two lines connecting them. There are also non-vacuum diagrams with a single truncated line. The simplest one is the $n = 1$ diagram with one vertex and a truncated line. There are also $n = 3$ and $n = 5$ diagrams obtained by adding a truncated line to the above vacuum diagrams. One can go on and construct diagrams with two truncated lines; the same procedure gives examples with $n = 2, 4, 6$.

Every connected vacuum Feynman diagram may be decomposed as a tree, where the vertices of the tree are line-irreducible diagrams with amputated lines, and the edges of the tree are the bridges. Each constituent line-irreducible diagram is called a *lump*. A lump can consist of a single point.

Write $M^{\odot}$ for the species of line-irreducible diagrams. The line-irreducible diagrams on label set U consist of a set partition of U (the vertices) and a perfect matching of a subset of U (the lines) with the condition that there is no bridge.

The species $M^{\odot}$ decomposes into two parts $M^{\odot} = M_0^{\odot} + M_1^{\odot}$. The part $M_0^{\odot}$ gives line-irreducible diagrams with no one-point vertices. If a line-irreducible diagram has one-point vertices, then it consists of a single one-point vertex. Thus the other part $M_1^{\odot}$ consists of line-irreducible diagrams consisting of a single label point belonging to a one-point vertex.

There is another species M related to the physics concept of 1-particle-irreducible diagram. The diagrams are obtained from line-irreducible Feynman diagrams by replacing each amputated line label with an ordinary label together with a line to a leg label. In the following construction these legs are used to construct bridges. For that reason, the species M will be called the species of *line-irreducible Feynman diagrams with bridge legs*. A leg label that is adjoined by a line to the original line-irreducible diagram is called a *bridge leg label*. The label set $U = U_0 + U_\beta$, where the labels in U_0 belong to vertices, while each bridge leg label in U_β is attached to a label in U_0 by a line.

The species $M = M_0 + M_1$ also decomposes into two parts. The part M_0 gives diagrams with no one-point vertices. The part M_1 consists of diagrams with a line from a label belonging to a one-point vertex to a bridge leg label.

4.2. **Bridge identities.** The context for the following results is the species F of connected Feynman diagrams and variations on this species. One variation is the species of connected Feynman diagrams with a designated bridge. This produces ordered pairs consisting of a Feynman diagram and a bridge of the diagram. Another variation is the species of connected Feynman diagram with a designated lump. This produces ordered pairs consisting of a Feynman diagram and a lump of the diagram. These species are less symmetrical than the species F; in compensation they enter in recursive structures.

Lemma 4.1. *Let $F^{\leftrightarrow}$ be the species of connected Feynman diagrams with a designated bridge. Let L be the line indicator species. For each y let $F_y^{\odot}$ be the species of connected diagrams with an amputated line of color y. Then*

$$F^{\leftrightarrow} = L \circ F^{\odot}. \tag{28}$$

To prove this lemma, start with $(L \circ F^{\odot})[a]$. The elements of this weighted set are indexed by two-element partitions $\Delta = \{V, W\}$ of U together with colorings $b : \Delta \to \mathcal{X}$. Consider a partition Δ. A typical element is a diagram γ on V with some amputated line label j of color $a_j = b(V)$ and a diagram δ on W with some amputated line label k of color $a_k = b(W)$. The weight is $C^{b(V)b(W)}\mathrm{Feyn}(a_V, \gamma)\mathrm{Feyn}(a_W, \delta)$. The values $b(V)$ and $b(W)$ are arbitrary, so one may think of a typical element as a diagram γ on V with some amputated line label j and a diagram δ on W with some amputated line label k. The weight is $C^{a_j a_k}\mathrm{Feyn}(a_V, \gamma)\mathrm{Feyn}(a_W, \delta)$. Map this pair of diagrams γ, δ to a combined diagram σ on U with an additional line from j to k. Its weight $\mathrm{Feyn}(a, \sigma)$ is the same. The designated bridge is the unique line that connected the partition. The conclusion is that one gets the weighted set $F^{\leftrightarrow}[a]$. The map is reversible; given σ and a bridge, there is a partition $\Delta = \{V, W\}$ determined by the connected diagrams γ, δ on each side of the bridge. The exponential generating function identity is $F^{\leftrightarrow}(z) = \frac{1}{2}\sum_x \sum_y F_x^{\odot}(z) C^{xy} F_y^{\odot}(z)$.

Lemma 4.2. *Let $F^{\bigcirc\leftrightarrow}$ be the species of connected vacuum Feynman diagrams with a designated incident lump-bridge pair. Then*

$$F^{\bigcirc\leftrightarrow} = F^{\odot} * F^{\uparrow} = F^{\odot} * (L' \circ F^{\odot}). \tag{29}$$

In the statement of the lemma the right hand side should be interpreted as $\sum_x F_x^{\odot} * F^{\uparrow x} = \sum_x F_x^{\odot} * (L'^x \circ F^{\odot})$. On a colored set

$a : U \to \mathcal{X}$. this gives an ordered pair of subsets V, W that are disjoint and exhaust U. Furthermore, a typical object is given by a connected diagram γ on V with amputated line label i and a connected diagram δ on W with external leg label. This external leg label may be identified with i. The only x that contributes is $x = a_i$. The weight of this pair of diagrams is $\mathrm{Feyn}(a_V, \gamma)\mathrm{Feyn}(a_W, \delta)$. The two diagrams are mapped to a combined diagram σ on all of U. The weight of this diagram is $\mathrm{Feyn}(a, \sigma)$, which is the same. The V subset is the one that has the lump as part of it; the bridge connects V with W. The mapping may be reversed, given σ and the lump-bridge pair, let the connected diagram on the lump side of the bridge define V and let W be the complement. The exponential generating function identity is $F^{\bigcirc\leftrightarrow}(z) = \sum_x F_x^{\odot}(z) F^{\uparrow x}(z) = \sum_x \sum_y F_x^{\odot}(z) C^{xy} F_y^{\odot}(z)$.

4.3. **Lump identities.** Let M be the species of line-irreducible Feynman diagrams with bridge legs. Regard M as a species with color palette $\mathcal{X} \times \{0, \beta\}$ with another dimension. The 0 and β are used to denote ordinary labels and bridge leg labels. A coloring is a pair $a : U \to \mathcal{X}$ and $c : U \to \{0, \beta\}$. Specifying c is the same as specifying a disjoint union decomposition $U = U_0 + U_\beta$. Then $M[a, c]$ consists of all line-irreducible Feynman diagrams with bridge legs such that the ordinary labels are in U_0 and the bridge leg labels are in U_β. Since there are two kinds of labels, there are two kinds of variables. Write them as $z_{x0} = z_x$ and $z_{x\beta} = K_x$. The exponential generating function $M(z, K)$ depends on both kinds of variables.

As before, let $F_y^{\odot}$ denote the species of connected Feynman diagrams with an amputated line of color y. Define a family of species F^* indexed by $\mathcal{X} \times \{0, \beta\}$ by $F^*_{y\beta} = F_y^{\odot}$ and $F^*_{y0} = X_y$.

Lemma 4.3. *Let $F^{\bigcirc}$ be the species of connected vacuum Feynman diagrams with a designated lump. Let F^* be the two-sort family that consists either of the species of connected Feynman diagrams with one amputated line or of the one-point indicator species. Then*

$$F^{\bigcirc} = M \circ F^*. \tag{30}$$

The idea of the proof is that a connected vacuum Feynman diagram together with a specified lump gives a set partition of the underlying set. The lump defines a collection of bridges to connected diagrams supported on a corresponding collection of disjoint subsets. The partition consists of one-point blocks that correspond to the labels in the lump and of other blocks that correspond to the subsets that support connected diagrams.

The proof begins by fixing a colored set $a : U \to \mathcal{X}$. Consider $(M \circ F^*)[a]$. Each partition Δ of U and each coloring $b : \Delta \to \mathcal{X}$ and $c : \Delta \to \{0, \beta\}$ gives a product space $M[b, c] \times \prod_{V \in \Delta} F^*_{b(V)c(V)}[a_V]$. This product space may be written as $M[b, c] \times \prod_{V \in \Delta_0} X_{b(V)}[a_V] \times \prod_{V \in \Delta_\beta} F^{\odot}_{b(V)}[a_V]$. This product space is empty unless the blocks $V \in \Delta_0$ are one point sets $V = \{i\}$ with $b(V) = a_i$. It is also empty unless the blocks $V \in \Delta_\beta$ each give a collection of connected diagrams with corresponding amputated line labels j in V of color $a_j = b(V)$. Identify the blocks $V = \{i\} \in \Delta_0$ with the corresponding i, and identify the blocks $V \in \Delta_\beta$ with the corresponding j. An element of $M[b, c]$, originally a line-irreducible diagram on Δ_0 with bridge leg labels $V \in \Delta_\beta$, is identified with a line-irreducible diagram γ with ordinary labels consisting of the points i with $\{i\}$ in Δ_0 and with bridge leg labels $j \in V \in \Delta_\beta$. An element of the product space also has a diagram δ_V for each V in Δ_β. Putting all these diagrams together gives a connected vacuum Feynman diagram τ on U with the correct weight. The colored partition also singles out a lump of this diagram. The result is that these objects build the weighted set $F^{\bigcirc}[a]$. On the level of exponential generating functions the lemma says that $F^{\bigcirc}(z) = M(z, F^{\odot}(z))$.

4.4. **The dissymmetry theorem for connected Feynman diagrams.** The next topic is symmetry about the center of a tree. A tree is a connected graph without cycles. A leaf of a tree is a vertex of degree zero. The center of a tree is defined recursively by pruning leaves. Remove each leaf along with its incident edges to produce a smaller tree. This process terminates when all the vertices are leaves. There remains a center vertex or two vertices with a center edge. There is always a center element, either a vertex or an edge.

There are combinatorial objects that violate this symmetry. Specifying a distinguished vertex or a distinguished edge in a tree introduces a lack of symmetry, since this distinguished object may not be at the center. The following results are based on a general dissymmetry theorem for trees [2]. This theorem relates the tree to these other less symmetric objects.

Let T be the species of trees; for each set U there is an associated set $T[U]$ of trees with vertex set U. Let $T^\bullet$ be the species of trees with distinguished vertex; in this case for each U there is set $T^\bullet[U]$ consisting of ordered pairs: a point in U and a tree on U. Similarly, let T^- be the species of trees with a distinguished edge. Finally, let $T^{\bullet-}$ be the species of trees with a distinguished incident vertex-edge pair. The dissymmetry theorem for trees states that $T + T^{\bullet-} = T^\bullet + T^-$.

Here is the proof. For the moment refer to an edge or a vertex of the tree as an element. Let $T^\star$ be the species of trees with a distinguished element. It is clear that $T^\star = T^\bullet + T^-$. Let $T^{\star\neq c}$ be the species of trees with a distinguished element other than the center. Then $T^{\star\neq c} = T^{\bullet -}$. The map between these species maps each element other than the center to the pair consisting of the element and the incident element that is closer to the center. To reverse the correspondence, map each pair of incident elements to the element in the pair farthest from the center. On the other hand, $T + T^{\star\neq c} = T^\star$, since specifying a tree automatically specifies its center. These equations combine to give the desired result $T + T^{\bullet -} = T + T^{\star\neq c} = T^\star = T^\bullet + T^-$.

The dissymmetry theorem for connected Feynman diagrams is a variation on the dissymmetry theorem for trees. In this case the vertices of the trees are the lumps, and the edges of the tree are the bridges. Since a connected Feynman diagram determines its center lump or edge, this gives a relation for the species of connected Feynman diagrams.

Proposition 4.1. *Let F be the species of connected vacuum Feynman diagrams. There is a species isomorphism*

$$F + F^{\bigcirc\leftrightarrow} = F^{\bigcirc} + F^{\leftrightarrow}. \tag{31}$$

In a previous section it was shown that these species with designated structures were equivalent to certain species defined as compositions. This leads to the following statement of the dissymmetry theorem.

Theorem 4.1 (Dissymmetry theorem)**.** *The species F of connected vacuum Feynman diagrams satisfies the identity*

$$F + F^{\odot} * (L' \circ F^{\odot}) = M \circ F^* + L \circ F^{\odot}. \tag{32}$$

In terms of exponential generating functions this says that

$$\begin{aligned} F(z) + \sum_x \sum_y F_x^{\odot}(z) C^{xy} F_y^{\odot}(z) &= M(z, F^{\odot}(z)) \\ &\quad + \frac{1}{2} \sum_x \sum_y F_x^{\odot}(z) C^{xy} F_y^{\odot}(z). \end{aligned} \tag{33}$$

The dissymmetry theorem may be expressed in a more detailed form. Recall that $M = M_0 + M_1$, where M_1 has a special form. Namely, $M_1[a, c]$ is empty unless $U = \{i, j\}$ with $c_i = 0$ and $c_j = \beta$. In that case it consists of diagrams with a one-point vertex at i and a bridge leg label at j. The weight of this diagram is $S_{a_i} C^{a_i a_j}$. The exponential generating function of the species M_1 is $M_1(z, K) = \sum_x \sum_y z_x S_x C^{xy} K_y$.

Corollary 4.1. *The species of connected vacuum Feynman diagrams satisfies*

$$F + F^{\odot} * (L' \circ F^{\odot}) = M_0 \circ F^* + M_1 \circ F^* + L \circ F^{\odot}. \tag{34}$$

Proposition 4.2. *The composition $M_1 \circ F^*$ acting on a colored set $a : U \to \mathcal{X}$ gives all connected diagrams for which there is a distinguished point i that belongs to a one-point vertex.*

The corollary thus leads to the exponential generating function identity

$$\begin{aligned} F(z) + \sum_x \sum_y F_x^{\odot}(z) C^{xy} F_y^{\odot}(z) &= M_0(z, F^{\odot}(z)) \\ &+ \sum_x z_x S_x C^{xy} F_y^{\odot}(z) + \frac{1}{2} \sum_x \sum_y F_x^{\odot}(z) C^{xy} F_y^{\odot}(z). \end{aligned} \tag{35}$$

4.5. **The fixed point equation for connected Feynman diagrams.** Recall that $F^{\uparrow x}$ is the species of connected Feynman diagrams with an external leg label of color x. Similarly, $M^{\uparrow x}$ is defined as the species of line-irreducible diagrams with bridge legs and also with a leg to an external leg label of color x. Let $a : U \to \mathcal{X}$ and $c : U \to \{0, \beta\}$. A diagram γ contributing to $M^{\uparrow x}[a, c]$ consists of a partition Γ of the set U_0 where $c = 0$ and a perfect matching σ of $U + \{*\}$. The bridge points in the set U_β where $c = \beta$ are matched to points in U_0, and the extra point $*$ is also matched with a point in U_0. The amputated diagram with the bridge points and the extra point removed is required to be line irreducible. The weight of the diagram is $\mathrm{Feyn}(a^x, \gamma)$. The following theorem is based on the observation that a connected diagram with a leg to an external point singles out a lump with a leg to an external point and with bridge legs each of which is attached to a connected diagram with amputated line.

Theorem 4.2 (Fixed point equation)**.** *The species of connected Feynman diagrams with an external leg satisfies the fixed point equation*

$$F^{\uparrow x} = L'^x \circ F^{\odot} = M^{\uparrow x} \circ F^*. \tag{36}$$

Here is a proof sketch. To analyze the composition, begin as usual with $a : U \to \mathcal{X}$. The weighted set $(M^{\uparrow x} \circ F^*)[a]$ is then a disjoint sum over set partitions Δ of U and colorings $b : \Delta \to \mathcal{X}$ and $c : \Delta \to \{0, \beta\}$ of products $M^{\uparrow x}[b, c] \times \prod_{V \in \Delta_0} X_{b(V)}[a_V] \times \prod_{V \in \Delta_\beta} F^{\odot}_{b(V)}[a_V]$. Identify each block $V = \{i\} \in \Delta_0$ with the corresponding i, and identify each block $V \in \Delta_\beta$ with the corresponding amputated line label j in V. An element of $M^{\uparrow x}[b, c]$ is identified with a line-irreducible diagram γ with ordinary labels consisting of the points i with $\{i\}$ in Δ_0, with bridge leg labels $j \in V \in \Delta_\beta$, and with one more leg label $*$. There is a line from some k to $*$. An element of the product space also has a diagram δ_V for each V in Δ_β. Putting all these diagrams together gives a connected vacuum Feynman diagram τ on $U + \{*\}$ with weight that includes a factor C^{xa_k}. Such diagrams build the weighted set $F^{\uparrow x}[a]$. In terms of exponential generating functions the theorem says that

$$F^{\uparrow x}(z) = \sum_x C^{xy} F^{\odot}_y(z) = M^{\uparrow x}(z, F^{\odot}(z)). \tag{37}$$

The decomposition $M = M_0 + M_1$ gives a corresponding decomposition $M^{\uparrow x} = M_0^{\uparrow x} + M_1^{\uparrow x}$. Here $M_0^{\uparrow x}$ is the species of line-irreducible diagrams with bridge legs that include a leg to an external point of color x and that have no one-point vertices. The other species $M_1^{\uparrow x}$ is the species that only gives a single line from a point i belonging to a one-point vertex to an external point $*$ of color x. Namely, $M_1^{\uparrow x}[a, c]$ is empty unless $U = \{i\}$ with $c_i = 0$. In that case it consists of a line with a one-point vertex at i and an external leg label at $*$ of color x. The weight of this diagram is $S_{a_i} C^{a_i x}$. There are no bridge leg labels, and so composition with this species is trivial.

Proposition 4.3. *The composition $M_1^{\uparrow x} \circ F^*$ acting on a colored set $a : U \to \mathcal{X}$ returns the empty set, unless $U = \{i\}$ is a one-point vertex. It that case it give a diagram on $U + \{*\}$ with a line from i to *. The weight of the diagram is $S_{a_i} C^{a_i x}$.*

With a slight abuse of notation one may denote $M_1^{\uparrow x} \circ F^*$ by $M_1^{\uparrow x}$. The exponential generating function is just $M_1^{\uparrow x}(z) = \sum_y C^{xy} S_y z_y$.

Corollary 4.2. *The species of connected Feynman diagrams with an external leg satisfies the fixed point equation*

$$F^{\uparrow x} = L'^{x} \circ F^{\odot} = M_0^{\uparrow x} \circ F^* + M_1^{\uparrow x}. \tag{38}$$

In terms of exponential generating functions this says that

$$F^{\uparrow x}(z) = \sum_x C^{xy} F^{\odot}_y(z) = M_0^{\uparrow x}(z, F^{\odot}(z)) + \sum_y C^{xy} S_y z_y. \tag{39}$$

5. The Legendre transform

5.1. The Legendre transform for connected Feynman diagrams. The results above give combinatorial identities for connected vacuum Feynman diagrams. In order to make the connection with the Legendre transform, it is useful to replace one-point vertices by leg labels, so there is no longer a factor S_x attached to such a vertex.

Furthermore, the leg labels are specified as part of the color structure. The resulting species provides a more detailed combinatorial description. Henceforth F denotes this species of connected Feynman diagrams with legs.

Given $a : U \to \mathcal{X}$ and $c : U \to \{0, \lambda\}$, the corresponding weighted set $F[a, c]$ consists of all connected Feynman diagrams with ordinary labels on the subset U_0 where $c = 0$ and with leg labels on the subset U_λ where $c = \lambda$. A diagram γ consists of a set partition Γ of U_0 into blocks of size two or more and a perfect matching σ of U, with the constraint that these structures connect the label set U. The weight of such a diagram is $\mathrm{Feyn}(a, \gamma) = S(a, \Gamma)C(a, \sigma)$. The reason this works is that a partition of U is equivalent to a subset of U together with a partition of the complement into blocks of size two or more. The sum over subsets arising from the various colorings c together with the sum over the partitions of the complement into blocks of size two or more give the sum over all partitions.

Denote the variables for the colors by $z_{x0} = z_x$ and by $z_{x\lambda} = J_x$. The J_x variables that occur in the exponential generating function compensate the missing S_x parameters in the action, so the effect is to replace every $S_x z_x$ by J_x. The derivative species of F with respect to a leg label of color x will be denoted F'^x. This replaces the species $F^{\uparrow x}$ considered before. The value of F'^x on a colored set is a connected Feynman diagram on the set U augmented by an external point $*$ of color x that is a leg label. That is, $F'^x[a, c] = F[a^x, c^\lambda]$, where a^x takes value x and c^λ takes value λ on $*$. It follows that the exponential generating functions are $F(z, J)$ and $F'^x(z, J) = \partial F(z, J)/\partial J_x$.

The species $F^\odot_y$ of connected Feynman diagrams with a single amputated leg label of color y is defined similarly. Give a and c the corresponding diagrams are given by a choice of j in U with $a_j = y$, a partition Γ of U_0, and a perfect matching of $U - \{j\}$.

In the dissymmetry theorem there are now two kinds of leg labels, and in the appropriate places it is natural to employ a color palette $\mathcal{X} \times \{0, \lambda, \beta\}$. The leg labels corresponding to λ come from the one-point vertices, while the leg labels corresponding to β are the bridge leg labels that come from the decomposition into line-irreducible diagrams. The relevant species is M, the species of line-irreducible diagrams with bridge legs. This is defined for a colored set with colorings $a : U \to \mathcal{X}$ and $c : U \to \{0, \lambda, \beta\}$. The coloring c defines a decomposition $U = U_0 + U_\lambda + U_\beta$. The resulting diagrams each consist of a partition Γ of U_0 and a perfect matching of U. Each bridge label in U_β is connected by a line to $U_0 + U_\lambda$. Each diagram is line-irreducible on $U_0 + U_\lambda$.

The remarkable fact is that $M = M_0 + M_1$, where the two component species have entirely different characters. The species M_0 gives line-irreducible diagrams with bridge legs for which U_λ is empty, while the species M_1 gives line-irreducible diagrams with bridge legs for which U_λ is non-empty. However the only M_1 diagrams are those that involve a line from a λ leg label to a β bridge label. The weight of such a diagram with $c_i = \lambda$, $c_j = \beta$ is $C^{a_i\, a_j}$.

Set $K_y = F^{\odot}_y$. Then K^* is a family of species such that $K^*_{y0} = X_{y0}$, $K^*_{y\lambda} = X_{y\lambda}$, and $K^*_{y\beta} = K_y$. The following result is a restatement of the previous dissymmetry theorem, except that Feynman diagrams are on a label set including λ leg labels.

Theorem 5.1 (Dissymmetry theorem). *The species F of connected Feynman diagrams with legs satisfies*

$$F + K * (L' \circ K) = M_0 \circ K^* + M_1 \circ K^* + L \circ K, \tag{40}$$

where $K = F^{\odot}$.

In terms of exponential generating functions this says that

$$F(z, J) + \sum_x \sum_y K_x C^{xy} K_y = M_0(z, K) + \sum_x J_x C^{xy} K_y + \frac{1}{2} \sum_x \sum_y K_x C^{xy} K_y, \tag{41}$$

where $K_y = F^{\odot}_y(z, J)$. This equation exhibits the duality that underlies the Legendre transform. The J_x parameters play the role of the one-point vertices. The K_y represent the contribution of connected graphs with an amputated line. The JCK term in the formula is the dual pairing of these quantities. The duality is mediated by the line corresponding to covariance C^{xy}.

If F is the species of connected Feynman diagrams with legs, then F'^x is the species of connected Feynman diagrams with legs and with an additional line to an external leg label of color x. Similarly, $F^{\odot}_y$ is the species of connected Feynman diagrams with legs and with one amputated line label of color y. The species M of line-irreducible diagrams with legs and bridge legs has a derivative species M'^x whose value on a, c is the value of M on the a^x, c^β that color the extra point * with colors x, β. This in turn has a decomposition $M'^x = M'^x_0 + M'^x_1$. The species M'^x_0 is the species of line-irreducible diagrams with bridge legs (but no other legs) that include a line to an external bridge leg label of color x. Similarly, M'^x_1 is the species of lines from a leg label to an external bridge leg label of color x. The weight of such a single line diagram with $c_i = \lambda$ is $C^{a_i\,x}$. The relation between these species is given by the following result.

Theorem 5.2. *The species of connected Feynman diagrams with legs satisfies the fixed point equation*

$$F'^x = L'^x \circ K = M_0'^x \circ K^* + M_1'^x, \tag{42}$$

where $K = F^\odot$.

In terms of exponential generating functions this says that

$$\frac{\partial F(z, J)}{\partial J_x} = \sum_x C^{xy} K_y(z, J) = \frac{\partial M_0(z, K(z, J))}{\partial K_x} + \sum_y C^{xy} J_y, \tag{43}$$

where $K_y = F_y^\odot(z, J)$.

5.2. **The Legendre transform for exponential generating functions.** The formulas in the physics literature are obtained from the exponential generating function identities by setting each $z_x = 1$ and then suppressing these variables from the notation. The dissymmetry theorem then takes the form

$$F(J) = M_0(K) + JCK - \frac{1}{2} KCK. \tag{44}$$

The fixed point theorem is

$$\frac{\partial F(J)}{\partial J_x} = \sum_y C^{xy} K_y = \frac{\partial M_0(K)}{\partial K_x} + \sum_y C^{xy} J_y. \tag{45}$$

Define

$$W(K) = \frac{1}{2} KCK - M_0(K). \tag{46}$$

The Legendre transformation in terms of covariant variables is

$$F(J) + W(K) = JCK. \tag{47}$$

where

$$\frac{\partial F(J)}{\partial J_x} = \sum_y C^{xy} K_y. \tag{48}$$

and

$$\frac{\partial W(K)}{\partial K_x} = \sum_y C^{xy} J_y. \tag{49}$$

To go from $F(J)$ to $W(K)$ one needs to solve for J in terms of K by inverting the first equation. To go from $W(K)$ to $F(J)$ one needs to solve for K in terms of J by inverting the second equation.

It is common to use variables $\phi = CK$. Define the effective potential $S(\phi) = W(K)$. If we write $I(\phi) = M_0(K)$, then this gives

$$S(\phi) = \frac{1}{2}\phi C^{-1}\phi - I(\phi). \tag{50}$$

One possible reason for hesitating to use these variables is to avoid the awkward inverse covariance C^{-1}. The Legendre transformation in terms of these variables is

$$F(J) + S(\phi) = J\phi. \tag{51}$$

where

$$\frac{\partial F(J)}{\partial J_x} = \phi^x. \tag{52}$$

and

$$\frac{\partial S(\phi)}{\partial \phi^x} = J_x. \tag{53}$$

To go from $F(J)$ to $S(\phi)$ one needs to solve for J in terms of ϕ by inverting the first equation. To go from $S(\phi)$ to $F(J)$ one needs to solve for ϕ in terms of J by inverting the second equation.

6. Appendix: Abstract Feynman diagrams

6.1. **Set partitions and integer partitions.** The kind of Feynman diagram considered in the main text of this article is what might be called a labeled Feynman diagram. This appendix is a brief description of the passage from labeled Feynman diagrams to abstract Feynman diagrams, that is, to isomorphism classes of Feynman diagrams. This will help make a comparison with other approaches, such as that of [1] and [7]. For the sake of exposition, the basic definitions will be repeated, so this appendix will be largely self-contained. It starts with some combinatorial background.

Let U be a set with n elements. A *set partition* Γ of U is a collection of non-empty subsets with no overlaps and whose union is U. Each subset V in Γ is called a *block*. The number of partitions of U with v blocks is the *Stirling set partition number* $S(n, v)$ (also known as the Stirling number of the second kind). The total number of partitions of U is the *Bell number* $B(n)$.

Let $n \geq 0$ be a natural number. An *integer partition* of n is a multiset of natural numbers $k \geq 1$ with sum n. Each such number k is called a *part* of n. An integer partition is naturally specified by its *type*: a sequence of numbers p_k for $k = 1, 2, 3, \ldots$ such that

$$1p_1 + 2p_2 + 3p_3 + \cdots + kp_k + \cdots = n. \tag{54}$$

Thus p_k represents the number of times the number $k \geq 1$ is used in the sum that gives n. If the integer partition has v parts, then

$$p_1 + p_2 + p_3 + \cdots + p_k + \cdots = v. \tag{55}$$

The number of integer partitions of n into v parts is denoted $P(n, v)$. The total number of integer partitions of n is denoted $P(n)$.

For each set partition Γ there is corresponding integer partition type p. As we shall see, the number of set partitions with type p is

$$O(p) = \frac{n!}{\prod_k p_k!(k!)^{p_k}}. \tag{56}$$

It follows that the Stirling set partition number $S(n, v)$ is given by

$$S(n, v) = \sum_p O(p), \tag{57}$$

where the sum is over types p with $1p_1 + 2p_2 + 3p_3 + \cdots + kp_k + \cdots = n$ and $p_1 + p_2 + p_3 + \cdots + p_k + \cdots = v$. The number of terms in this sum is $P(n, v)$.

In the following sections these ideas are illustrated in detail for $n = 4$ and $n = 6$. Here is a summary for quick reference. For $n = 4$ the numbers $P(4, v)$ for $v = 1, 2, 3, 4$ are $1, 2, 1, 1$ with sum $P(4) = 5$. The corresponding numbers $S(4, v)$ are $1, 7, 6, 1$ with sum $B(4) = 15$. For $n = 6$ the numbers $P(6, v)$ for $v = 1, 2, 3, 4, 5, 6$ are $1, 3, 3, 2, 1, 1$ with sum $P(4) = 11$. The corresponding numbers $S(6, v)$ are $1, 31, 90, 65, 15, 1$ with sum $B(6) = 203$.

6.2. **Feynman diagrams.** For simplicity the discussion will concentrate on vacuum Feynman diagrams; the term Feynman diagram with no modifier will mean vacuum Feynman diagram. However, at the end there will be a brief reference to the more general concept of truncated Feynman diagram.

Fix a set U with n elements. A *Feynman diagram* with label set U has vertices and lines. The vertex part consists of a set partition Γ of U. The line part consists of a perfect matching σ of U. (A perfect matching is a set partition into two-element subsets.) Thus n must be even, and U consists of all possible end points of lines. The case $n = 0$ is allowed; in that case both Γ and σ are empty, and there is only one diagram. A Feynman diagram built over U may be pictured as a set U of points collected into bags that correspond to the blocks of the partition, together with lines between pairs of points given by the perfect matching.

Let F be the set of Feynman diagrams built over U. The number of Feynman diagrams is the number of set partitions times the number of perfect matchings. The number of set partitions is $B(n)$, the nth Bell number. Perfect matchings are set partitions corresponding to the integer partition with type given by $p_2 = n/2$ and all other $p_k = 0$. The general formula for $O(p)$ gives the number of perfect matchings as

$$m_n = \frac{n!}{\left(\frac{n}{2}\right)!\, 2^{\frac{n}{2}}}. \tag{58}$$

A Feynman diagram defines a graph, where the blocks of the set partition are the vertices of the graph, and the lines of the perfect matching are the edges of the graph. In other words, each block is shrunk to a single vertex. If a block has k points, then the corresponding vertex has degree k, that is, it can have k edges attached. The graph may have multiple edges joining a pair of vertices. Also, the graph may have loops, that is, edges from a vertex to itself.

The contribution of a Feynman graph is calculated from data associated with a modification of a Gaussian process. The set $\mathcal{X}$ is the index set for a Gaussian process. For a finite set W consider a function $a : W \to \mathcal{X}$; this is an assignment of indices. For each such index assignment there is a corresponding set of occupation numbers $N_a(x) = \#\{j \mid a_j = x\}$. The process is determined by interaction vertex coefficients S_a and line covariance coefficients C^a. These only depend on a through the occupation number coefficients N_a. In the case of a line covariance coefficient C^a the set W has exactly two points. Thus when $N_a(x) = 2$ this is the variance C^{xx}, while when $N_a(x) = N_a(y) = 1$ this is the covariance C^{xy}.

If $\gamma = (\Gamma, \sigma)$ is a Feynman diagram in F, and $a : U \to \mathcal{X}$ is given, the contribution has two main factors. The factor $S(a, \Gamma) = \prod_{V \in \Gamma} S_{a_V}$ is the product of interaction factors associated with the vertices. Here $a_V : V \to \mathcal{X}$ denotes the restriction of a to $V \subseteq U$. If a vertex V is of large enough degree, then such a factor is supposed to be zero (or at least small). This means that the contribution (or the main contribution) is from graphs with limited degree of the vertices. The factor $C(a, \sigma) = \prod_{W \in \sigma} C^{a_W}$ is the product of the covariances of the Gaussian process over the pairs in the perfect matching. In other words, it is a product of factors associated with the lines. The total contribution is obtained by taking product of these two factors and then summing over the index assignments, giving

$$t(\gamma) = \sum_{a : U \to \mathcal{X}} S(a, \Gamma) C(a, \sigma). \tag{59}$$

The contribution to the partition function of the Feynman diagrams with given U (representing n endpoints of lines) is

$$Z_n = \frac{1}{n!} \sum_{\gamma \in F} t(\gamma). \tag{60}$$

The purpose of the discussion in the following sections is to relate this quantity to abstract Feynman diagrams, defined as equivalence classes of Feynman diagrams that capture the graphical structure.

Before turning to examples, it may help to recall graphical ideas associated with Feynman diagrams on a set U. First, it makes sense to say that a diagram is connected. This holds if U is non-empty and cannot be partitioned in a way that decomposes the diagram into a pair of diagrams. A stronger condition is to say that the diagram is line-irreducible. This says that the diagram is connected and furthermore there is no bridge in the diagram, that is no line whose removal disconnects the diagram.

Here are the two one-line ($n = 2$) abstract Feynman diagrams.

- One vertex. Loop.
- Two vertices. Bridge.

These are both connected, but the only line-irreducible diagram is the loop.

Here are the four connected two-line ($n = 4$) abstract Feynman diagrams g and the corresponding number $O(g)$ of labeled diagrams.

- One vertex. Double loop. $O(g) = 3$
- Two vertices. Bridge-loop. $O(g) = 12$
- Two vertices. Double parallel lines. $O(g) = 6$
- Three vertices. Bridge-bridge. $O(g) = 12$

The only ones that are line-irreducible are the double loop and the double parallel lines. Each abstract Feynman diagram comes from a number of different labeled Feynman diagrams. To see how this works, start with a label set with four elements. First pick the set partitions that define a suitable vertex set. Then find the perfect matchings that create the lines of the appropriate diagram. The results for the four diagrams are $1 \cdot 3 = 3$, $4 \cdot 3 = 12$, $3 \cdot 2 = 6$ and $6 \cdot 2 = 12$. This gives the statistics in the above list.

6.3. **Vertex sets.** As a preliminary step, examine the vertex structure alone. Let P be the collection of set partitions of U. The permutation group G of U is a group of order $n!$. It acts in a natural way on P. Let $\mathcal{P} = P/G$ be the collection of orbits of this action. Each orbit p is characterized by numbers p_k, where a set partition Γ in p has p_k blocks of cardinality k. These numbers specify an integer partition of the integer $n \geq 0$ into parts, where there are $p_k \geq 0$ parts of size $k \geq 1$. Furthermore, if Γ has v blocks, then the integer partition is into v parts. In the Feynman diagram context an integer partition is considered as a *vertex set*, where the number of vertices that accommodate k lines is p_k. Each vertex in the vertex set is pictured as a point with k half-lines emerging from it. The vertex set itself is a collection of such vertices. A vertex set also could be called an *abstract pre-Feynman diagram*, since it is related to a similar concept introduced in [1]. When the half-lines are paired up and joined to create a graph, the result is an abstract Feynman diagram. The total number of vertices in a vertex set is v. The number of half-lines emerging from the vertices is the number n of potential end points of lines.

Consider a vertex set p in $\mathcal{P}$ and set partition Γ in p. Let G_Γ be the subgroup of G that leaves Γ invariant. This is the stabilizer (isotropy subgroup) of Γ. If Γ has p_k blocks of size k, then the order of this group is

$$A(p) = |G_\Gamma| = \prod_k p_k!\,(k!)^{p_k}. \tag{61}$$

This is because for each fixed k one can permute the p_k vertices, and within each vertex one can permute the k lines that emanate from this vertex.

If Γ belongs to p in $\mathcal{P}$, then $p = G\Gamma$ is the orbit of Γ under the group G. There is a natural map from G/G_Γ to $G\Gamma$. In particular, the size of the orbit $O(p) = |G\Gamma|$ is the number of cosets in G/G_Γ, which in turn is

$$O(p) = |G\Gamma| = \frac{|G|}{|G_\Gamma|} = \frac{n!}{A(p)}. \tag{62}$$

In other words, there are $O(p)$ set partitions for each integer partition p.

Consider the case $n = 4$. There are five vertex sets (integer partitions of 4). The total number of set partitions is $B_4 = 1+(4+3)+6+1 = 15$. If we group according to the number of blocks in the set partition, this is the expression for the Bell number $B_4 = 1 + 7 + 6 + 1 = 15$ as a sum of Stirling set partition numbers. The statistics of the corresponding five vertex sets are shown in Figure 1. The sum in the $O(p)$ column is of course 15.

For the case $n = 6$ there are 11 different vertex sets (integer partitions of 6). The total number of set partitions is $B_6 = 1 + (6 + 15 + 10) + (15 + 60 + 15) + (45 + 20) + 15 + 1 = 203$. If we group according to the number of blocks in the set partition, this is the expression for the Bell number $B_6 = 1 + 31 + 90 + 65 + 15 + 1 = 203$ as a sum of Stirling

v	p_1	p_2	p_3	p_4	$A(p)$	$O(p)$
1	0	0	0	1	24	1
2	1	0	1	0	6	4
2	0	2	0	0	8	3
3	2	1	0	0	4	6
4	4	0	0	0	24	1

TABLE 1. Statistics of two-line vertex sets

v	p_1	p_2	p_3	p_4	p_5	p_6	$A(p)$	$O(p)$
1	0	0	0	0	0	1	720	1
2	1	0	0	0	1	0	120	6
2	0	1	0	1	0	0	48	15
2	0	0	2	0	0	0	72	10
3	2	0	0	1	0	0	48	15
3	1	1	1	0	0	0	12	60
3	0	3	0	0	0	0	48	15
4	2	2	0	0	0	0	16	45
4	3	0	1	0	0	0	36	20
5	4	1	0	0	0	0	48	15
6	6	0	0	0	0	0	720	1

TABLE 2. Statistics of three-line vertex sets

set partition numbers. The statistics are given in Figure 2. The sum in the $O(p)$ column is 203.

6.4. **Abstract Feynman diagrams.** A Feynman diagram γ on the label set U (endpoints of lines) consists of a set partition Γ of U (a collection of vertices) and a perfect matching σ of U (a collection of lines). The permutation group G of U acts in a natural way on the set F of Feynman diagrams on U. Let $\mathcal{F} = F/G$ be the collection of orbits of this action. An element of $\mathcal{F}$ is called an *abstract Feynman diagram.*

The values of $t(\gamma)$ are constant for γ belonging to a particular orbit g in $\mathcal{F}$. It is thus consistent to denote the contribution of a diagram in the orbit g by $t(g)$. Let $O(g)$ be the number of points in the orbit of the abstract Feynman diagram g, that is, the number of labeled Feynman diagrams with the same graphical structure. In this notation we have

$$Z_n = \frac{1}{n!} \sum_{g \in \mathcal{F}} O(g) t(g). \tag{63}$$

This expresses Z_n as a sum over abstract Feynman diagrams of corresponding contributions.

Consider an abstract Feynman diagram g in $\mathcal{F}$ and diagram γ in g. Let G_γ be the subgroup of the permutation group G on U that leaves γ invariant. This is the stabilizer (isotropy subgroup) of the diagram γ. If γ, γ' belong to the same orbit g, then G_γ and $G_{\gamma'}$ are conjugate. The isomorphism class of such stabilizers constitutes the abstract *automorphism group* of the abstract Feynman diagram g. Let $A(g) = |G_\gamma|$ for $\gamma \in g$ be the order of this group.

If γ belongs to g in $\mathcal{F}$, then $g = G\gamma$ is the orbit of γ under the group G. There is a natural map from G/G_γ to $G\gamma$. In particular, the size of the orbit

$$O(g) = |G\gamma| \tag{64}$$

is the number of cosets in G/G_γ, which in turn is $|G|/|G_\gamma| = n!/A(g)$. This gives the result

$$A(g) = |G_\gamma| = \frac{n!}{|G\gamma|} = \frac{n!}{O(g)}. \tag{65}$$

This leads to the remarkable identity

$$Z_n = \sum_{g \in \mathcal{F}} \frac{1}{A(g)} t(g). \tag{66}$$

This identity says that Z_n is the sum of the weights of the abstract Feynman graphs divided by their symmetry factors.

It is easy to pass from an abstract Feynman diagram to a Feynman diagram; all that is required is to label the end points of the lines. Then an automorphism of this labeled diagram is a map from the label set to itself such that lines are taken to lines and vertices are taken to vertices. A more abstract point of view is to think of a map from end points of lines to end points of lines such that lines are taken to lines and vertices are taken to vertices.

Example: Consider the abstract Feynman diagram g with two vertices and two parallel lines. The two vertices may be labeled with 1,2 and with 3,4 in such a way that one line runs from 1 to 3 and the other from 2 to 4. The stabilizer group of the diagram is of order $A(g) = 4$. There is the identity automorphism, the automorphism $(12)(34)$, the automorphism $(13)(24)$, and the automorphism $(14)(23)$. From the abstract point of view these do nothing, interchange the lines, interchange the vertices, and interchange the vertices and lines.

In this same example, one can compute the orbit of this diagram. These are the other diagrams with the same abstract structure that are obtained by permuting the end points of lines. The number of such diagrams is $O(g) = 6$. There are three possibilities for grouping the end points into vertices: 12 and 34, 13 and 24, 14 and 23. For each of these vertices one can join the vertices with lines in two different ways. So the number of such diagrams is $O(g) = 3 \cdot 2 = 6$.

Example: The automorphism group acts on vertices, and it also acts on lines. However behind it all the automorphism group acts on the end points of the lines. An extreme example is the loop on one vertex. The automorphism group swaps the end points, so it has order $A(g) = 2$. The same thing happens for the double loop on one vertex. The automorphism group includes the operation of interchanging the loops, but it also includes swapping the end points of the individual loops. Thus for this example $A(g) = 8$.

Another way of approaching the classification of abstract Feynman diagrams is to fix the partition Γ belonging to the vertex set p. The stabilizer group G_Γ of Γ is a group of order $A(p) = \prod_k p_k!(k!)^{p_k}$. The stabilizer group of a Feynman diagram γ is a subgroup of this group. The quotient G_Γ/G_γ is the orbit $G_\Gamma\gamma$ of the perfect matchings under this smaller group. These orbits are smaller and relatively easy to compute. They are just all the ways of attaching lines to a given vertex set to make the desired diagram. Define

$$O(g|p) = |G_\Gamma\gamma|. \tag{67}$$

So this gives another way of computing $A(g) = |G_\gamma|$, namely,

$$A(g) = \frac{|G_\Gamma|}{|G_\Gamma\gamma|} = \frac{A(p)}{O(g|p)}. \tag{68}$$

In particular, $A(g)$ must divide $A(p)$, which in turn divides $n!$. An alternative way of doing the computation of $A(g) = n!/O(g)$ is to use

$$O(g) = O(p)O(g|p). \tag{69}$$

Example: It is illuminating to look at the example of the diagram with two vertices and two parallel lines. The vertex set p consists of the two vertices, each of degree 2. Say that these vertices are labeled 12 and 34. The vertex set stabilizer group is of order $A(p) = 8$. It switches the end points within each vertex and also switches the vertices. These permutations also act on the Feynman diagrams. Consider the Feynman diagram g with a 13 and a 24 line. The stabilizer of g has order $A(g) = 4$. The orbit of this diagram under the vertex set stabilizer group has only two diagrams: the original diagram and the one with lines switched, that is, with the same vertices but with a 14 and a 23

v	p_1	p_2	p_3	p_4	$A(p)$	$O(p)$	$O(g\|p)$	$O(g)$	$A(g)$
1	0	0	0	1	24	1	3	3	8
2	1	0	1	0	6	4	3	12	2
2	0	2	0	0	8	3	2	6	4
2	0	2	0	0	8	3	1	3	8
3	2	1	0	0	4	6	2	12	2
3	2	1	0	0	4	6	1	6	4
4	4	0	0	0	24	1	3	3	8

TABLE 3. Statistics of two-line abstract Feynman diagrams

line. This shows that $O(g|p) = 2$. Since $O(p) = 3$, the product is $O(g) = 3 \cdot 2 = 6$, as it should be.

6.5. **Two-line diagrams.** It helps to look at examples more systematically. Here are the seven two-line ($n = 4$) abstract Feynman diagrams, together with their decompositions into connected components.

- One vertex. Double loop.
- Two vertices. Bridge-loop.
- Two vertices. Double parallel lines.
- Two vertices. Loop + loop.
- Three vertices. Bridge-bridge.
- Three vertices. Bridge + loop.
- Four vertices. Bridge + bridge.

Four of these are connected. Of these, the double loop and the double parallel line diagrams are the only line-irreducible diagrams.

Since there are $B_4 = 15$ set partitions and $m_4 = 3$ perfect matchings, there are 45 Feynman diagrams. There are 7 abstract Feynman diagrams. The total number of diagrams grouped by vertex type is $3 + 12 + (6 + 3) + (12 + 6) + 3 = 45$. For each of the five vertex sets, the sum of the sizes of the corresponding orbits is 3. The statistics for these abstract Feynman diagrams are displayed in Figure 3. The sum in the $O(g|p)$ column is the number of vertex sets times the number of perfect matchings, that is, $5 \cdot 3 = 15$. The sum in the $O(g)$ column is the number of set partitions times the number of perfect matchings, which is $15 \cdot 3 = 45$.

6.6. **Three-line diagrams.** A nice test of the theory is to classify all three-line ($n = 6$) Feynman diagrams. The number of perfect matchings is $m_6 = 15$, so the total number of Feynman diagrams is $203 \cdot 15 = 3045$. However they only lead to 23 three-line abstract Feynman diagrams. Here is the list.

- One vertex. Triple loop.
- Two vertices. Bridge-double loop.
- Two vertices. Double parallel lines with loop.
- Two vertices. Loop + double loop.
- Two vertices. Triple parallel lines.
- Two vertices. Loop-bridge-loop.
- Three vertices. Bridge-loop-bridge.
- Three vertices. Bridge + double loop.
- Three vertices. Bridge-double parallel lines.
- Three vertices. Bridge-bridge-loop.
- Three vertices. Loop + bridge-loop.
- Three vertices. Triple line triangle.
- Three vertices. Loop + Double parallel lines.
- Three vertices. Loop + loop + loop.
- Four vertices. Bridge-bridge-bridge.
- Four vertices. Bridge-bridge + loop.
- Four vertices. Bridge + double parallel lines.
- Four vertices. Bridge + loop + loop.
- Four vertices. Triple bridge star.
- Four vertices. Bridge + bridge-loop.
- Five vertices. Bridge + bridge-bridge.
- Five vertices. Bridge + bridge + loop.
- Six vertices. Bridge + bridge + bridge.

The statistics of these 23 diagrams are summarized in Figure 4. The sum in the $O(g|p)$ column is the number of vertex sets times the number of perfect matchings: $11 \cdot 15 = 165$, and the sum in the $O(g)$ column is the number of set partitions times the number of perfect matchings: $203 \cdot 15 = 3045$.

6.7. **Line-irreducible graphs.** Here are remarks on the further decomposition of Feynman diagrams. For this purpose a more general concept is needed. An amputated Feynman diagram on U consists of a set partition Γ of U and a perfect matching σ of a subset of U. Again the blocks of the set partition are regarded as vertices, and the matching pairs are regarded as lines that run between vertices. The unmatched points in U are each regarded as a potential end point of a line—an amputated line. A connected Feynman diagram is line-irreducible if it has no bridge, that is, it has no line whose removal disconnects the diagram. When there are bridges, their removal produces components that are line-irreducible amputated diagrams.

There are only two one-line abstract Feynman diagrams with $n = 2$. The one-vertex loop is line-irreducible, while the two-vertex bridge

v	p_1	p_2	p_3	p_4	p_5	p_6	$A(p)$	$O(p)$	$O(g\|p)$	$O(g)$	$A(g)$
1	0	0	0	0	0	1	720	1	15	15	48
2	1	0	0	0	1	0	120	6	15	90	8
2	0	1	0	1	0	0	48	15	12	180	4
2	0	1	0	1	0	0	48	15	3	45	16
2	0	0	2	0	0	0	72	10	6	60	12
2	0	0	2	0	0	0	72	10	9	90	8
3	2	0	0	1	0	0	48	15	12	180	4
3	2	0	0	1	0	0	48	15	3	45	16
3	1	1	1	0	0	0	12	60	6	360	2
3	1	1	1	0	0	0	12	60	6	360	2
3	1	1	1	0	0	0	12	60	3	180	4
3	0	3	0	0	0	0	48	15	8	120	6
3	0	3	0	0	0	0	48	15	6	90	8
3	0	3	0	0	0	0	48	15	1	15	48
4	2	2	0	0	0	0	16	45	8	360	2
4	2	2	0	0	0	0	16	45	4	180	4
4	2	2	0	0	0	0	16	45	2	90	8
4	2	2	0	0	0	0	16	45	1	45	16
4	3	0	1	0	0	0	36	20	6	120	6
4	3	0	1	0	0	0	36	20	9	180	4
5	4	1	0	0	0	0	48	15	12	180	4
5	4	1	0	0	0	0	48	15	3	45	16
6	6	0	0	0	0	0	720	1	15	15	48

TABLE 4. Statistics of three-line abstract Feynman diagrams

decomposes into two $n = 1$ diagrams each consisting of a vertex with an amputated line. The four connected two-line abstract Feynman diagrams with $n = 4$ provide a richer variety of examples. The one-vertex double loop diagram is line-irreducible. The two-vertex bridge-loop diagram decomposes into a $n = 1$ vertex with amputated line diagram plus a $n = 3$ vertex with loop and amputated line diagram. The two-vertex double parallel line diagram is line-irreducible. The three-vertex bridge-bridge diagram decomposes into three parts: two $n = 1$ vertex with amputated line diagrams and one $n = 2$ vertex with double amputated lines diagram.

6.8. **Related work.** The article of Abdesselam [1] is in a somewhat different setting. His Feynman diagrams have oriented lines, which corresponds in physics to bosons with charge. He uses the Feynman diagram formulation to represent various kinds of functional composition, and this is directional by nature. Nevertheless, there are elements in common with the present treatment. The principal difference is that in the main part of the present treatment the species of Feynman diagrams is formulated as a functor on colored sets. This means that the weight of a Feynman diagram does not have a sum over colors; that sum arises only when the exponential generating function is computed. By contrast, in the treatment by Abdesselam the species consists of abstract Feynman diagrams. These are regarded as functors on sets, and the weight of an abstract Feynman diagram involves a sum. The transition from one point of view to the other was presented earlier in this appendix. The article [1] has a rich variety of other combinatorial ideas, including a symbolic calculus of functional integration.

The Brydges–Leroux article [3] does not consider Feynman diagrams; the subject is ordinary graphs. In the statistical mechanics of particles these would be called Mayer graphs. As is well-known [11], there is a decomposition of connected graphs into 2-vertex-connected subgraphs that is important for statistical mechanics. In their article the graphs are defined over sets, so color does not play a role. The interesting feature is that the authors study the decomposition of connected graphs into 2-edge-connected subgraphs. This is a close analog to what is done in the decomposition of Feynman diagrams into line-irreducible graphs. It is not clear to the present author whether this decomposition could also play a role in statistical mechanics.

Another fascinating direction is higher Legendre transforms. This subject is treated in the articles of Cooper, Feldman, and Rosen (see for instance [5]) and in the references they cite. There is also a book by Vasiliev [12] that gives a systematic account.

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