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# DECOMPOSABLE FUNCTORS AND THE EXPONENTIAL PRINCIPLE, II

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Abstract. We develop a new setting for the exponential principle in the context of multisort species, where indecomposable objects are generated intrinsically instead of being given in advance. Our approach uses the language of functors and natural transformations (composition operators), and we show that, somewhat surprisingly, a single axiom for the composition already suffices to guarantee validity of the exponential formula. We provide various illustrations of our theory, among which are applications to the enumeration of (semi-)magic squares.

## 1. Introduction

One of the corner stones of combinatorial enumeration is a theory which runs under several different names, for instance *theory of species* [6], *theory of exponential structures* [28, Ch. 5], *theory of exponential families* [29], *symbolic method* [11, Part A], *théorie du composé partitionnel* [12], *theory of prefabs* [4], all of which are more or less equivalent. It is probably fair to say that the most elaborate of these theories is the theory of species, as formulated by Joyal [16] (with the functorial concept of species of structures going back to Ehresmann [10]) and further developed by many other authors. It provides the most general framework for such a theory, at the expense of employing a rather abstract language, namely that of category theory.

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A fundamental theorem in each of these theories is the so-called *exponential formula*. Roughly speaking, given a family $\mathcal{G}$ of labelled combinatorial objects ("components"), one produces a larger family $\mathcal{F}$ ("composite objects") whose objects are obtained by putting together various elements of $\mathcal{G}$. The theorem then states that the (exponential) generating function for $\mathcal{F}$ equals the exponential of the (exponential) generating function for $\mathcal{G}$.

The aim of the present article is to develop a setting, where one starts with a family $\mathcal{F}$ of labelled combinatorial objects ("composite objects") and a composition of such objects, and then identifies, in an intrinsic way, a subfamily $\mathcal{G}$ of indecomposable objects ("components"), such that each element of $\mathcal{F}$ can be decomposed into objects from $\mathcal{G}$, and such that the exponential formula holds for $\mathcal{F}$ and $\mathcal{G}$. The main point here is that, in contrast to the usual set-up for the exponential formula, indecomposable objects are *not* given in advance, but are defined *inherently* via the composition operation. In particular, our theory leads to a uniform definition of the property to be *indecomposable* for arbitrary labelled combinatorial objects equipped with a composition operator. Interestingly, we show that a single axiom for the composition operator suffices to guarantee validity of the exponential formula. The natural language for formulating a corresponding theory is that of functors and natural transformations. Consequently, our presentation will be in the context of species theory.

For "ordinary" species (1-sort species), such a theory has been presented in [9] on the basis of two axioms for the composition operator. In the present article, we extend this approach to *weighted multisort species*. Moreover, we show that, actually, one of the axioms in [9] can be derived from the other, and that also in our multivariate setting a single axiom suffices. Strictly speaking, our presentation does not cover weighted species (in the sense of [16, Sec. 6], [6, p. 104]) in full generality; rather, we restrict ourselves to the case where the defining functor maps to a category of finite sets, thus avoiding unnecessary technicalities. However, extension to the general case of weighted multisort species is completely straightforward, and is left to the interested reader (see also Footnote 4).

An exponential principle in a wider context, that encompasses multisort species as a particular case, has been defined by Menni in [21] (see [23] for further work in this direction). Indeed, Menni's work and ours partially overlap. In order to explain the relation between the two, recall that — as already pointed out by Joyal [16, Sec. 7.1] — multisort species have the structure of a *symmetric monoidal category*. Now, in the focus of [21] there are *simple commutative monoids* in a given symmetric monoidal category. Menni defines an exponential principle in this set-up, and he proves this principle to hold for a large family of symmetric monoidal categories (see [21, Prop. 1.4]). He shows that this provides a uniform framework for the exponential principle for numerous variations of species that had appeared earlier in the literature, including multisort species (see [21, Ex. 3.2 and Ex. 3.5 with $I = 1 + 1 + \cdots + 1$]).[1] However, as it turns out, in the case of multisort species, Menni's theory does not cover the setting of our paper. It does apply whenever the considered multisort species, together with the product induced by our composition operator, forms a simple commutative monoid. This does not need to be the case, as Example 3 in Section 7 shows (see

[1]Strictly speaking, weights are not discussed in [21]. However, it would not be difficult to include them in the theory developed in [21].

also Section 9 for more detailed elaboration on these matters). So, one could say that Menni's theory exhibits the structural essentials of the exponential principle in a wide categorical framework, whereas our paper presents a "minimalistic" axiomatic setting for the exponential principle that is specific for multisort species but, as a bonus, includes a wider set of examples than Menni's theory does (in the case of multisort species). It is conceivable that our setting can be adapted to work for some other kinds of species, but it is unlikely that it can be lifted to the level of generality of Menni's theory.

In the next section, we develop the general set-up for our theory. It is formulated within the theory of multisort species, for which we define certain composition operators $\boldsymbol{\eta}$ that are subject to a single axiom, which, in order to be consistent with [9], we call (D1). Furthermore, in the same section, we present our main results. These are two exponential formulae, see Theorems 1 and 4. Theorem 4 refines Theorem 1 by introducing another variable whose powers keep track of the number of "components." The proof of Theorem 4 requires two general facts about our composition operators $\boldsymbol{\eta}$, which are presented in Propositions 2 and 3. (It is the latter, which, in the less general context of [9], had been assumed as a separate axiom, (D2). As our proof of Proposition 3 shows, this was actually not necessary since, within the general framework, (D2) follows from (D1). It is interesting to note that, in the context of [21], Menni also observed that the axiom (D2) was not necessary; see [21, Ex. 3.5]. Our derivation of (D2) from (D1) provides the reason why this is the case: we show that (D2) is implied by — as we call it — "$m$-permutability" of a species together with a composition operator; see the proof of Proposition 3, and Lemmas 11 and 15. This "$m$-permutability" comes for free in the context of [21] since the underlying species is assumed to form a commutative monoid in the category of species and, thus, satisfies stronger forms of "permutatility;" see also the more detailed explanations at the beginning of Section 9, and in particular the paragraph containing (9.4) and (9.5).) The proofs of Theorems 1 and 4, and of Propositions 2 and 3, are given in Sections 4 and 6, respectively. They require a number of auxiliary results, which are established in Sections 3 and 5, respectively.

Sections 7 and 8 offer illustrations for the theory developed in Sections 2–6. Section 7 presents three simple examples highlighting different aspects of combinatorial situations covered by Theorems 1 and 4. In Section 8, we show how to apply our results to obtain generating function identities for (semi-)magic squares, thereby generalising previous results in the literature.

The final section, Section 9, discusses the relationship between Menni's theory in [21] and ours. Moreover, there we make Menni's characterisation (see [21, Sec. 2.6]) of the simple commutative monoids that are covered by his result [21, Prop. 1.4] on the (general) exponential principle explicit for the special case of multisort species. As we show, this provides, in the language of our paper, a characterisation of composition operators that are pointwise associative and commutative (see (9.4) and (9.5)). Menni's characterisation implies that a family $\mathcal{F}$ of labelled combinatorial objects equipped with such a composition operator can be re-constructed, in a sense made precise in Theorem 22, from the standard operation of forming the disjoint union in $E(\mathcal{G})$ (the species of sets of objects from $\mathcal{G}$), where $\mathcal{G}$ denotes again the family of indecomposable objects in $\mathcal{F}$. We conclude our paper by "twisting" this construction (see Theorem 23), thereby obtaining a large family of examples that fit under our theory but not under Menni's.

## 2. Set-up and main results

Denote by $\widehat{\mathbf{Set}}$ the category of finite sets and injective mappings, and by $\mathbf{Set}$ the subcategory consisting of finite sets and bijective maps. Moreover, for a positive integer $r$, let $\mathfrak{D}_r$ be the full subcategory of $\mathbf{Set}^r \times \mathbf{Set}^r$ whose objects are given by

$$\mathrm{Ob}(\mathfrak{D}_r) = \Big\{(\mathbf{\Omega}_1, \mathbf{\Omega}_2) \in \mathrm{Ob}(\mathbf{Set}^r \times \mathbf{Set}^r) :\ \mathbf{\Omega}_1 \cap \mathbf{\Omega}_2 = \boldsymbol{\emptyset}\Big\},$$

where $\boldsymbol{\emptyset} = (\emptyset, \dots, \emptyset)$ is the element of $\mathrm{Ob}(\mathbf{Set}^r)$ all of whose components are empty, and intersection is componentwise.

The ingredients needed for our theory are *$r$-sort species* and certain *composition operators* defined on them. Recall from [16] (or see [6, Def. 4 on p. 102] for a definition avoiding the language of category theory) that, for a positive integer $r$, an $r$-sort species is a covariant functor $F : \mathbf{Set}^r \to \mathbf{Set}$. Given $r$ and an $r$-sort species $F$, the composition operators we have in mind are certain natural transformations $\boldsymbol{\eta}$ from the functor[2]

$$F \times F :\ \mathfrak{D}_r \xrightarrow{(F,F)} \mathbf{Set} \times \mathbf{Set} \xrightarrow{\times} \mathbf{Set} \xrightarrow{\iota} \widehat{\mathbf{Set}}$$

to the functor

$$F \circ \amalg :\ \mathfrak{D}_r \xrightarrow{\amalg} \mathbf{Set}^r \xrightarrow{F} \mathbf{Set} \xrightarrow{\iota} \widehat{\mathbf{Set}};$$

that is, families $\boldsymbol{\eta} = (\eta_{(\mathbf{\Omega}_1,\mathbf{\Omega}_2)})_{(\mathbf{\Omega}_1,\mathbf{\Omega}_2)\in \mathrm{Ob}(\mathfrak{D}_r)}$ of injective maps,

$$\eta_{(\mathbf{\Omega}_1,\mathbf{\Omega}_2)} : F[\mathbf{\Omega}_1] \times F[\mathbf{\Omega}_2] \hookrightarrow F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2], \tag{2.1}$$

such that, for every morphism $\boldsymbol{f} : (\mathbf{\Omega}_1, \mathbf{\Omega}_2) \longrightarrow (\widetilde{\mathbf{\Omega}}_1, \widetilde{\mathbf{\Omega}}_2)$ of $\mathfrak{D}_r$, the diagram

$$\begin{tikzcd} F[\mathbf{\Omega}_1] \times F[\mathbf{\Omega}_2] \arrow[r, "\eta_{(\mathbf{\Omega}_1,\mathbf{\Omega}_2)}"] \arrow[d, "(F\times F)[\boldsymbol{f}]"'] & F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2] \arrow[d, "(F\circ\amalg)[\boldsymbol{f}]"] \\ F[\widetilde{\mathbf{\Omega}}_1] \times F[\widetilde{\mathbf{\Omega}}_2] \arrow[r, "\eta_{(\widetilde{\mathbf{\Omega}}_1,\widetilde{\mathbf{\Omega}}_2)}"'] & F[\widetilde{\mathbf{\Omega}}_1 \amalg \widetilde{\mathbf{\Omega}}_2] \end{tikzcd} \tag{2.2}$$

commutes. Here, $\times$ is the natural product (Cartesian product) in the category of sets, $\amalg$ is the natural coproduct (componentwise disjoint union) in the category $\mathbf{Set}^r$ and in the category $\mathbf{Set}$ (relying on the context to clarify the intended meaning), and $\iota : \mathbf{Set} \to \widehat{\mathbf{Set}}$ is the inclusion functor. In what follows, the set-theoretic operations $\cap, \cup, -$ as well as the inclusion relation $\subseteq$ and $|$ (restriction of morphisms) in $\mathbf{Set}^r$ are all understood to be componentwise.[3] We shall most of the time drop the indices of $\boldsymbol{\eta}$-maps when they are clear from the context, thus writing $\eta(F[\mathbf{\Omega}_1] \times F[\mathbf{\Omega}_2])$ instead of $\eta_{(\mathbf{\Omega}_1,\mathbf{\Omega}_2)}(F[\mathbf{\Omega}_1] \times F[\mathbf{\Omega}_2])$, for example. We shall think of the elements of a set $\eta(F[\mathbf{\Omega}_1] \times F[\mathbf{\Omega}_2])$ as *composite objects* within $F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2]$.

Given an $r$-sort species $F$ and a composition operator $\boldsymbol{\eta}$ as above, the next step is to identify the subset $F_{\boldsymbol{\eta}}[\mathbf{\Omega}]$ of *"indecomposable" elements* of a set $F[\mathbf{\Omega}]$. It is most

[2]The introduction of the category $\mathfrak{D}_r$ corrects a slight imprecision in the set-up of [9].

[3]Throughout this paper, we use the symbol $-$ to denote the difference of sets.

natural to define $F_{\boldsymbol{\eta}} : \mathrm{Ob}(\mathbf{Set}^r) \to \mathrm{Ob}(\mathbf{Set})$ via

$$F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}] := \begin{cases} F[\boldsymbol{\Omega}] - \bigcup\limits_{\substack{(\boldsymbol{I},\boldsymbol{J}) \in \mathrm{Ob}(\mathfrak{D}_r) \\ \boldsymbol{I} \amalg \boldsymbol{J} = \boldsymbol{\Omega} \\ \boldsymbol{I} \neq \boldsymbol{\emptyset} \neq \boldsymbol{J}}} \eta\big(F[\boldsymbol{I}] \times F[\boldsymbol{J}]\big), & \boldsymbol{\Omega} \neq \boldsymbol{\emptyset} \\ \emptyset, & \boldsymbol{\Omega} = \boldsymbol{\emptyset} \end{cases}, \qquad \boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r).$$

At this point, $F_{\boldsymbol{\eta}}$ is just defined as a *map* from $\mathrm{Ob}(\mathbf{Set}^r)$ to $\mathrm{Ob}(\mathbf{Set})$. In Lemma 14 in Section 3 we shall show that $F_{\boldsymbol{\eta}}$ is in fact a functor, that is, an $r$-sort species.

Not every natural transformation $\boldsymbol{\eta}$ is suited for giving rise to an exponential principle. We present the single axiom which is needed for this purpose next. Given $F$, we call a natural transformation $\boldsymbol{\eta} : F \times F \to F \circ \amalg$ a *composition operator* of $F$, if Axiom (D1) below holds.

(D1) *For each* $\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$ *and any two partitions* $(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2), (\widetilde{\boldsymbol{\Omega}}_1, \widetilde{\boldsymbol{\Omega}}_2) \in \mathrm{Ob}(\mathfrak{D}_r)$ *of* $\boldsymbol{\Omega}$ *into disjoint parts,*

$$\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2 = \boldsymbol{\Omega} = \widetilde{\boldsymbol{\Omega}}_1 \amalg \widetilde{\boldsymbol{\Omega}}_2,$$

*we have that*

$$\eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big) \cap \eta\big(F[\widetilde{\boldsymbol{\Omega}}_1] \times F[\widetilde{\boldsymbol{\Omega}}_2]\big) = \eta\Big(\eta\big(F[\boldsymbol{\Omega}_{11}] \times F[\boldsymbol{\Omega}_{12}]\big) \times \eta\big(F[\boldsymbol{\Omega}_{21}] \times F[\boldsymbol{\Omega}_{22}]\big)\Big), \tag{2.3}$$

*where* $\boldsymbol{\Omega}_{ij} := \boldsymbol{\Omega}_i \cap \widetilde{\boldsymbol{\Omega}}_j$ *for* $i, j \in \{1, 2\}$.

An $r$-sort species $F$ will be called *decomposable*, if $F \neq \emptyset$ (that is, $F[\boldsymbol{\Omega}] \neq \emptyset$ for some $\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$), and if $F$ admits some composition operator $\boldsymbol{\eta}$.

Next, we define *weights* on $(F, \boldsymbol{\eta})$. Fix a commutative ring $\Lambda$ which contains the rational numbers. A family $\boldsymbol{w} = (w_{\boldsymbol{\Omega}})_{\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)}$ of maps $w_{\boldsymbol{\Omega}} : F[\boldsymbol{\Omega}] \to \Lambda$ is termed a $\Lambda$-*weight* on $(F, \boldsymbol{\eta})$, if the following three conditions hold:

(W0) *For all* $x \in F[\boldsymbol{\emptyset}]$, *we have* $w_{\boldsymbol{\emptyset}}(x) = 1$.

(W1) *For each morphism* $\boldsymbol{f} : \boldsymbol{\Omega}_1 \to \boldsymbol{\Omega}_2$ *of* $\mathbf{Set}^r$, *the diagram*

$$\begin{tikzcd} F[\boldsymbol{\Omega}_1] \arrow[r, "w_{\boldsymbol{\Omega}_1}"] \arrow[d, "F[\boldsymbol{f}]"'] & \Lambda \arrow[d, "\mathrm{id}_\Lambda"] \\ F[\boldsymbol{\Omega}_2] \arrow[r, "w_{\boldsymbol{\Omega}_2}"'] & \Lambda \end{tikzcd}$$

*commutes.*

(W2) *For each pair* $(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_r)$, *the diagram*

$$\begin{tikzcd} F_\eta[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2] \arrow[r, "\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}|_{F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]}"] \arrow[d, "w_{\boldsymbol{\Omega}_1}|_{F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1]} \times w_{\boldsymbol{\Omega}_2}"'] & F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2] \arrow[d, "w_{\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2}"] \\ \Lambda \times \Lambda \arrow[r, "\text{multiplication in } \Lambda"'] & \Lambda \end{tikzcd}$$

*commutes.*

Here, (W0) and (W1) make $F$ a *weighted* $r$-sort species (cf. [6, p. 104]), whereas (W2) demands (in a weak form) the $\Lambda$-weight $\boldsymbol{w}$ to be compatible with the composition operator $\boldsymbol{\eta}$.[4] In Section 9, we shall also need the concept of a *weak* $\Lambda$*-weight*, by which we mean a collection $\boldsymbol{w} = (w_{\boldsymbol{\Omega}})_{\boldsymbol{\Omega}\in\mathrm{Ob}(\mathbf{Set}^r)}$ of mappings as above satisfying (W0) and (W1), but not necessarily (W2).

Given a $\Lambda$-weight $\boldsymbol{w} = (w_{\boldsymbol{\Omega}})_{\boldsymbol{\Omega}\in\mathrm{Ob}(\mathbf{Set}^r)}$ on $(F,\boldsymbol{\eta})$, we define the corresponding *exponential generating functions* for $F$ and $F_{\boldsymbol{\eta}}$, respectively, by[5]

$$\mathrm{GF}_F(z_1,\dots,z_r) := \sum_{n_1,\dots,n_r\ge 0} \sum_{x\in F[([n_1],\dots,[n_r])]} w_{([n_1],\dots,[n_r])}(x)\frac{z_1^{n_1}\cdots z_r^{n_r}}{n_1!\cdots n_r!},$$
$$\mathrm{GF}_{F_{\boldsymbol{\eta}}}(z_1,\dots,z_r) := \sum_{n_1,\dots,n_r\ge 0} \sum_{x\in F_{\boldsymbol{\eta}}[([n_1],\dots,[n_r])]} w_{([n_1],\dots,[n_r])}(x)\frac{z_1^{n_1}\cdots z_r^{n_r}}{n_1!\cdots n_r!},$$

where we suppress the dependence on $\boldsymbol{w}$ in the notation for better readability.

We are now ready to state our first main result, an *exponential principle*, which generalises Part (a) of the main result in [9].

**Theorem 1.** *Let $r$ be a positive integer, $F:\mathbf{Set}^r\to\mathbf{Set}$ an $r$-sort species, and let $\boldsymbol{\eta}: F\times F\to F\circ\amalg$ be a natural transformation. If $F$ is decomposable and $\boldsymbol{\eta}$ is a composition operator for $F$, then the generating functions $\mathrm{GF}_F$ and $\mathrm{GF}_{F_{\boldsymbol{\eta}}}$ are connected via the relation*

$$\mathrm{GF}_F(z_1,\dots,z_r) = \exp\big(\mathrm{GF}_{F_{\boldsymbol{\eta}}}(z_1,\dots,z_r)\big). \tag{2.4}$$

The proof of Theorem 1 is given in Section 4. It requires several preparatory results, which are established in the next section.

In analogy to [9], there is a refinement of Theorem 1 in the spirit of [18, 24], which we explain next. Making use of the map $F_{\boldsymbol{\eta}}$ defined above, we define a sequence of mappings

$$F_{\boldsymbol{\eta}}^{(k)}:\mathrm{Ob}(\mathbf{Set}^r)\to\mathrm{Ob}(\mathbf{Set}),\quad k\ge 0,$$

with the property that $F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}]\subseteq F[\boldsymbol{\Omega}]$ by induction on $k$ via

$$F_{\boldsymbol{\eta}}^{(0)}[\boldsymbol{\Omega}] := \begin{cases} F[\boldsymbol{\emptyset}], & \boldsymbol{\Omega}=\boldsymbol{\emptyset}\\ \emptyset, & \boldsymbol{\Omega}\ne\boldsymbol{\emptyset}\end{cases}$$

and

$$F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}] := \bigcup_{\substack{\boldsymbol{\Omega}_1\in\mathrm{Ob}(\mathbf{Set}^r)\\ \boldsymbol{\Omega}_1\subseteq\boldsymbol{\Omega}}} \eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1]\times F_{\boldsymbol{\eta}}^{(k-1)}[\boldsymbol{\Omega}-\boldsymbol{\Omega}_1]\big),\quad k\ge 1. \tag{2.5}$$

As the definition suggests, one should think of $F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}]$ as the subset of objects in $F[\boldsymbol{\Omega}]$ consisting of exactly $k$ "indecomposable" elements (components).

An immediate induction on

$$\|\boldsymbol{\Omega}\| = \|(\Omega^{(1)},\Omega^{(2)},\dots,\Omega^{(r)})\| := \sum_{j=1}^r |\Omega^{(j)}| \tag{2.6}$$

[4]As already remarked in the introduction, it would be easy to generalise our set-up to cover weighted multisort species in full generality, by relaxing the condition that $F[\boldsymbol{\Omega}]$ needs to be finite, and requiring instead that each preimage $w_{\boldsymbol{\Omega}}^{-1}(\lambda)$ is finite and that the ring $\Lambda$ is *multiplicatively finite*, in the sense that the number of different product representations of $\lambda\in\Lambda$ is always finite.

[5]For a non-negative integer $n$, we write $[n]$ for the standard set $\{1,2,\dots,n\}$ of cardinality $n$.

shows that

$$F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}] = \emptyset, \quad k > \|\boldsymbol{\Omega}\|. \tag{2.7}$$

Again, by definition, $F_{\boldsymbol{\eta}}^{(k)}$ is just a *map* from $\mathrm{Ob}(\mathbf{Set}^r)$ to $\mathrm{Ob}(\mathbf{Set})$. In Lemma 16 in Section 5 we shall show that $F_{\boldsymbol{\eta}}^{(k)}$ is in fact a functor, that is, an $r$-sort species.

It is not difficult to see that, for any $\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$, the sets $F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}]$ with $k = 0, 1, 2, \dots$ cover all of $F[\boldsymbol{\Omega}]$.

**Proposition 2.** *For every $\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$, we have*

$$F[\boldsymbol{\Omega}] = \bigcup_{k \geq 0} F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}]. \tag{2.8}$$

The proof of Proposition 2 can be found in Section 6.

In [9], a second axiom, (D2), was imposed on the composition operators $\boldsymbol{\eta}$ for obtaining a refined exponential principle that takes into account the filtration given by the sets $F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}]$ appearing on the right-hand side of (2.8) (for the case of 1-sort species): it required pairwise disjointness of the sets $F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}]$ for $k = 0, 1, 2, \dots$. The proposition below says that, actually, this assertion is a consequence of Axiom (D1) (within the general set-up), and this is even true for multisort species.

**Proposition 3.** *If $F : \mathbf{Set}^r \to \mathbf{Set}$ is an $r$-sort species and $\boldsymbol{\eta}$ is a composition operator for $F$, then we have, for*[6] *$k, \ell \in \mathbb{N}_0$ and $k \neq \ell$,*

$$F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}] \cap F_{\boldsymbol{\eta}}^{(\ell)}[\boldsymbol{\Omega}] = \emptyset, \quad \boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r). \tag{2.9}$$

Proposition 3 is also proved in Section 6. Its proof depends crucially on the fact that "$\boldsymbol{\eta}$-bracketings" of $F$-sets and $F_{\boldsymbol{\eta}}$-sets do not depend on the order of the terms $F[\boldsymbol{\Omega}]$ respectively $F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}]$ involved, nor on the type of bracketing used; see Lemmas 11 and 15 in Sections 3 and 5, respectively. From a technical point of view, this is the decisive improvement over the results in [9], and it is the reason that the dependence of Axiom (D2) from Axiom (D1) was not observed there.

Given a $\Lambda$-weight $\boldsymbol{w}$ on $(F, \boldsymbol{\eta})$, the above propositions allow us to refine the weighting to

$$\widetilde{w}_{\boldsymbol{\Omega}}(x) := y^k w_{\boldsymbol{\Omega}}(x), \qquad x \in F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}].$$

We then can define the refined generating function

$$\begin{aligned}\widetilde{\mathrm{GF}}_F(z_1, \dots, z_r, y) &:= \sum_{n_1, \dots, n_r \geq 0} \sum_{x \in F[([n_1], \dots, [n_r])]} \widetilde{w}_{([n_1], \dots, [n_r])}(x) \, \frac{z_1^{n_1} \cdots z_r^{n_r}}{n_1! \cdots n_r!} \\ &= \sum_{n_1, \dots, n_r \geq 0} \sum_{k \geq 0} \sum_{x \in F_{\boldsymbol{\eta}}^{(k)}[([n_1], \dots, [n_r])]} y^k \, w_{([n_1], \dots, [n_r])}(x) \, \frac{z_1^{n_1} \cdots z_r^{n_r}}{n_1! \cdots n_r!}.\end{aligned}$$

With the above notation, we have the following refinement of Theorem 1, which is our second main result.

**Theorem 4.** *Under the hypotheses of Theorem 1, we have*

$$\widetilde{\mathrm{GF}}_F(z_1, \dots, z_r, y) = \exp\big(y \, \mathrm{GF}_{F_{\boldsymbol{\eta}}}(z_1, \dots, z_r)\big), \tag{2.10}$$

[6] We denote by $\mathbb{N}_0$ the set of non-negative integers.

*as well as*

$$\widetilde{\mathrm{GF}}_F(z_1,\dots,z_r,y) = \big(\,\mathrm{GF}_F(z_1,\dots,z_r)\big)^y. \tag{2.11}$$

The proof of Theorem 4 is given in Section 6, as a simple consequence of (the proof of) Proposition 3.

## 3. Auxiliary results, I

The purpose of this section is to establish several lemmas, which will be needed in the next section in the proof of Theorem 1. At the same time, they also form the basis for the proofs of the auxiliary results in Section 5, which eventually will lead to proofs of Propositions 2 and 3, and of Theorem 4, in Section 6. In all of this section, we assume that $F$ is a decomposable $r$-sort species with composition operator $\boldsymbol{\eta}$.

**Lemma 5.** *We have* $|F[\boldsymbol{\emptyset}]| = 1$.

*Proof.* By the injectivity of $\eta_{(\boldsymbol{\emptyset},\boldsymbol{\emptyset})} : F[\boldsymbol{\emptyset}]\times F[\boldsymbol{\emptyset}] \to F[\boldsymbol{\emptyset}]$, the set $F[\boldsymbol{\emptyset}]$ is either empty or a 1-set. Suppose that $F[\boldsymbol{\emptyset}] = \emptyset$. Choose $\boldsymbol{\Omega}_1 = (\Omega_1^{(1)},\dots,\Omega_1^{(r)}) \in \mathrm{Ob}(\mathbf{Set}^r)$ with $F[\boldsymbol{\Omega}_1] \ne \emptyset$, and $\boldsymbol{\Omega}_2 = (\Omega_2^{(1)},\dots,\Omega_2^{(r)}) \in \mathrm{Ob}(\mathbf{Set}^r)$ such that $\boldsymbol{\Omega}_1 \cap \boldsymbol{\Omega}_2 = \boldsymbol{\emptyset}$ and $|\Omega_1^{(i)}| = |\Omega_2^{(i)}|$ for $1 \le i \le r$. Now consider (D1) for the partition $\boldsymbol{\Omega} := \boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2$, $\widetilde{\boldsymbol{\Omega}}_i := \boldsymbol{\Omega}_i$ for $i = 1,2$. By the functoriality of $F$, we also have $F[\boldsymbol{\Omega}_2] \ne \emptyset$ and, consequently, the left-hand side of (2.3) is non-empty, whereas the right-hand side of (2.3) would be empty in case $F[\boldsymbol{\emptyset}] = \emptyset$, a contradiction. $\square$

**Lemma 6** (Commutativity for $(F,\boldsymbol{\eta})$)**.** *For every pair* $(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_r)$, *we have*

$$\eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big) = \eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1]\big). \tag{3.1}$$

*Proof.* Applying Axiom (D1) to the partitions

$$\boldsymbol{\Omega} := \boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2 = \boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_1,$$

we find that

$$\begin{aligned}\mathfrak{I} &:= \eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big) \cap \eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1]\big)\\ &= \eta\big(\eta\big(F[\boldsymbol{\Omega}_1\cap\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1]\big) \times \eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1\cap\boldsymbol{\Omega}_2]\big)\big).\end{aligned}$$

By Lemma 5 and injectivity of the $\boldsymbol{\eta}$-maps, the map

$$\eta_{(\boldsymbol{\emptyset},\boldsymbol{\Omega}_1)} : F[\boldsymbol{\emptyset}] \times F[\boldsymbol{\Omega}_1] \to F[\boldsymbol{\Omega}_1]$$

is surjective; that is,

$$\eta\big(F[\boldsymbol{\Omega}_1\cap\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1]\big) = F[\boldsymbol{\Omega}_1].$$

Similarly, we have

$$\eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1\cap\boldsymbol{\Omega}_2]\big) = F[\boldsymbol{\Omega}_2].$$

Thus,

$$\mathfrak{I} = \eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big).$$

By an analogous application of (D1) and Lemma 5 to the partitions

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_1 = \boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2,$$

we find that

$$\mathfrak{I} = \eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1]\big),$$

and the proof is complete. $\square$

**Lemma 7** (3-Associativity for $(F, \boldsymbol{\eta})$)**.** *For pairwise disjoint* $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \boldsymbol{\Omega}_3 \in \mathrm{Ob}(\mathbf{Set}^r)$*, we have*

$$\eta\big(\eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big) \times F[\boldsymbol{\Omega}_3]\big) = \eta\big(F[\boldsymbol{\Omega}_1] \times \eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_3]\big)\big). \tag{3.2}$$

*Proof.* We shall show that both sides of (3.2) equal the intersection

$$\mathfrak{I} := \eta\big(F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_3]\big) \cap \eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_3]\big).$$

Applying (D1) to the partitions

$$\boldsymbol{\Omega} := (\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2) \amalg \boldsymbol{\Omega}_3 = \boldsymbol{\Omega}_1 \amalg (\boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_3),$$

we find that

$$\mathfrak{I} = \eta\big(\eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big) \times \eta\big(F[\boldsymbol{\Omega}_1 \cap \boldsymbol{\Omega}_3] \times F[\boldsymbol{\Omega}_3]\big)\big);$$

and, arguing as in the proof of Lemma 6, this equation simplifies to

$$\mathfrak{I} = \eta\big(\eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big) \times F[\boldsymbol{\Omega}_3]\big).$$

The same argument, when applied to the partitions

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_1 \amalg (\boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_3) = (\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2) \amalg \boldsymbol{\Omega}_3,$$

yields

$$\mathfrak{I} = \eta\big(F[\boldsymbol{\Omega}_1] \times \eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_3]\big)\big),$$

whence (3.2). □

For the sake of convenience, for pairwise disjoint elements $\boldsymbol{\Omega}_1, \dots, \boldsymbol{\Omega}_m \in \mathrm{Ob}(\mathbf{Set}^r)$, let us call expressions formed by applying $\boldsymbol{\eta}$-maps to $F[\boldsymbol{\Omega}_1]$, …, $F[\boldsymbol{\Omega}_m]$ (in any order), with each set $F[\boldsymbol{\Omega}_i]$ occurring exactly once, an *$\boldsymbol{\eta}$-bracketing of* $F[\boldsymbol{\Omega}_1]$, …, $F[\boldsymbol{\Omega}_m]$. More formally, for any $\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$, we call the set $F[\boldsymbol{\Omega}]$ an $\boldsymbol{\eta}$-bracketing of $F[\boldsymbol{\Omega}]$; and, given an $\boldsymbol{\eta}$-bracketing $B_1$ of $F[\boldsymbol{\Omega}_{i_1}]$, …, $F[\boldsymbol{\Omega}_{i_k}]$ and an $\boldsymbol{\eta}$-bracketing $B_2$ of $F[\boldsymbol{\Omega}_{i_{k+1}}]$, …, $F[\boldsymbol{\Omega}_{i_m}]$, with $1 \leq k \leq n-1$ and $\{i_1, i_2, \dots, i_m\} = \{1, 2, \dots, m\}$, the expression $\eta(B_1 \times B_2)$ is, by definition, an $\boldsymbol{\eta}$-bracketing of $F[\boldsymbol{\Omega}_1]$, …, $F[\boldsymbol{\Omega}_m]$. For example, the left-hand side and the right-hand side of (3.2) are two possible $\boldsymbol{\eta}$-bracketings of $F[\boldsymbol{\Omega}_1]$, $F[\boldsymbol{\Omega}_2]$, $F[\boldsymbol{\Omega}_3]$, as are

$$\eta\big(\eta\big(F[\boldsymbol{\Omega}_3] \times F[\boldsymbol{\Omega}_2]\big) \times F[\boldsymbol{\Omega}_1]\big) \quad \text{and} \quad \eta\big(F[\boldsymbol{\Omega}_3] \times \eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big)\big).$$

A simple consequence of Lemma 6 and (the proof of) Lemma 7 is the following fact.

**Corollary 8.** *All $\boldsymbol{\eta}$-bracketings of* $F[\boldsymbol{\Omega}_1]$*,* $F[\boldsymbol{\Omega}_2]$*,* $F[\boldsymbol{\Omega}_3]$ *are equal to*

$$\eta\big(F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_3]\big) \cap \eta\big(F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_3] \times F[\boldsymbol{\Omega}_2]\big) \cap \eta\big(F[\boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_3] \times F[\boldsymbol{\Omega}_1]\big). \tag{3.3}$$

*Proof.* From the proof of Lemma 7, we know that

$$\eta\big(\eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big) \times F[\boldsymbol{\Omega}_3]\big) = \eta\big(F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_3]\big) \cap \eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_3]\big).$$

If, in this equation, we interchange $\boldsymbol{\Omega}_1$ and $\boldsymbol{\Omega}_2$ and use Lemma 6 (commutativity), then we obtain

$$\eta\big(\eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big) \times F[\boldsymbol{\Omega}_3]\big) = \eta\big(F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_3]\big) \cap \eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_3]\big).$$

Both equations together, plus another application of Lemma 6, imply our claim. □

**Lemma 9** (4-Permutability for $(F, \boldsymbol{\eta})$)**.** *If* $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \boldsymbol{\Omega}_3, \boldsymbol{\Omega}_4$ *are pairwise disjoint elements of* $\mathrm{Ob}(\mathbf{Set}^r)$*, then all $\boldsymbol{\eta}$-bracketings of* $F[\boldsymbol{\Omega}_1]$*,* $F[\boldsymbol{\Omega}_2]$*,* $F[\boldsymbol{\Omega}_3]$*,* $F[\boldsymbol{\Omega}_4]$ *are equal to each other.*

*Proof.* The possible $\boldsymbol{\eta}$-bracketings of $F[\mathbf{\Omega}_1]$, $F[\mathbf{\Omega}_2]$, $F[\mathbf{\Omega}_3]$, $F[\mathbf{\Omega}_4]$ are

$$\eta\big(\eta\big(\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_2]\big)\times F[\mathbf{\Omega}_3]\big)\times F[\mathbf{\Omega}_4]\big), \tag{3.4}$$

$$\eta\big(\eta\big(F[\mathbf{\Omega}_1]\times \eta\big(F[\mathbf{\Omega}_2]\times F[\mathbf{\Omega}_3]\big)\big)\times F[\mathbf{\Omega}_4]\big), \tag{3.5}$$

$$\eta\big(\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_2]\big)\times \eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big)\big), \tag{3.6}$$

$$\eta\big(F[\mathbf{\Omega}_1]\times \eta\big(\eta\big(F[\mathbf{\Omega}_2]\times F[\mathbf{\Omega}_3]\big)\times F[\mathbf{\Omega}_4]\big)\big), \tag{3.7}$$

$$\eta\big(F[\mathbf{\Omega}_1]\times \eta\big(F[\mathbf{\Omega}_2]\times \eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big)\big)\big), \tag{3.8}$$

together with all expressions arising from the above by permuting $F[\mathbf{\Omega}_1]$, $F[\mathbf{\Omega}_2]$, $F[\mathbf{\Omega}_3]$, $F[\mathbf{\Omega}_4]$. By Lemma 7 (3-associativity), the bracketing (3.4) equals the bracketing (3.5), and the bracketing (3.7) equals the bracketing (3.8). It suffices therefore to prove the equality of (3.4), (3.6), (3.8), and all expressions arising from these three by permuting $F[\mathbf{\Omega}_1]$, $F[\mathbf{\Omega}_2]$, $F[\mathbf{\Omega}_3]$, $F[\mathbf{\Omega}_4]$.

Applying (D1) to the partitions

$$\mathbf{\Omega} := (\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2) \amalg (\mathbf{\Omega}_3 \amalg \mathbf{\Omega}_4) = (\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_3) \amalg (\mathbf{\Omega}_2 \amalg \mathbf{\Omega}_4),$$

we find that

$$\eta\big(F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2]\times F[\mathbf{\Omega}_3 \amalg \mathbf{\Omega}_4]\big)\cap \eta\big(F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_3]\times F[\mathbf{\Omega}_2 \amalg \mathbf{\Omega}_4]\big) \\ = \eta\big(\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_2]\big)\times \eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big)\big).$$

Using this equation, as well as the one which arises by interchanging $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ and applying Lemma 6 (commutativity) on the resulting right-hand side, we obtain

$$\eta\big(\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_2]\big)\times \eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big)\big) \\ = \eta\big(F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2]\times F[\mathbf{\Omega}_3 \amalg \mathbf{\Omega}_4]\big)\cap \eta\big(F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_3]\times F[\mathbf{\Omega}_2 \amalg \mathbf{\Omega}_4]\big) \\ \cap \eta\big(F[\mathbf{\Omega}_2 \amalg \mathbf{\Omega}_3]\times F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_4]\big). \tag{3.9}$$

On the other hand, by Corollary 8, the expression

$$\eta\big(\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_2]\big)\times F[\mathbf{\Omega}_3]\big)$$

equals the expression (3.3). Therefore, "bracketing" these two expressions by $\eta(\,.\,\times F[\mathbf{\Omega}_4])$, using the injectivity of $\eta$, and applying Lemma 7 (3-associativity) to the expression resulting from (3.3), we arrive at

$$\eta\big(\eta\big(\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_2]\big)\times F[\mathbf{\Omega}_3]\big)\times F[\mathbf{\Omega}_4]\big) \\ = \eta\big(F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2]\times \eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big)\big)\cap \eta\big(F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_3]\times \eta\big(F[\mathbf{\Omega}_2]\times F[\mathbf{\Omega}_4]\big)\big) \\ \cap \eta\big(F[\mathbf{\Omega}_2 \amalg \mathbf{\Omega}_3]\times \eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_4]\big)\big). \tag{3.10}$$

Since $\eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big) \subseteq F[\mathbf{\Omega}_3 \amalg \mathbf{\Omega}_4]$, and similar inclusions hold for other combinations of the $\mathbf{\Omega}_i$'s, we have

$$\eta\big(F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2]\times \eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big)\big) \subseteq \eta\big(F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2]\times F[\mathbf{\Omega}_3 \amalg \mathbf{\Omega}_4]\big),$$
$$\eta\big(F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_3]\times \eta\big(F[\mathbf{\Omega}_2]\times F[\mathbf{\Omega}_4]\big)\big) \subseteq \eta\big(F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_3]\times F[\mathbf{\Omega}_2 \amalg \mathbf{\Omega}_4]\big),$$
$$\eta\big(F[\mathbf{\Omega}_2 \amalg \mathbf{\Omega}_3]\times \eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_4]\big)\big) \subseteq \eta\big(F[\mathbf{\Omega}_2 \amalg \mathbf{\Omega}_3]\times F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_4]\big).$$

Altogether, these inclusions imply that the right-hand side of (3.10) is contained in the right-hand side of (3.9). We infer that

$$\eta\big(\eta\big(\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_2]\big)\times F[\mathbf{\Omega}_3]\big)\times F[\mathbf{\Omega}_4]\big)\subseteq \eta\big(\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_2]\big)\times \eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big)\big). \tag{3.11}$$

However, by injectivity of the $\boldsymbol{\eta}$-maps, both sides of (3.11) have cardinality

$$|F[\mathbf{\Omega}_1]|\cdot|F[\mathbf{\Omega}_2]|\cdot|F[\mathbf{\Omega}_3]|\cdot|F[\mathbf{\Omega}_4]|.$$

Hence, they must be equal, which proves the equality of (3.4) and (3.6).

An analogous argument proves equality of (3.6) and (3.8).

Since, by Lemma 6 (commutativity), the right-hand side of (3.9) is invariant under permutation of $\mathbf{\Omega}_1,\mathbf{\Omega}_2,\mathbf{\Omega}_3,\mathbf{\Omega}_4$, the proof is complete. $\square$

The (proof of the) above lemma, combined with Lemma 6 and Corollary 8, leads to the following observation.

**Corollary 10.** *All $\boldsymbol{\eta}$-bracketings of $F[\mathbf{\Omega}_1]$, $F[\mathbf{\Omega}_2]$, $F[\mathbf{\Omega}_3]$, $F[\mathbf{\Omega}_4]$ are equal to*

$$\begin{aligned}\eta\big(F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_2]\times F[\mathbf{\Omega}_3\amalg\mathbf{\Omega}_4]\big)\cap\eta\big(F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_2\amalg\mathbf{\Omega}_4]\big)&\\ \cap\,\eta\big(F[\mathbf{\Omega}_2\amalg\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_4]\big)&\\ \cap\,\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_2\amalg\mathbf{\Omega}_3\amalg\mathbf{\Omega}_4]\big)\cap\eta\big(F[\mathbf{\Omega}_2]\times F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_3\amalg\mathbf{\Omega}_4]\big)&\\ \cap\,\eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_2\amalg\mathbf{\Omega}_4]\big)\cap\eta\big(F[\mathbf{\Omega}_4]\times F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_2\amalg\mathbf{\Omega}_3]\big)&.\end{aligned} \tag{3.12}$$

*Proof.* By Corollary 8, we have

$$\begin{aligned}&\eta\big(F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_2]\times\eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big)\big)\\ &\quad=\eta\big(F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_2\amalg\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big)\cap\eta\big(F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_2\amalg\mathbf{\Omega}_4]\times F[\mathbf{\Omega}_3]\big)\\ &\qquad\cap\,\eta\big(F[\mathbf{\Omega}_3\amalg\mathbf{\Omega}_4]\times F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_2]\big).\end{aligned}$$

Analogous identities hold for the other two terms on the right-hand side of (3.10). If we combine this with Lemma 6 (commutativity) and Lemma 9 (4-permutability), then the claim follows immediately. $\square$

Before we state the general permutability result, let us introduce the following short-hand notation, which will be used in its proof. Given a subset $I$ of $[m]$, where $I=\{i_1,i_2,\dots,i_k\}$ with $i_1<i_2<\dots<i_k$, we let $\mathbf{\Omega}_I$ stand for $\mathbf{\Omega}_{i_1}\amalg\dots\amalg\mathbf{\Omega}_{i_k}$.

**Lemma 11** ($m$-Permutability for $(F,\boldsymbol{\eta})$)**.** *If $\mathbf{\Omega}_1,\dots,\mathbf{\Omega}_m$ are pairwise disjoint elements of* $\mathrm{Ob}(\mathbf{Set}^r)$*, then all $\boldsymbol{\eta}$-bracketings of $F[\mathbf{\Omega}_1]$, …, $F[\mathbf{\Omega}_m]$ are equal to each other.*

*Proof.* We shall prove by induction on $m$ that, for each $m\ge 2$, all $\boldsymbol{\eta}$-bracketings of $F[\mathbf{\Omega}_1]$, …, $F[\mathbf{\Omega}_m]$ equal

$$\bigcap_{\substack{I,J\subseteq[m]\\ I\cup J=[m],\ I\cap J=\emptyset\\ I\ne\emptyset\ne J}}\eta\big(F[\mathbf{\Omega}_I]\times F[\mathbf{\Omega}_J]\big). \tag{3.13}$$

For $m=2$, the assertion follows from Lemma 6 (commutativity). For $m=3$, the assertion is equivalent to Corollary 8 (again, modulo Lemma 6), and for $m=4$, the assertion is equivalent to Corollary 10.

Now let $m \ge 5$, and let us suppose that the assertion is true up to $m-1$. Consider an $\boldsymbol{\eta}$-bracketing of $\mathbf{\Omega}_1, \dots, \mathbf{\Omega}_m$. There are three possibilities. Either this bracketing has the form

$$\eta\big(\eta\big(E_1\big) \times \eta\big(E_2\big)\big), \tag{3.14}$$

where $E_1$ and $E_2$ are expressions involving $\boldsymbol{\eta}$-maps and distinct $\mathbf{\Omega}_i$'s, each of them involving at least two $\mathbf{\Omega}_i$'s, or the form

$$\eta\big(F[\mathbf{\Omega}_i] \times \eta\big(E_3\big)\big), \tag{3.15}$$

where $E_3$ involves $\boldsymbol{\eta}$-maps and $\mathbf{\Omega}_1, \dots, \mathbf{\Omega}_{i-1}, \mathbf{\Omega}_{i+1}, \dots, \mathbf{\Omega}_m$, for some $i$, or the form

$$\eta\big(\eta\big(E_4\big) \times F[\mathbf{\Omega}_i]\big), \tag{3.16}$$

where $E_4$ involves $\boldsymbol{\eta}$-maps and $\mathbf{\Omega}_1, \dots, \mathbf{\Omega}_{i-1}, \mathbf{\Omega}_{i+1}, \dots, \mathbf{\Omega}_m$, for some $i$.

We start by considering (3.14). Let us assume that $E_1$ involves all $\mathbf{\Omega}_r$'s for $r \in R$, and $E_2$ involves all $\mathbf{\Omega}_s$'s for $s \in S$, with $R \cup S = [m]$, $R \cap S = \emptyset$, $R \ne \emptyset \ne S$. By the inductive hypothesis, we know that

$$\eta(E_1) = \bigcap_{\substack{I_1, J_1 \subseteq R \\ I_1 \cup J_1 = R,\ I_1 \cap J_1 = \emptyset \\ I_1 \ne \emptyset \ne J_1}} \eta\big(F[\mathbf{\Omega}_{I_1}] \times F[\mathbf{\Omega}_{J_1}]\big) \tag{3.17}$$

and

$$\eta(E_2) = \bigcap_{\substack{I_2, J_2 \subseteq S \\ I_2 \cup J_2 = S,\ I_2 \cap J_2 = \emptyset \\ I_2 \ne \emptyset \ne J_2}} \eta\big(F[\mathbf{\Omega}_{I_2}] \times F[\mathbf{\Omega}_{J_2}]\big). \tag{3.18}$$

If we substitute (3.17) and (3.18) in (3.14) and use injectivity of the $\boldsymbol{\eta}$-maps, then we obtain

$$\bigcap_{\substack{I_1, J_1 \subseteq R \\ I_1 \cup J_1 = R,\ I_1 \cap J_1 = \emptyset \\ I_1 \ne \emptyset \ne J_1}} \bigcap_{\substack{I_2, J_2 \subseteq S \\ I_2 \cup J_2 = S,\ I_2 \cap J_2 = \emptyset \\ I_2 \ne \emptyset \ne J_2}} \eta\big(\eta\big(F[\mathbf{\Omega}_{I_1}] \times F[\mathbf{\Omega}_{J_1}]\big) \times \eta\big(F[\mathbf{\Omega}_{I_2}] \times F[\mathbf{\Omega}_{J_2}]\big)\big).$$

We may now apply Corollary 10 to each of the terms on the right-hand side of this equation. It is not difficult to see that, together with Lemma 6 (commutativity), we obtain (3.13).

Next we consider (3.15). By the inductive hypothesis, we know that

$$\eta(E_3) = \bigcap_{\substack{I, J \subseteq [m]-\{i\} \\ I \cup J = [m]-\{i\},\ I \cap J = \emptyset \\ I \ne \emptyset \ne J}} \eta\big(F[\mathbf{\Omega}_I] \times F[\mathbf{\Omega}_J]\big).$$

If we substitute this in (3.15) and use injectivity of the $\boldsymbol{\eta}$-maps, then we obtain

$$\bigcap_{\substack{I, J \subseteq [m]-\{i\} \\ I \cup J = [m]-\{i\},\ I \cap J = \emptyset \\ I \ne \emptyset \ne J}} \eta\big(F[\mathbf{\Omega}_i] \times \eta\big(F[\mathbf{\Omega}_I] \times F[\mathbf{\Omega}_J]\big)\big).$$

We may now apply Corollary 8 to each of the terms on the right-hand side of this equation. It is not difficult to see that, together with Lemma 6 (commutativity), we obtain (3.13).

The argument for (3.16) is analogous. This completes the proof of the lemma. □

For $\boldsymbol{\Omega} = (\Omega^{(1)}, \dots, \Omega^{(r)}) \in \mathrm{Ob}(\mathbf{Set}^r)$ and an integer $\rho \in [r]$, we write $(\omega, \rho) \in \boldsymbol{\Omega}$ to mean $\omega \in \Omega^{(\rho)}$. This is the concept of *base point* needed in the present context.

**Lemma 12.** *For non-empty* $\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$ *and every choice of base point* $(\omega, \rho) \in \boldsymbol{\Omega}$, *we have*

$$F[\boldsymbol{\Omega}] = \bigcup_{\substack{\boldsymbol{\Omega}_1 \in \mathrm{Ob}(\mathbf{Set}^r)\\ (\omega,\rho) \in \boldsymbol{\Omega}_1 \subseteq \boldsymbol{\Omega}}} \eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega} - \boldsymbol{\Omega}_1]\big). \tag{3.19}$$

*Proof.* Let $x \in F[\boldsymbol{\Omega}]$ be an arbitrary element, and consider the totality of all $\boldsymbol{\Omega}_1 \in \mathrm{Ob}(\mathbf{Set}^r)$ such that $(\omega, \rho) \in \boldsymbol{\Omega}_1 \subseteq \boldsymbol{\Omega}$ and $x \in \eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega} - \boldsymbol{\Omega}_1]\big)$. Such $\boldsymbol{\Omega}_1$'s do exist; for instance $\boldsymbol{\Omega}_1 = \boldsymbol{\Omega}$ has these properties, since the map

$$\eta_{(\boldsymbol{\Omega},\boldsymbol{\emptyset})} : F[\boldsymbol{\Omega}] \times F[\boldsymbol{\emptyset}] \hookrightarrow F[\boldsymbol{\Omega}]$$

is surjective. Among these $\boldsymbol{\Omega}_1$'s we choose one of minimal norm $\|\boldsymbol{\Omega}_1\|$ (recall the definition in (2.6)), say $\boldsymbol{\Omega}_1(x)$. Now suppose that $x \notin \eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1(x)] \times F[\boldsymbol{\Omega} - \boldsymbol{\Omega}_1(x)]\big)$. Then, by the definition of $F_{\boldsymbol{\eta}}$, the injectivity of $\eta$, and the choice of $\boldsymbol{\Omega}_1(x)$, we must have

$$\begin{aligned} x \in \eta\Big(\big(F[\boldsymbol{\Omega}_1(x)] - F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1(x)]\big) \times F[\boldsymbol{\Omega} - \boldsymbol{\Omega}_1(x)]\Big) \\ = \bigcup_{\substack{(\boldsymbol{I},\boldsymbol{J}) \in \mathrm{Ob}(\mathfrak{D}_r)\\ \boldsymbol{I} \amalg \boldsymbol{J} = \boldsymbol{\Omega}_1(x)\\ \boldsymbol{I} \neq \boldsymbol{\emptyset} \neq \boldsymbol{J}}} \eta\Big(\eta\big(F[\boldsymbol{I}] \times F[\boldsymbol{J}]\big) \times F[\boldsymbol{\Omega} - \boldsymbol{\Omega}_1(x)]\Big). \end{aligned}$$

Consequently, there exists $(\boldsymbol{I}_1, \boldsymbol{J}_1) \in \mathrm{Ob}(\mathfrak{D}_r)$ such that $\boldsymbol{I}_1 \amalg \boldsymbol{J}_1 = \boldsymbol{\Omega}_1(x)$, $\boldsymbol{I}_1 \neq \boldsymbol{\emptyset} \neq \boldsymbol{J}_1$, and

$$x \in \eta\Big(\eta\big(F[\boldsymbol{I}_1] \times F[\boldsymbol{J}_1]\big) \times F[\boldsymbol{\Omega} - \boldsymbol{\Omega}_1(x)]\Big).$$

Using Corollary 8 (3-permutability), we see that the latter set is contained in both $\eta\big(F[\boldsymbol{I}_1] \times F[\boldsymbol{\Omega} - \boldsymbol{I}_1]\big)$ and $\eta\big(F[\boldsymbol{J}_1] \times F[\boldsymbol{\Omega} - \boldsymbol{J}_1]\big)$. The base point $(\omega, \rho)$ is contained in $\boldsymbol{I}_1$ or $\boldsymbol{J}_1$; to fix ideas, say $(\omega, \rho) \in \boldsymbol{I}_1$. Hence, we arrive at the assertion that

$$x \in \eta\big(F[\boldsymbol{I}_1] \times F[\boldsymbol{\Omega} - \boldsymbol{I}_1]\big),\ (\omega, \rho) \in \boldsymbol{I}_1 \subseteq \boldsymbol{\Omega},\ \|\boldsymbol{I}_1\| < \|\boldsymbol{\Omega}_1(x)\|,$$

contradicting the choice of $\boldsymbol{\Omega}_1(x)$. We conclude that $x$ is indeed contained in

$$\eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1(x)] \times F[\boldsymbol{\Omega} - \boldsymbol{\Omega}_1(x)]\big),$$

and (3.19) is proven. □

**Lemma 13.** *The right-hand side of* (3.19) *is a disjoint union.*

*Proof.* In the context of Lemma 12, let $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2 \in \mathrm{Ob}(\mathbf{Set}^r)$ be such that $(\omega, \rho) \in \boldsymbol{\Omega}_i \subseteq \boldsymbol{\Omega}$ and $\boldsymbol{\Omega}_1 \neq \boldsymbol{\Omega}_2$, say $\boldsymbol{\Omega}_1 \not\subseteq \boldsymbol{\Omega}_2$. It is enough to show that

$$\mathfrak{I} := \eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega} - \boldsymbol{\Omega}_1]\big) \cap \eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega} - \boldsymbol{\Omega}_2]\big)$$

has an empty intersection with $\eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega} - \boldsymbol{\Omega}_1]\big)$. But, by (D1), we have

$$\mathfrak{I} \subseteq \eta\Big(\eta\big(F[\boldsymbol{\Omega}_1 \cap \boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1 - \boldsymbol{\Omega}_2]\big) \times F[\boldsymbol{\Omega} - \boldsymbol{\Omega}_1]\Big),$$

and, by definition of $F_{\boldsymbol\eta}$ and the fact that $(\omega,\rho)\in\boldsymbol\Omega_1\cap\boldsymbol\Omega_2\ne\boldsymbol\emptyset\ne\boldsymbol\Omega_1-\boldsymbol\Omega_2$, we have

$$F_{\boldsymbol\eta}[\boldsymbol\Omega_1]\cap\eta\big(F[\boldsymbol\Omega_1\cap\boldsymbol\Omega_2]\times F[\boldsymbol\Omega_1-\boldsymbol\Omega_2]\big)=\emptyset.$$

Consequently, by the injectivity of $\eta$, we must indeed have

$$\begin{aligned}&\eta\big(F_{\boldsymbol\eta}[\boldsymbol\Omega_1]\times F[\boldsymbol\Omega-\boldsymbol\Omega_1]\big)\cap\mathfrak{I}\\ &\quad\subseteq\eta\big(F_{\boldsymbol\eta}[\boldsymbol\Omega_1]\times F[\boldsymbol\Omega-\boldsymbol\Omega_1]\big)\cap\eta\Big(\eta\big(F[\boldsymbol\Omega_1\cap\boldsymbol\Omega_2]\times F[\boldsymbol\Omega_1-\boldsymbol\Omega_2]\big)\times F[\boldsymbol\Omega-\boldsymbol\Omega_1]\Big)=\emptyset,\end{aligned}$$

as required. □

**Lemma 14** (Functoriality of $F_{\boldsymbol\eta}$)**.** *Let $\boldsymbol\Omega,\widetilde{\boldsymbol\Omega}\in\mathrm{Ob}(\mathbf{Set}^r)$, and let $\boldsymbol f:\boldsymbol\Omega\to\widetilde{\boldsymbol\Omega}$ be a morphism. Then*

$$F[\boldsymbol f](F_{\boldsymbol\eta}[\boldsymbol\Omega])=F_{\boldsymbol\eta}[\widetilde{\boldsymbol\Omega}];$$

*that is, setting $F_{\boldsymbol\eta}[\boldsymbol f]:=F[\boldsymbol f]|_{F_{\boldsymbol\eta}[\boldsymbol\Omega]}$, we get a functor $F_{\boldsymbol\eta}:\mathbf{Set}^r\to\mathbf{Set}$.*

*Proof.* The assertion is obvious if $\boldsymbol\Omega=\boldsymbol\emptyset$, so we may suppose that $\boldsymbol\Omega\ne\boldsymbol\emptyset$. Then, using the naturality of $\boldsymbol\eta$ (that is, the diagram (2.2)), we have

$$\begin{aligned}F[\boldsymbol f](F_{\boldsymbol\eta}[\boldsymbol\Omega])&=F[\boldsymbol f]\left(F[\boldsymbol\Omega]-\bigcup_{\substack{(\boldsymbol I_1,\boldsymbol J_1)\in\mathrm{Ob}(\mathfrak{D}_r)\\ \boldsymbol I_1\amalg\boldsymbol J_1=\boldsymbol\Omega\\ \boldsymbol I_1\ne\boldsymbol\emptyset\ne\boldsymbol J_1}}\eta\big(F[\boldsymbol I_1]\times F[\boldsymbol J_1]\big)\right)\\
&=F[\widetilde{\boldsymbol\Omega}]-\bigcup_{\substack{(\boldsymbol I_1,\boldsymbol J_1)\in\mathrm{Ob}(\mathfrak{D}_r)\\ \boldsymbol I_1\amalg\boldsymbol J_1=\boldsymbol\Omega\\ \boldsymbol I_1\ne\boldsymbol\emptyset\ne\boldsymbol J_1}}\big(F\circ\amalg\big)\big[(\boldsymbol f|_{\boldsymbol I_1},\boldsymbol f|_{\boldsymbol J_1})\big]\big(\eta\big(F[\boldsymbol I_1]\times F[\boldsymbol J_1]\big)\big)\\
&=F[\widetilde{\boldsymbol\Omega}]-\bigcup_{\substack{(\boldsymbol I_1,\boldsymbol J_1)\in\mathrm{Ob}(\mathfrak{D}_r)\\ \boldsymbol I_1\amalg\boldsymbol J_1=\boldsymbol\Omega\\ \boldsymbol I_1\ne\boldsymbol\emptyset\ne\boldsymbol J_1}}\eta\big(F[\boldsymbol f|_{\boldsymbol I_1}](F[\boldsymbol I_1])\times F[\boldsymbol f|_{\boldsymbol J_1}](F[\boldsymbol J_1])\big)\\
&=F[\widetilde{\boldsymbol\Omega}]-\bigcup_{\substack{(\boldsymbol I_2,\boldsymbol J_2)\in\mathrm{Ob}(\mathfrak{D}_r)\\ \boldsymbol I_2\amalg\boldsymbol J_2=\widetilde{\boldsymbol\Omega}\\ \boldsymbol I_2\ne\boldsymbol\emptyset\ne\boldsymbol J_2}}\eta\big(F[\boldsymbol I_2]\times F[\boldsymbol J_2]\big)\\
&=F[\widetilde{\boldsymbol\Omega}].\end{aligned}$$

□

## 4. Proof of Theorem 1

For convenience, let us "extend" the $\Lambda$-weight $\boldsymbol w$ to subsets of $F[\boldsymbol\Omega]$, for all $\boldsymbol\Omega\in\mathrm{Ob}(\mathbf{Set}^r)$. To be precise, for $A\subseteq F[\boldsymbol\Omega]$, we define

$$w_{\boldsymbol\Omega}(A):=\sum_{x\in A}w_{\boldsymbol\Omega}(x).$$

Given disjoint $\mathbf{\Omega}_1, \mathbf{\Omega}_2 \in \mathrm{Ob}(\mathbf{Set}^r)$ and subsets $A_1 \subseteq F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1]$ and $A_2 \subseteq F[\mathbf{\Omega}_2]$, Axiom (W2) says that

$$w_{\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2}(\eta(A_1 \times A_2)) = w_{\mathbf{\Omega}_1}(A_1) \cdot w_{\mathbf{\Omega}_2}(A_2). \tag{4.1}$$

In the sequel, we shall suppress the indices of weights $w$ for better readability, the indices always being clear from the context.

As a direct consequence of Lemmas 12 and 13, of the injectivity of the $\boldsymbol{\eta}$-maps, and of (4.1), we have that, for $n_1, n_2, \dots, n_r \in \mathbb{N}_0$ and $n_\rho > 0$,

$$\begin{aligned} &w\big(F[([n_1], \dots, [n_r])]\big) \\ &\quad = \sum_{\substack{\Omega_1^{(i)} \subseteq [n_i]\ (1 \le i \le r) \\ 1 \in \Omega_1^{(\rho)}}} w\big(F_{\boldsymbol{\eta}}[(\Omega_1^{(1)}, \dots, \Omega_1^{(r)})]\big) \cdot w\big(F[([n_1] - \Omega_1^{(1)}, \dots, [n_r] - \Omega_1^{(r)})]\big). \end{aligned} \tag{4.2}$$

Using the functoriality of $F$ and $F_{\boldsymbol{\eta}}$, together with Axiom (W1), each $\mathbf{\Omega}_1$ with

$$\mathbf{\Omega}_1 = (\Omega_1^{(1)}, \dots, \Omega_1^{(r)}) \subseteq ([n_1], \dots, [n_r]),$$

$(1, \rho) \in \mathbf{\Omega}_1$, and cardinalities $|\Omega_1^{(i)}| = \mu_i$ $(1 \le i \le r)$, is seen to contribute

$$w\big(F_{\boldsymbol{\eta}}[([\mu_1], \dots, [\mu_r])]\big) \cdot w\big(F[([n_1 - \mu_1], \dots, [n_r - \mu_r])]\big) \tag{4.3}$$

to the right-hand side of (4.2). We observe that (4.3) does not depend upon $\mathbf{\Omega}_1$ itself, but only on the cardinalities $\mu_1, \dots, \mu_r$ of the components $\Omega_1^{(1)}, \dots, \Omega_1^{(r)}$. Therefore, the

$$\frac{\mu_\rho}{n_\rho} \prod_{1 \le i \le r} \binom{n_i}{\mu_i}$$

elements $\mathbf{\Omega}_1 \in \mathrm{Ob}(\mathbf{Set}^r)$ with $\mathbf{\Omega}_1 \subseteq ([n_1], \dots, [n_r])$, $(1, \rho) \in \mathbf{\Omega}_1$, and such that $|\Omega_1^{(i)}| = \mu_i$ for $1 \le i \le r$, contribute

$$\frac{\mu_\rho}{n_\rho} \Bigg( \prod_{1 \le i \le r} \binom{n_i}{\mu_i} \Bigg) w\big(F_{\boldsymbol{\eta}}[([\mu_1], \dots, [\mu_r])]\big) \cdot w\big(F[([n_1 - \mu_1], \dots, [n_r - \mu_r])]\big)$$

to the right-hand side of (4.2), and we obtain

$$\begin{aligned} w\big(F[([n_1], \dots, [n_r])]\big) = \sum_{0 \le \mu_i \le n_i\ (1 \le i \le r)} \frac{\mu_\rho}{n_\rho} \Bigg( \prod_{1 \le i \le r} \binom{n_i}{\mu_i} \Bigg) & w\big(F_{\boldsymbol{\eta}}[([\mu_1], \dots, [\mu_r])]\big) \\ & \cdot w\big(F[([n_1 - \mu_1], \dots, [n_r - \mu_r])]\big), \\ & \qquad\qquad n_\rho > 0, \end{aligned}$$

or, equivalently,

$$\begin{aligned} n_\rho \cdot w\big(F[([n_1], \dots, [n_r])]\big) = \sum_{0 \le \mu_i \le n_i\ (1 \le i \le r)} \mu_\rho \Bigg( \prod_{1 \le i \le r} \binom{n_i}{\mu_i} \Bigg) & w\big(F_{\boldsymbol{\eta}}[([\mu_1], \dots, [\mu_r])]\big) \\ & \cdot w\big(F[([n_1 - \mu_1], \dots, [n_r - \mu_r])]\big), \end{aligned} \tag{4.4}$$

as long as $n_\rho > 0$. However, Equation (4.4) holds as well for $n_\rho = 0$, with both sides vanishing, so that we are allowed to drop the restriction $n_\rho > 0$.

Fix $\rho \in [r]$, multiply both sides of (4.4) by

$$z_1^{n_1} \cdots z_{\rho-1}^{n_{\rho-1}} z_\rho^{n_\rho - 1} z_{\rho+1}^{n_{\rho+1}} \cdots z_r^{n_r} / (n_1! \cdots n_r!),$$

and sum over all tuples $(n_1, \dots, n_r) \in \mathbb{N}_0^r$, to get

$$\sum_{n_1,\dots,n_r \ge 0} w\big(F[([n_1],\dots,[n_r])]\big) \frac{z_1^{n_1} \cdots z_{\rho-1}^{n_{\rho-1}} z_\rho^{n_\rho - 1} z_{\rho+1}^{n_{\rho+1}} \cdots z_r^{n_r}}{n_1! \cdots n_{\rho-1}! \, (n_\rho - 1)! \, n_{\rho+1}! \cdots n_r!}$$
$$= \sum_{n_1,\dots,n_r \ge 0} \sum_{0 \le \mu_i \le n_i \, (1 \le i \le r)} \frac{w\big(F_{\boldsymbol\eta}[([\mu_1],\dots,[\mu_r])]\big)}{\mu_1! \cdots \mu_{\rho-1}! \, (\mu_\rho - 1)! \, \mu_{\rho+1}! \cdots \mu_r!}$$
$$\cdot \frac{w\big(F[([n_1-\mu_1],\dots,[n_r-\mu_r])]\big)}{(n_1-\mu_1)! \cdots (n_r-\mu_r)!} z_1^{n_1} \cdots z_{\rho-1}^{n_{\rho-1}} z_\rho^{n_\rho - 1} z_{\rho+1}^{n_{\rho+1}} \cdots z_r^{n_r}, \quad (4.5)$$

where $1/(-1)!$ has to be interpreted as 0. The left-hand side equals

$$\frac{\partial \, \mathrm{GF}_F}{\partial z_\rho},$$

while the right-hand side is identified as

$$\frac{\partial \, \mathrm{GF}_{F_{\boldsymbol\eta}}}{\partial z_\rho} \, \mathrm{GF}_F;$$

whence the equations

$$\frac{\partial \, \mathrm{GF}_F}{\partial z_\rho} = \frac{\partial \, \mathrm{GF}_{F_{\boldsymbol\eta}}}{\partial z_\rho} \, \mathrm{GF}_F, \qquad 1 \le \rho \le r. \quad (4.6)$$

Set

$$Q(z_1,\dots,z_r) := \mathrm{GF}_F(z_1,\dots,z_r) \exp\big(-\mathrm{GF}_{F_{\boldsymbol\eta}}(z_1,\dots,z_r)\big).$$

Then, in view of Equations (4.6), the series $Q$ satisfies

$$\frac{\partial Q}{\partial z_\rho} = 0, \qquad 1 \le \rho \le r.$$

These last equations force $Q$ to be independent of $z_1, z_2, \dots, z_r$. However, since $\mathrm{GF}_{F_{\boldsymbol\eta}}(0,\dots,0) = 0$ by definition of $F_{\boldsymbol\eta}$, and since $\mathrm{GF}_F(0,\dots,0) = 1$ by Axiom (W0) and Lemma 5, direct inspection shows that

$$Q(0,0,\dots,0) = 1,$$

and (2.4) follows. □

## 5. Auxiliary results, II

In this section, we complement the results of Section 3 by establishing several further results which will be needed in the proofs of Proposition 3 and Theorem 4, to be given in the next section. The first lemma provides the analogue of Lemma 11 for $F_{\boldsymbol\eta}$, namely that arbitrary permutability holds also for $\boldsymbol\eta$-bracketings of $F_{\boldsymbol\eta}$-images (see the subsequent paragraph for the precise definition). All the remaining lemmas concern the maps $F_{\boldsymbol\eta}^{(k)}$. In all of this section, we assume that $F$ is a decomposable $r$-sort species with composition operator $\boldsymbol\eta$.

In complete analogy with the corresponding definition in Section 3, we define the concept of an $\boldsymbol{\eta}$*-bracketing of* $F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1], \dots, F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]$ for pairwise disjoint elements $\boldsymbol{\Omega}_1, \dots, \boldsymbol{\Omega}_m \in \mathrm{Ob}(\mathbf{Set}^r)$: simply replace $F$ by $F_{\boldsymbol{\eta}}$ everywhere in the definition just after the proof of Lemma 7.

**Lemma 15** ($m$-Permutability for $(F_{\boldsymbol{\eta}}, \boldsymbol{\eta})$)**.** *If* $\boldsymbol{\Omega}_1, \dots, \boldsymbol{\Omega}_m$ *are pairwise disjoint elements of* $\mathrm{Ob}(\mathbf{Set}^r)$*, then all* $\boldsymbol{\eta}$*-bracketings of* $F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1]$*, ...,* $F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]$ *are equal to each other.*

*Proof.* Since $F_{\boldsymbol{\eta}}[\boldsymbol{\emptyset}] = \emptyset$ by definition of $F_{\boldsymbol{\eta}}$, our claim holds if at least one of $\boldsymbol{\Omega}_1, \dots, \boldsymbol{\Omega}_m$ equals $\boldsymbol{\emptyset}$. Hence, we may assume that all of $\boldsymbol{\Omega}_1, \dots, \boldsymbol{\Omega}_m$ are non-empty.

Next we note that, for sets $M_1, \dots, M_m, A_1, \dots, A_m$, we have

$$\begin{aligned}&(M_1 - A_1) \times (M_2 - A_2) \times \cdots \times (M_m - A_m)\\ &= M_1 \times M_2 \times \cdots \times M_m - \left(\bigcup_{k=1}^{m} M_1 \times \cdots \times M_{k-1} \times A_k \times M_{k+1} \times \cdots \times M_m\right). \end{aligned}\tag{5.1}$$

Now assume that we are given two $\boldsymbol{\eta}$-bracketings of $F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1], \dots, F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]$, say

$$B_{\boldsymbol{\eta}}\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1], \dots, F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]\big) \quad \text{and} \quad \bar{B}_{\boldsymbol{\eta}}\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1], \dots, F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]\big).$$

Substituting the definition of $F_{\boldsymbol{\eta}}$ into $B_{\boldsymbol{\eta}}\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1], \dots, F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]\big)$, and applying Identity (5.1) plus injectivity of $\boldsymbol{\eta}$-maps, we find that

$$\begin{aligned}&B_{\boldsymbol{\eta}}\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1], \dots, F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]\big) = B_{\boldsymbol{\eta}}\big(F[\boldsymbol{\Omega}_1], \dots, F[\boldsymbol{\Omega}_m]\big)\\ &\quad - \bigcup_{k=1}^{m} B_{\boldsymbol{\eta}}\left(F[\boldsymbol{\Omega}_1], \dots, F[\boldsymbol{\Omega}_{k-1}], \bigcup_{\substack{(\boldsymbol{I},\boldsymbol{J}) \in \mathrm{Ob}(\mathfrak{D}_r)\\ \boldsymbol{I} \amalg \boldsymbol{J} = \boldsymbol{\Omega}_k\\ \boldsymbol{I} \neq \boldsymbol{\emptyset} \neq \boldsymbol{J}}} \eta\big(F[\boldsymbol{I}] \times F[\boldsymbol{J}]\big), F[\boldsymbol{\Omega}_{k+1}], \dots, F[\boldsymbol{\Omega}_m]\right)\\ &= B_{\boldsymbol{\eta}}\big(F[\boldsymbol{\Omega}_1], \dots, F[\boldsymbol{\Omega}_m]\big)\\ &\quad - \bigcup_{k=1}^{m} \bigcup_{\substack{(\boldsymbol{I},\boldsymbol{J}) \in \mathrm{Ob}(\mathfrak{D}_r)\\ \boldsymbol{I} \amalg \boldsymbol{J} = \boldsymbol{\Omega}_k\\ \boldsymbol{I} \neq \boldsymbol{\emptyset} \neq \boldsymbol{J}}} B_{\boldsymbol{\eta}}\big(F[\boldsymbol{\Omega}_1], \dots, F[\boldsymbol{\Omega}_{k-1}], \eta\big(F[\boldsymbol{I}] \times F[\boldsymbol{J}]\big), F[\boldsymbol{\Omega}_{k+1}], \dots, F[\boldsymbol{\Omega}_m]\big).\end{aligned}$$

The same argument shows that $\bar{B}_{\boldsymbol{\eta}}\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1], \dots, F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]\big)$ equals the last expression where every occurrence of $B_{\boldsymbol{\eta}}$ is replaced by $\bar{B}_{\boldsymbol{\eta}}$. By Lemma 11 ($m$-permutability for $(F, \boldsymbol{\eta})$), we have

$$B_{\boldsymbol{\eta}}\big(F[\boldsymbol{\Omega}_1], \dots, F[\boldsymbol{\Omega}_m]\big) = \bar{B}_{\boldsymbol{\eta}}\big(F[\boldsymbol{\Omega}_1], \dots, F[\boldsymbol{\Omega}_m]\big)$$

and

$$\begin{aligned}&B_{\boldsymbol{\eta}}\big(F[\boldsymbol{\Omega}_1], \dots, F[\boldsymbol{\Omega}_{k-1}], \eta\big(F[\boldsymbol{I}] \times F[\boldsymbol{J}]\big), F[\boldsymbol{\Omega}_{k+1}], \dots, F[\boldsymbol{\Omega}_m]\big)\\ &\qquad\qquad = \bar{B}_{\boldsymbol{\eta}}\big(F[\boldsymbol{\Omega}_1], \dots, F[\boldsymbol{\Omega}_{k-1}], \eta\big(F[\boldsymbol{I}] \times F[\boldsymbol{J}]\big), F[\boldsymbol{\Omega}_{k+1}], \dots, F[\boldsymbol{\Omega}_m]\big),\end{aligned}$$

hence

$$B_{\boldsymbol{\eta}}\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1], \dots, F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]\big) = \bar{B}_{\boldsymbol{\eta}}\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1], \dots, F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]\big),$$

which establishes our claim. □

**Lemma 16** (Functoriality of $F_{\boldsymbol\eta}^{(k)}$). *For each morphism $\boldsymbol f : \boldsymbol\Omega \to \widetilde{\boldsymbol\Omega}$ in $\mathbf{Set}^r$ and every integer $k \ge 0$, we have*

$$F[\boldsymbol f](F_{\boldsymbol\eta}^{(k)}[\boldsymbol\Omega]) = F_{\boldsymbol\eta}^{(k)}[\widetilde{\boldsymbol\Omega}].$$

*Proof.* We use induction on $k$, our claim being obvious for $k = 0$. Suppose that the assertion holds for $0 \le k < K$ with some $K \ge 1$. Then, using the definition of $F_{\boldsymbol\eta}^{(k)}$, the functoriality of $F_{\boldsymbol\eta}$ already demonstrated in Lemma 14, the inductive hypothesis, as well as the naturality of $\boldsymbol\eta$, we find that

$$\begin{aligned}
F[\boldsymbol f](F_{\boldsymbol\eta}^{(K)}[\boldsymbol\Omega]) &= F[\boldsymbol f]\Bigg(\bigcup_{\substack{\boldsymbol\Omega'\in \mathrm{Ob}(\mathbf{Set}^r)\\ \boldsymbol\Omega'\subseteq\boldsymbol\Omega}} \eta\big(F_{\boldsymbol\eta}[\boldsymbol\Omega'] \times F_{\boldsymbol\eta}^{(K-1)}[\boldsymbol\Omega - \boldsymbol\Omega']\big)\Bigg)\\
&= \bigcup_{\substack{\boldsymbol\Omega'\in \mathrm{Ob}(\mathbf{Set}^r)\\ \boldsymbol\Omega'\subseteq\boldsymbol\Omega}} \big(F \circ \amalg\big)\big[(\boldsymbol f|_{\boldsymbol\Omega'}, \boldsymbol f|_{\boldsymbol\Omega-\boldsymbol\Omega'})\big]\big(\eta\big(F_{\boldsymbol\eta}[\boldsymbol\Omega'] \times F_{\boldsymbol\eta}^{(K-1)}[\boldsymbol\Omega - \boldsymbol\Omega']\big)\big)\\
&= \bigcup_{\substack{\boldsymbol\Omega'\in \mathrm{Ob}(\mathbf{Set}^r)\\ \boldsymbol\Omega'\subseteq\boldsymbol\Omega}} \eta\big(F[\boldsymbol f|_{\boldsymbol\Omega'}](F_{\boldsymbol\eta}[\boldsymbol\Omega']) \times F[\boldsymbol f|_{\boldsymbol\Omega-\boldsymbol\Omega'}](F_{\boldsymbol\eta}^{(K-1)}[\boldsymbol\Omega - \boldsymbol\Omega'])\big)\\
&= \bigcup_{\substack{\widetilde{\boldsymbol\Omega}'\in \mathrm{Ob}(\mathbf{Set}^r)\\ \widetilde{\boldsymbol\Omega}'\subseteq\widetilde{\boldsymbol\Omega}}} \eta\big(F_{\boldsymbol\eta}[\widetilde{\boldsymbol\Omega}'] \times F_{\boldsymbol\eta}^{(K-1)}[\widetilde{\boldsymbol\Omega} - \widetilde{\boldsymbol\Omega}']\big)\\
&= F_{\boldsymbol\eta}^{(K)}[\widetilde{\boldsymbol\Omega}].
\end{aligned}$$

□

**Lemma 17.** *The functors $F_{\boldsymbol\eta}^{(1)}$ and $F_{\boldsymbol\eta}$ coincide.*

*Proof.* It suffices to show that $F_{\boldsymbol\eta}^{(1)}[\boldsymbol\Omega] = F_{\boldsymbol\eta}[\boldsymbol\Omega]$ for every $\boldsymbol\Omega \in \mathrm{Ob}(\mathbf{Set}^r)$. By (2.7), this holds if $\boldsymbol\Omega = \boldsymbol\emptyset$, so assume that $\boldsymbol\Omega \ne \boldsymbol\emptyset$. Then, using the definition of $F_{\boldsymbol\eta}^{(k)}$, the injectivity of $\boldsymbol{\eta}$-maps, Lemma 7 (3-associativity), and Lemma 5, we have

$$\begin{aligned}
F_{\boldsymbol{\eta}}^{(1)}[\boldsymbol{\Omega}] &= \bigcup_{\substack{\boldsymbol{\Omega}_1\in\mathrm{Ob}(\mathbf{Set}^r)\\ \boldsymbol{\Omega}_1\subseteq\boldsymbol{\Omega}}} \eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1]\times F_{\boldsymbol{\eta}}^{(0)}[\boldsymbol{\Omega}-\boldsymbol{\Omega}_1]\big)\\
&= \eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}]\times F[\boldsymbol{\emptyset}]\big)\\
&= \eta\left(\left(F[\boldsymbol{\Omega}] - \bigcup_{\substack{(\boldsymbol{I},\boldsymbol{J})\in\mathrm{Ob}(\mathfrak{D}_r)\\ \boldsymbol{I}\amalg\boldsymbol{J}=\boldsymbol{\Omega}\\ \boldsymbol{I}\neq\boldsymbol{\emptyset}\neq\boldsymbol{J}}} \eta\big(F[\boldsymbol{I}]\times F[\boldsymbol{J}]\big)\right)\times F[\boldsymbol{\emptyset}]\right)\\
&= \eta\big(F[\boldsymbol{\Omega}]\times F[\boldsymbol{\emptyset}]\big) - \bigcup_{\substack{(\boldsymbol{I},\boldsymbol{J})\in\mathrm{Ob}(\mathfrak{D}_r)\\ \boldsymbol{I}\amalg\boldsymbol{J}=\boldsymbol{\Omega}\\ \boldsymbol{I}\neq\boldsymbol{\emptyset}\neq\boldsymbol{J}}} \eta\Big(\eta\big(F[\boldsymbol{I}]\times F[\boldsymbol{J}]\big)\times F[\boldsymbol{\emptyset}]\Big)\\
&= F[\boldsymbol{\Omega}] - \bigcup_{\substack{(\boldsymbol{I},\boldsymbol{J})\in\mathrm{Ob}(\mathfrak{D}_r)\\ \boldsymbol{I}\amalg\boldsymbol{J}=\boldsymbol{\Omega}\\ \boldsymbol{I}\neq\boldsymbol{\emptyset}\neq\boldsymbol{J}}} \eta\big(F[\boldsymbol{I}]\times F[\boldsymbol{J}]\big)\\
&= F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}],
\end{aligned}$$

proving our claim. □

In the next lemma, we require again the concept of a base point, which was introduced just before Lemma 12.

**Lemma 18.** *For every non-empty $\boldsymbol{\Omega}\in\mathrm{Ob}(\mathbf{Set}^r)$, each choice of base point $(\omega,\rho)\in\boldsymbol{\Omega}$, and every integer $k\geq 1$, we have*

$$F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}] = \coprod_{\substack{\boldsymbol{\Omega}_1\in\mathrm{Ob}(\mathbf{Set}^r)\\ (\omega,\rho)\in\boldsymbol{\Omega}_1\subseteq\boldsymbol{\Omega}}} \eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1]\times F_{\boldsymbol{\eta}}^{(k-1)}[\boldsymbol{\Omega}-\boldsymbol{\Omega}_1]\big). \tag{5.2}$$

*Proof.* The fact that the terms on the right-hand side of (5.2) are pairwise disjoint follows from Lemma 13, since a term $\eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1]\times F_{\boldsymbol{\eta}}^{(k-1)}[\boldsymbol{\Omega}-\boldsymbol{\Omega}_1]\big)$ is contained in the larger set $\eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1]\times F[\boldsymbol{\Omega}-\boldsymbol{\Omega}_1]\big)$.

An immediate induction using the definition of $F_{\boldsymbol{\eta}}^{(m)}$ shows that, for all $m\geq 2$, we have

$$F_{\boldsymbol{\eta}}^{(m)}[\boldsymbol{\Omega}] = \bigcup_{\substack{\boldsymbol{\Omega}_1,\dots,\boldsymbol{\Omega}_m\in\mathrm{Ob}(\mathbf{Set}^r)\\ \boldsymbol{\Omega}_1\amalg\cdots\amalg\boldsymbol{\Omega}_m=\boldsymbol{\Omega}}} \eta\Big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1]\times\eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_2]\times\cdots\times\eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_{m-1}]\times F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]\big)\cdots\big)\Big). \tag{5.3}$$

We first consider the case where $k=1$. Here, by definition of $F_{\boldsymbol{\eta}}^{(0)}[\boldsymbol{\emptyset}]$, the only contribution to the union on the right-hand side of (5.2) arises for $\boldsymbol{\Omega}_1=\boldsymbol{\Omega}$. In that situation, we have

$$\eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}]\times F_{\boldsymbol{\eta}}^{(0)}[\boldsymbol{\emptyset}]\big) = \eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}]\times F[\boldsymbol{\emptyset}]\big).$$

However, this is also the only contribution on the right-hand side of the definition of $F_{\boldsymbol{\eta}}^{(1)}$ given in (2.5), thus proving (5.2) for $k = 1$.

Now we consider the case where $k \ge 2$. If $k \ge 3$, then, given $\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$, we substitute the right-hand side of (5.3) with $m = k-1$ in (5.2). As a result, we obtain

$$\bigcup_{\substack{\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2,\dots,\boldsymbol{\Omega}_k \in \mathrm{Ob}(\mathbf{Set}^r)\\ \boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2 \amalg \cdots \amalg \boldsymbol{\Omega}_k = \boldsymbol{\Omega}\\ (\omega,\rho)\in\boldsymbol{\Omega}_1}} \eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1] \times \eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_2] \times \cdots \times \eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_{k-1}] \times F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_k]\big) \cdots\big) \tag{5.4}$$

for the right-hand side of (5.2). We note that Expression (5.4) also agrees with the right-hand side of (5.2) for $k = 2$ (taking into account the fact that we already know that the union on the right-hand side of (5.2) is a disjoint union).

Expression (5.4) is almost (5.3) with $m = k$, except that $\boldsymbol{\Omega}_1$ is distinguished by having to contain the given base point $(\omega, \rho)$. However, by Lemma 15 ($m$-permutability for $(F_{\boldsymbol{\eta}}, \boldsymbol{\eta})$), the ordering of $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \dots, \boldsymbol{\Omega}_k$ in the $\boldsymbol{\eta}$-bracketing in the union on the right-hand side of (5.4) is of no relevance. Thus, the restriction that $(\omega, \rho) \in \boldsymbol{\Omega}_1$ can be dropped. This shows that the right-hand side of (5.2) equals $F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}]$, as claimed. $\square$

## 6. Proofs of Propositions 2 and 3, and of Theorem 4

We begin this section with the proof of Proposition 2. With Lemma 18 in hand, we are finally in the position to also establish Proposition 3. Theorem 4 is then a simple consequence of an identity on which the proof of Proposition 3 rests (see (6.1) below). Although the property expressed in this proposition is of a structural nature, our proof relies in fact on a counting argument. It would be desirable to find an alternative approach more in keeping with the actual nature of Proposition 3.

*Proof of Proposition* 2. We use induction on $\|\boldsymbol{\Omega}\|$, where $\|\,.\,\|$ has been defined in (2.6). By (2.7) and the definition of $F_{\boldsymbol{\eta}}^{(0)}$, the statement holds if $\|\boldsymbol{\Omega}\| = 0$, that is, if $\boldsymbol{\Omega} = \boldsymbol{\emptyset}$. Let $\boldsymbol{\Omega}$ be such that $\|\boldsymbol{\Omega}\| = N$ for some integer $N > 0$, and suppose that (2.8) holds for all $\mathbf{\Omega}' \in \mathrm{Ob}(\mathbf{Set}^r)$ of norm strictly less than $N$. Then we have $\mathbf{\Omega} \neq \boldsymbol{\emptyset}$, and therefore

$$\begin{aligned}
\bigcup_{k\ge 0} F_{\boldsymbol{\eta}}^{(k)}[\mathbf{\Omega}] &= \bigcup_{k\ge 1} F_{\boldsymbol{\eta}}^{(k)}[\mathbf{\Omega}] \\
&= \bigcup_{k\ge 1} \bigcup_{\substack{\mathbf{\Omega}_1 \in \mathrm{Ob}(\mathbf{Set}^r)\\ \mathbf{\Omega}_1 \subseteq \mathbf{\Omega}}} \eta\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times F_{\boldsymbol{\eta}}^{(k-1)}[\mathbf{\Omega} - \mathbf{\Omega}_1]\big) \\
&= \bigcup_{\substack{\mathbf{\Omega}_1 \in \mathrm{Ob}(\mathbf{Set}^r)\\ \mathbf{\Omega}_1 \subseteq \mathbf{\Omega}}} \eta\bigg(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times \Big(\bigcup_{k\ge 1} F_{\boldsymbol{\eta}}^{(k-1)}[\mathbf{\Omega} - \mathbf{\Omega}_1]\Big)\bigg) \\
&= \bigcup_{\substack{\mathbf{\Omega}_1 \in \mathrm{Ob}(\mathbf{Set}^r)\\ \boldsymbol{\emptyset} \neq \mathbf{\Omega}_1 \subseteq \mathbf{\Omega}}} \eta\bigg(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times \Big(\bigcup_{k\ge 0} F_{\boldsymbol{\eta}}^{(k)}[\mathbf{\Omega} - \mathbf{\Omega}_1]\Big)\bigg) \\
&= \bigcup_{\substack{\mathbf{\Omega}_1 \in \mathrm{Ob}(\mathbf{Set}^r)\\ \mathbf{\Omega}_1 \subseteq \mathbf{\Omega}}} \eta\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times F[\mathbf{\Omega} - \mathbf{\Omega}_1]\big) \\
&= F[\mathbf{\Omega}].
\end{aligned}$$

Here, we have used Lemma 12 for the last equality, and the inductive hypothesis in the second but last step (here it is important that $\mathbf{\Omega}_1 \neq \boldsymbol{\emptyset}$ in order to guarantee that $\|\mathbf{\Omega} - \mathbf{\Omega}_1\| < \|\mathbf{\Omega}\|$). $\square$

*Proof of Proposition* 3 *and of Theorem* 4. For $k \ge 0$, let us define the generating function for $F_{\boldsymbol{\eta}}^{(k)}$ by

$$\mathrm{GF}_{F_{\boldsymbol{\eta}}^{(k)}}(z_1,\dots,z_r) := \sum_{n_1,\dots,n_r\ge 0} \sum_{x \in F_{\boldsymbol{\eta}}^{(k)}[([n_1],\dots,[n_r])]} w_{([n_1],\dots,[n_r])}(x)\, \frac{z_1^{n_1}\cdots z_r^{n_r}}{n_1!\cdots n_r!}.$$

Again, in the sequel, we shall suppress the indices to weights $w$ for better readability, the indices always being clear from the context.

The first step consists in showing that

$$\mathrm{GF}_{F_{\boldsymbol{\eta}}^{(k)}}(z_1,\dots,z_r) = \frac{1}{k!}\big(\,\mathrm{GF}_{F_{\boldsymbol{\eta}}}(z_1,\dots,z_r)\big)^k. \tag{6.1}$$

By definition of $F_{\boldsymbol{\eta}}^{(0)}$, the left-hand side of (6.1) equals 1, so that (6.1) holds for $k=0$. Therefore, we may in the sequel assume that $k \ge 1$.

We now proceed in a manner similar to the proof of Theorem 1 given in Section 4. Here, however, we use Lemma 18 instead of Lemmas 12 and 13, and we also need the functoriality of $F_{\boldsymbol{\eta}}^{(m)}$ for $m = 0,1,2,\dots$ established in Lemma 16. In this way, we obtain from (5.2) the identity

$$\begin{aligned}
n_\rho \cdot w\big(F_{\boldsymbol{\eta}}^{(k)}[([n_1],\dots,[n_r])]\big) = \sum_{0\le \mu_i \le n_i\,(1\le i\le r)} \mu_\rho \bigg(\prod_{1\le i\le r} \binom{n_i}{\mu_i}\bigg) w\big(F_{\boldsymbol{\eta}}[([\mu_1],\dots,[\mu_r])]\big)\\
\cdot\, w\big(F_{\boldsymbol{\eta}}^{(k-1)}[([n_1-\mu_1],\dots,[n_r-\mu_r])]\big).
\end{aligned} \tag{6.2}$$

Fixing $\rho \in [r]$, multiplying both sides of (6.2) by

$$z_1^{n_1} \cdots z_{\rho-1}^{n_{\rho-1}} z_\rho^{n_\rho - 1} z_{\rho+1}^{n_{\rho+1}} \cdots z_r^{n_r} / (n_1! \cdots n_r!),$$

and summing over all tuples $(n_1, \dots, n_r) \in \mathbb{N}_0^r$, gives

$$\sum_{n_1,\dots,n_r \ge 0} w\big(F_{\boldsymbol\eta}^{(k)}[([n_1],\dots,[n_r])]\big) \frac{z_1^{n_1} \cdots z_{\rho-1}^{n_{\rho-1}} z_\rho^{n_\rho-1} z_{\rho+1}^{n_{\rho+1}} \cdots z_r^{n_r}}{n_1! \cdots n_{\rho-1}!\,(n_\rho - 1)!\,n_{\rho+1}! \cdots n_r!}$$
$$= \sum_{n_1,\dots,n_r \ge 0} \sum_{0 \le \mu_i \le n_i\,(1 \le i \le r)} \frac{w\big(F_{\boldsymbol\eta}[([\mu_1],\dots,[\mu_r])]\big)}{\mu_1! \cdots \mu_{\rho-1}!\,(\mu_\rho - 1)!\,\mu_{\rho+1}! \cdots \mu_r!}$$
$$\cdot \frac{w\big(F_{\boldsymbol\eta}^{(k-1)}[([n_1-\mu_1],\dots,[n_r-\mu_r])]\big)}{(n_1-\mu_1)! \cdots (n_r-\mu_r)!} z_1^{n_1} \cdots z_{\rho-1}^{n_{\rho-1}} z_\rho^{n_\rho-1} z_{\rho+1}^{n_{\rho+1}} \cdots z_r^{n_r}, \quad (6.3)$$

where, again, $1/(-1)!$ has to be interpreted as 0. The left-hand side equals

$$\frac{\partial\,\mathrm{GF}_{F_{\boldsymbol\eta}^{(k)}}}{\partial z_\rho},$$

while the right-hand side is identified as

$$\frac{\partial\,\mathrm{GF}_{F_{\boldsymbol\eta}}}{\partial z_\rho}\,\mathrm{GF}_{F_{\boldsymbol\eta}^{(k-1)}},$$

whence the equations

$$\frac{\partial\,\mathrm{GF}_{F_{\boldsymbol\eta}^{(k)}}}{\partial z_\rho} = \frac{\partial\,\mathrm{GF}_{F_{\boldsymbol\eta}}}{\partial z_\rho}\,\mathrm{GF}_{F_{\boldsymbol\eta}^{(k-1)}}, \qquad 1 \le \rho \le r. \quad (6.4)$$

Assuming inductively that

$$\mathrm{GF}_{F_{\boldsymbol\eta}^{(k-1)}} = \frac{1}{(k-1)!}\big(\mathrm{GF}_{F_{\boldsymbol\eta}}\big)^{k-1},$$

we infer from (6.4) that

$$\mathrm{GF}_{F_{\boldsymbol\eta}^{(k)}} = \frac{1}{k!}\big(\mathrm{GF}_{F_{\boldsymbol\eta}}\big)^{k} + C,$$

where $C$ is independent of $z_1, z_2, \dots, z_r$. Making use of the facts that $\mathrm{GF}_{F_{\boldsymbol\eta}^{(k)}}(0,\dots,0) = 0$ (since $k \ge 1$) and that $\mathrm{GF}_{F_{\boldsymbol\eta}}(0,\dots,0) = 0$, we see that $C = 0$, which proves (6.1).

On the other hand, by Theorem 1, we know that

$$\mathrm{GF}_F(z_1, z_2, \dots, z_r) = \exp\big(\mathrm{GF}_{F_{\boldsymbol\eta}}(z_1, z_2, \dots, z_r)\big),$$

or, equivalently,

$$\mathrm{GF}_F(z_1, z_2, \dots, z_r) = \sum_{k \ge 0} \frac{1}{k!}\big(\mathrm{GF}_{F_{\boldsymbol\eta}}(z_1, z_2, \dots, z_r)\big)^k. \quad (6.5)$$

If there were a non-empty intersection between $F_{\boldsymbol\eta}^{(k_1)}[\boldsymbol\Omega]$ and $F_{\boldsymbol\eta}^{(k_2)}[\boldsymbol\Omega]$, for some $k_1, k_2$ with $k_1 < k_2$ and some $\boldsymbol\Omega \in \mathrm{Ob}(\mathbf{Set}^r)$, then Proposition 2 would contradict (6.5) and (6.1). This proves the assertion of Proposition 3.

The proof of Theorem 4 is now easily completed. By definition of $\widetilde{\mathrm{GF}}_F(z_1,\dots,z_r,y)$, we have

$$\widetilde{\mathrm{GF}}_F(z_1,\dots,z_r,y)=\sum_{k\ge 0} y^k\,\mathrm{GF}_{F^{(k)}_{\boldsymbol\eta}}(z_1,\dots,z_r).$$

If we now substitute (6.1), then we immediately obtain (2.10). Identity (2.11) results from using Theorem 1 to express $\mathrm{GF}_{F_{\boldsymbol\eta}}(z_1,\dots,z_r)$ in terms of $\mathrm{GF}_F(z_1,\dots,z_r)$ and substituting the result in (2.10). □

## 7. Illustrations, I: Three examples

We give here three illustrations for the application of our theory. In the first and second example below, bipartite graphs are considered. Example 1 is, in some sense, "standard," since it addresses the case where the composition operator $\boldsymbol\eta$ consists in "putting objects together," so that the combinatorial objects in our (in this case, 2-sort) species are sets of indecomposable objects, a situation which is well covered by classical species theory. In Example 2, however, the composition operator $\boldsymbol\eta$ is different, "non-standard," so that classical species theory does not apply, but our (extension of species) theory does. On the other hand, we shall see in Section 9 that this composition operator is pointwise associative and commutative (for the precise definition see (9.4) and (9.5)), and that this is equivalent to the fact that this case is also covered by Menni's theory in [21]. As a consequence, this family of composition operators is closely related to the classical operation of "putting objects together." (See Theorem 22 for the precise statement.) Our last example in this section, Example 3, presents an example of a composition operator that is neither pointwise associative nor pointwise commutative, in other words, a composition operator that is not covered by Menni's theory in [21]. A particular aspect demonstrated by Examples 1 and 2 that we want to highlight is that composition operators need not be unique.

*Example* 1 (Bipartite graphs I). Let the 2-sort species $F:\mathbf{Set}^2\to\mathbf{Set}$ be defined by

$$F[\boldsymbol\Omega]=F[(\Omega^{(1)},\Omega^{(2)})]:=2^{\Omega^{(1)}\times\Omega^{(2)}},\quad \boldsymbol\Omega=(\Omega^{(1)},\Omega^{(2)})\in\mathrm{Ob}(\mathbf{Set}^2).$$

Thus, $F[(\Omega^{(1)},\Omega^{(2)})]$ can be considered as set of all bipartite graphs, where the set of "white" vertices is $\Omega^{(1)}$ and the set of "black" vertices is $\Omega^{(2)}$. For $(\boldsymbol\Omega_1,\boldsymbol\Omega_2)\in\mathrm{Ob}(\mathfrak{D}_2)$, $b_1\in F[\boldsymbol\Omega_1]$ and $b_2\in F[\boldsymbol\Omega_2]$, put

$$\eta_{(\boldsymbol\Omega_1,\boldsymbol\Omega_2)}\big((b_1,b_2)\big):=b_1\amalg b_2.$$

This means that $\eta_{(\boldsymbol\Omega_1,\boldsymbol\Omega_2)}$ merely forms the disjoint union of the bipartite graphs $b_1$ and $b_2$. Then it is not difficult to see that $\boldsymbol\eta$ is a natural transformation satisfying (D1). Moreover, $F_{\boldsymbol\eta}[\boldsymbol\Omega]$ consists of the *connected* bipartite graphs with bipartition $\boldsymbol\Omega=(\Omega^{(1)},\Omega^{(2)})$.

For a weight, we choose $\Lambda=\mathbb{Z}[t]$ and

$$w_{\boldsymbol\Omega}(b):=t^{|b|},\quad b\in F[\boldsymbol\Omega].$$

Again, it is not difficult to see that $\boldsymbol w$ satisfies Axioms (W0)–(W2); that is, $\boldsymbol w$ is a $\Lambda$-weight on $(F,\boldsymbol\eta)$.

Theorem 1 then says that

$$\mathrm{GF}_F(z_1,z_2)=\exp\big(\mathrm{GF}_{F_{\boldsymbol\eta}}(z_1,z_2)\big),\tag{7.1}$$

where

$$\mathrm{GF}_F(z_1,z_2)=\sum_{n_1,n_2\ge 0}\ \sum_{b\in F[([n_1],[n_2])]} t^{|b|}\frac{z_1^{n_1}z_2^{n_2}}{n_1!\,n_2!}$$

and

$$\mathrm{GF}_{F_{\boldsymbol\eta}}(z_1,z_2)=\sum_{n_1,n_2\ge 0}\ \sum_{b\in F_{\boldsymbol\eta}[([n_1],[n_2])]} t^{|b|}\frac{z_1^{n_1}z_2^{n_2}}{n_1!\,n_2!}.$$

However, by straightforward counting, one sees that

$$\mathrm{GF}_F(z_1,z_2)=\sum_{n_1,n_2\ge 0}(1+t)^{n_1n_2}\frac{z_1^{n_1}z_2^{n_2}}{n_1!\,n_2!}.$$

From (7.1), it then follows that the generating function for connected bipartite graphs is given by

$$\mathrm{GF}_{F_{\boldsymbol\eta}}(z_1,z_2)=\log\left(\sum_{n_1,n_2\ge 0}(1+t)^{n_1n_2}\frac{z_1^{n_1}z_2^{n_2}}{n_1!\,n_2!}\right),$$

while (2.11) implies that

$$\begin{aligned}\widetilde{\mathrm{GF}}_F(z_1,z_2)&:=\sum_{n_1,n_2\ge 0}\ \sum_{b\in F[([n_1],[n_2])]} t^{|b|}y^{\#(\text{connected components of }b)}\frac{z_1^{n_1}z_2^{n_2}}{n_1!\,n_2!}\\&=\left(\sum_{n_1,n_2\ge 0}(1+t)^{n_1n_2}\frac{z_1^{n_1}z_2^{n_2}}{n_1!\,n_2!}\right)^y.\end{aligned}\tag{7.2}$$

This example can be considered as a 2-dimensional analogue of the example in [9, Sec. 3] (with the first of the two composition operators considered there). The knowledgeable reader will recognise (7.2) as the exponential generating function for the Tutte polynomials of complete bipartite graphs (cf. e.g. [24, Eq. (3.10)]).

*Example* 2 (Bipartite graphs II). Let $F:\mathbf{Set}^2\to\mathbf{Set}$ be as in Example 1. Here, for $(\boldsymbol\Omega_1,\boldsymbol\Omega_2)\in\mathrm{Ob}(\mathfrak{D}_2)$, where $\boldsymbol\Omega_i=(\Omega_i^{(1)},\Omega_i^{(2)})$, $i=1,2$, for $b_1\in F[\boldsymbol\Omega_1]$ and $b_2\in F[\boldsymbol\Omega_2]$, we put

$$\eta'_{(\boldsymbol\Omega_1,\boldsymbol\Omega_2)}\big((b_1,b_2)\big):=b_1\amalg b_2\amalg\left(\Omega_1^{(1)}\times\Omega_2^{(2)}\right)\amalg\left(\Omega_1^{(2)}\times\Omega_2^{(1)}\right).$$

The graph $\eta'_{(\boldsymbol\Omega_1,\boldsymbol\Omega_2)}\big((b_1,b_2)\big)$ can be considered as a kind of bipartite completion of the disjoint union of $b_1$ and $b_2$. Again, it is not difficult to see that $\boldsymbol\eta'$ is a natural transformation satisfying (D1). Moreover, $F_{\boldsymbol\eta'}(\boldsymbol\Omega)$ consists of the *complements* of connected bipartite graphs with bipartition $\boldsymbol\Omega=(\Omega^{(1)},\Omega^{(2)})$, where the complement $b^c$ of a bipartite graph $b\in F[\boldsymbol\Omega]$ is defined as $b^c:=\big(\Omega^{(1)}\times\Omega^{(2)}\big)-b$.

If we now were to choose the weight of Example 1, then Axiom (W2) would be violated. Instead, with $\Lambda=\mathbb{Z}[t]$, we set

$$w'_{\boldsymbol\Omega}(b):=t^{|\Omega^{(1)}|\cdot|\Omega^{(2)}|-|b|},\quad b\in F[\boldsymbol\Omega]=F\big[(\Omega^{(1)},\Omega^{(2)})\big].$$

Then it is not difficult to see that $\boldsymbol w'$ does satisfy Axioms (W0)–(W2); that is, $\boldsymbol w'$ is a $\Lambda$-weight on $(F,\boldsymbol\eta')$.

Theorem 1 then says that

$$\mathrm{GF}_F(z_1,z_2)=\exp\big(\mathrm{GF}_{F_{\boldsymbol\eta'}}(z_1,z_2)\big),\tag{7.3}$$

where

$$\mathrm{GF}_F(z_1,z_2)=\sum_{n_1,n_2\ge 0}\ \sum_{b\in F[([n_1],[n_2])]} t^{n_1n_2-|b|}\frac{z_1^{n_1}z_2^{n_2}}{n_1!\,n_2!}$$

and

$$\mathrm{GF}_{F_{\boldsymbol\eta'}}(z_1,z_2)=\sum_{n_1,n_2\ge 0}\ \sum_{b\in F_{\boldsymbol\eta'}([n_1],[n_2])} t^{n_1n_2-|b|}\frac{z_1^{n_1}z_2^{n_2}}{n_1!\,n_2!}.$$

Again, by straightforward counting, one sees that

$$\mathrm{GF}_F(z_1,z_2)=\sum_{n_1,n_2\ge 0}(1+t)^{n_1n_2}\frac{z_1^{n_1}z_2^{n_2}}{n_1!\,n_2!},$$

and we obtain the formulae

$$\mathrm{GF}_{F_{\boldsymbol\eta'}}(z_1,z_2)=\log\left(\sum_{n_1,n_2\ge 0}(1+t)^{n_1n_2}\frac{z_1^{n_1}z_2^{n_2}}{n_1!\,n_2!}\right)$$

and

$$\begin{aligned}\widetilde{\mathrm{GF}}_F(z_1,z_2)&:=\sum_{n_1,n_2\ge 0}\ \sum_{b\in F[([n_1],[n_2])]} t^{n_1n_2-|b|}y^{\#(\text{connected components of } b^c)}\frac{z_1^{n_1}z_2^{n_2}}{n_1!\,n_2!}\\ &=\left(\sum_{n_1,n_2\ge 0}(1+t)^{n_1n_2}\frac{z_1^{n_1}z_2^{n_2}}{n_1!\,n_2!}\right)^y.\end{aligned}\tag{7.4}$$

This example can be viewed as a 2-dimensional analogue of the example in [9, Sec. 3] (with the second of the two composition operators considered there).

The alert reader will have noticed that the $\boldsymbol\eta'$-maps could have been alternatively defined by

$$\eta'_{(\boldsymbol\Omega_1,\boldsymbol\Omega_2)}\big((b_1,b_2)\big):=\big(b_1^c\amalg b_2^c\big)^c,\tag{7.5}$$

where the complements have to be taken in the appropriate complete bipartite graphs. This construction will be generalised in Section 9.

*Example* 3 (Binary functions). Let the (1-sort) species $F:\mathbf{Set}\to\mathbf{Set}$ be defined by

$$F[\boldsymbol\Omega]:=\{0,1\}^{\boldsymbol\Omega},\quad \boldsymbol\Omega\in\mathrm{Ob}(\mathbf{Set}).$$

For $(\boldsymbol\Omega_1,\boldsymbol\Omega_2)\in\mathrm{Ob}(\mathfrak{D}_1)$, $f_1\in F[\boldsymbol\Omega_1]$, and $f_2\in F[\boldsymbol\Omega_2]$, put

$$\big(\eta_{(\boldsymbol\Omega_1,\boldsymbol\Omega_2)}\big((f_1,f_2)\big)\big)(\omega):=\begin{cases}f_1(\omega), & \text{if } \omega\in\boldsymbol\Omega_1,\\ 1-f_2(\omega), & \text{if } \omega\in\boldsymbol\Omega_2.\end{cases}$$

Then it is easy to see that $\boldsymbol\eta$ is a natural transformation satisfying (D1). Moreover,

$$F_{\boldsymbol\eta}[\boldsymbol\Omega]=\begin{cases}\{0_{\boldsymbol\Omega},1_{\boldsymbol\Omega}\}, & \text{if } |\boldsymbol\Omega|=1,\\ \{\}, & \text{otherwise,}\end{cases}$$

where $0_{\boldsymbol\Omega}$ and $1_{\boldsymbol\Omega}$ are the constant functions on $\boldsymbol\Omega$ taking the value 0 and 1, respectively. We note that, in contrast to Examples 1 and 2, the $\boldsymbol\eta$-maps of the present example are *pointwise non-associative and non-commutative* (cf. Section 9); to be precise, in general we have

$$\eta_{(\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2, \mathbf{\Omega}_3)}\Big(\big(\eta_{(\mathbf{\Omega}_1,\mathbf{\Omega}_2)}\big((f_1,f_2)\big), f_3\big)\Big) \neq \eta_{(\mathbf{\Omega}_1,\mathbf{\Omega}_2 \amalg \mathbf{\Omega}_3)}\Big(\big(f_1, \eta_{(\mathbf{\Omega}_2,\mathbf{\Omega}_3)}\big((f_2,f_3)\big)\big)\Big)$$

and

$$\eta_{(\mathbf{\Omega}_1,\mathbf{\Omega}_2)}\big((f_1,f_2)\big) \neq \eta_{(\mathbf{\Omega}_2,\mathbf{\Omega}_1)}\big((f_2,f_1)\big).$$

For the sake of completeness, we remark that, choosing the trivial weighting

$$w_{\mathbf{\Omega}}(f) := 1, \quad f \in F[\mathbf{\Omega}],$$

Theorem 1 yields the trivial identity

$$\mathrm{GF}_F(z) = \sum_{n \ge 0} 2^n \frac{z^n}{n!} = \exp\big(\mathrm{GF}_{F_\eta}(z)\big) = \exp(2z).$$

The construction of this example can also be generalised to produce many more (multisort) species with pointwise non-associative and non-commutative composition operator, see Theorem 23 in Section 9.

## 8. Illustrations, II: Magic squares

The purpose of this section is to illustrate the increased flexibility of our present multivariate setting. We show that a number of generating function identities for combinatorial matrices found scattered throughout the literature can be uniformly explained, and generalised, in the context of our theory.

By a *combinatorial matrix* on $\mathbf{\Omega} = (\Omega^{(1)}, \Omega^{(2)}) \in \mathrm{Ob}(\mathbf{Set}^2)$ we shall mean any map

$$m : \Omega^{(1)} \times \Omega^{(2)} \to \mathbb{N}_0.$$

The pair of sets $\mathbf{\Omega}$ is called the *support* of $m$. Let $m_1, m_2$ be two combinatorial matrices with supports $\mathbf{\Omega}_1 = (\Omega_1^{(1)}, \Omega_1^{(2)})$ and $\mathbf{\Omega}_2 = (\Omega_2^{(1)}, \Omega_2^{(2)})$, respectively, and suppose that $\mathbf{\Omega}_1 \cap \mathbf{\Omega}_2 = \boldsymbol{\emptyset}$. Then we define their *direct sum* $m = m_1 \oplus m_2$ to be the combinatorial matrix with support $\mathbf{\Omega} := \mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2$ given by

$$m(\omega_1,\omega_2) := \begin{cases} m_1(\omega_1,\omega_2), & (\omega_1,\omega_2) \in \Omega_1^{(1)} \times \Omega_1^{(2)}, \\ m_2(\omega_1,\omega_2), & (\omega_1,\omega_2) \in \Omega_2^{(1)} \times \Omega_2^{(2)}, \\ 0, & \text{otherwise.} \end{cases}$$

A combinatorial matrix $m$ on $\mathbf{\Omega}$ is termed *$s$-magic*,[7] $s$ a positive integer, if

$$\sum_{\omega_2 \in \Omega^{(2)}} m(\omega_1,\omega_2) = s, \quad \omega_1 \in \Omega^{(1)},$$

and

$$\sum_{\omega_1 \in \Omega^{(1)}} m(\omega_1,\omega_2) = s, \quad \omega_2 \in \Omega^{(2)}.$$

Computing the sum of entries, we find that an $s$-magic matrix is necessarily square, $|\Omega^{(1)}| = |\Omega^{(2)}|$. The enumeration of $s$-magic squares has a long history, going back to MacMahon [20, §404–419]. A good account of the enumerative theory of magic squares

[7]Strictly speaking, the correct term here would be "$s$-semi-magic," since we do not require diagonals to sum up to $s$ as well, see e.g. [3]. However, we prefer the term "$s$-magic" for the sake of brevity.

can be found in [27, Sec. 4.6], with many pointers to further literature. For more recent work, see for instance [3, 8].

For $s \in \mathbb{N}$ and $\mathbf{\Omega} \in \mathrm{Ob}(\mathbf{Set}^2)$, denote by $F_s(\mathbf{\Omega})$ the set of all $s$-magic matrices on $\mathbf{\Omega}$, and by $\bar{F}_s(\mathbf{\Omega})$ the set of those $s$-magic matrices on $\mathbf{\Omega}$ which do not contain $s$ as an entry. We thus have mappings

$$F_s, \bar{F}_s : \mathrm{Ob}(\mathbf{Set}^2) \to \mathrm{Ob}(\mathbf{Set}),$$

which we turn into functors $F_s, \bar{F}_s : \mathbf{Set}^2 \to \mathbf{Set}$ by assigning to a morphism

$$\boldsymbol{f} = (f_1, f_2) : \mathbf{\Omega} \to \widetilde{\mathbf{\Omega}}$$

in $\mathbf{Set}^2$ the map (denoted $F_s[\boldsymbol{f}]$ respectively $\bar{F}_s[\boldsymbol{f}]$) sending a combinatorial matrix $m$ in the respective domain to $m \circ (f_1^{-1} \times f_2^{-1})$. Moreover, given $s$ and a finite set $\Omega$, let $F_s^*(\Omega)$ be the set of *symmetric* $s$-magic matrices on $\mathbf{\Omega} = (\Omega, \Omega)$; that is, combinatorial matrices satisfying

$$m(\omega_1, \omega_2) = m(\omega_2, \omega_1), \quad (\omega_1, \omega_2) \in \Omega^2;$$

and denote by $\bar{F}_s^*(\Omega)$ the subset of $F_s^*(\Omega)$ consisting of those matrices with no entry equal to $s$. Just as above, the maps

$$F_s^*, \bar{F}_s^* : \mathrm{Ob}(\mathbf{Set}) \to \mathrm{Ob}(\mathbf{Set})$$

become functors $F_s^*, \bar{F}_s^* : \mathbf{Set} \to \mathbf{Set}$ by assigning to a morphism $f : \Omega \to \widetilde{\Omega}$ in $\mathbf{Set}$ the map sending a combinatorial matrix $m$ in the respective domain to $m \circ (f^{-1} \times f^{-1})$.

Next, given $s \in \mathbb{N}$, a pair $(\mathbf{\Omega}_1, \mathbf{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_2)$, and a pair $(\Omega_1, \Omega_2) \in \mathrm{Ob}(\mathfrak{D}_1)$, the direct sum construction provides us with injective maps

$$\begin{aligned}
(\eta_s)_{(\mathbf{\Omega}_1, \mathbf{\Omega}_2)} &: F_s(\mathbf{\Omega}_1) \times F_s(\mathbf{\Omega}_2) \to F_s(\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2),\\
(\bar{\eta}_s)_{(\mathbf{\Omega}_1, \mathbf{\Omega}_2)} &: \bar{F}_s(\mathbf{\Omega}_1) \times \bar{F}_s(\mathbf{\Omega}_2) \to \bar{F}_s(\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2),\\
(\eta_s^*)_{(\Omega_1, \Omega_2)} &: F_s^*(\Omega_1) \times F_s^*(\Omega_2) \to F_s^*(\Omega_1 \amalg \Omega_2),\\
(\bar{\eta}_s^*)_{(\Omega_1, \Omega_2)} &: \bar{F}_s^*(\Omega_1) \times \bar{F}_s^*(\Omega_2) \to \bar{F}_s^*(\Omega_1 \amalg \Omega_2).
\end{aligned}$$

A certain amount of checking is required in order to convince oneself that these definitions fit into the framework of Theorems 1 and 4. The next lemma states the corresponding result. We leave its proof, which essentially amounts to a routine verification, to the reader.

**Lemma 19.** (i) *For each* $s \in \mathbb{N}$, *the collection of maps*

$$\boldsymbol{\eta}_s = \big((\eta_s)_{(\mathbf{\Omega}_1, \mathbf{\Omega}_2)}\big)_{(\mathbf{\Omega}_1, \mathbf{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_2)}$$

*is a natural transformation from the functor* $F_s \times F_s$ *to the functor* $F_s \circ \amalg$. *Analogous statements hold for the functors* $\bar{F}_s, F_s^*, \bar{F}_s^*$, *and the families of maps*

$$\begin{aligned}
\bar{\boldsymbol{\eta}}_s &= \big((\bar{\eta}_s)_{(\mathbf{\Omega}_1, \mathbf{\Omega}_2)}\big)_{(\mathbf{\Omega}_1, \mathbf{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_2)},\\
\boldsymbol{\eta}_s^* &= \big((\eta_s^*)_{(\Omega_1, \Omega_2)}\big)_{(\Omega_1, \Omega_2) \in \mathrm{Ob}(\mathfrak{D}_1)},\\
\bar{\boldsymbol{\eta}}_s^* &= \big((\bar{\eta}_s^*)_{(\Omega_1, \Omega_2)}\big)_{(\Omega_1, \Omega_2) \in \mathrm{Ob}(\mathfrak{D}_1)}.
\end{aligned}$$

(ii) *For each* $s$, *the pair* $(F_s, \boldsymbol{\eta}_s)$ *satisfies Axiom* (D1), *an analogous statement holding for each of the other pairs* $(\bar{F}_s, \bar{\boldsymbol{\eta}}_s)$, $(F_s^*, \boldsymbol{\eta}_s^*)$, *and* $(\bar{F}_s^*, \bar{\boldsymbol{\eta}}_s^*)$.

It follows from Lemma 19 and Theorem 4, that Equations (2.10) and (2.11) hold for each of the pairs $(F_s, \boldsymbol{\eta}_s)$, $(\bar F_s, \bar{\boldsymbol{\eta}}_s)$, $(F_s^*, \boldsymbol{\eta}_s^*)$, and $(\bar F^*, \bar{\boldsymbol{\eta}}_s^*)$; in particular, we find that

$$\widetilde{\mathrm{GF}}_{F_s}(z_1,z_2,y) = \exp\big(y\,\mathrm{GF}_{(F_s)_{\boldsymbol{\eta}_s}}(z_1,z_2)\big), \tag{8.1}$$

$$\widetilde{\mathrm{GF}}_{\bar F_s}(z_1,z_2,y) = \exp\big(y\,\mathrm{GF}_{(\bar F_s)_{\bar{\boldsymbol{\eta}}_s}}(z_1,z_2)\big), \tag{8.2}$$

$$\widetilde{\mathrm{GF}}_{F_s^*}(z,y) = \exp\big(y\,\mathrm{GF}_{(F_s^*)_{\boldsymbol{\eta}_s^*}}(z)\big), \tag{8.3}$$

$$\widetilde{\mathrm{GF}}_{\bar F_s^*}(z,y) = \exp\big(y\,\mathrm{GF}_{(\bar F_s^*)_{\bar{\boldsymbol{\eta}}_s^*}}(z)\big). \tag{8.4}$$

Note that in these identities the variable $y$ keeps track of the number of indecomposable matrices into which the matrices which are counted by the respective generating functions on the left-hand sides can be decomposed. Clearly, the generating functions occurring in (8.1) and (8.2) can be viewed as formal power series in $z_1z_2$ and $y$; that is, $z_1z_2$ could be replaced by a single variable. However, we prefer to keep $z_1$ and $z_2$ separate, since this is more in line with our general theory.

We note certain dependencies among the series $\widetilde{\mathrm{GF}}_{F_s}$, $\widetilde{\mathrm{GF}}_{\bar F_s}$, $\widetilde{\mathrm{GF}}_{F_s^*}$, $\widetilde{\mathrm{GF}}_{\bar F_s^*}$; for instance, we observe that an indecomposable $s$-magic matrix on $([n_1],[n_2])$ cannot contain an entry equal to $s$, unless $n_1=n_2=1$. It follows that

$$\big|(F_s)_{\boldsymbol{\eta}_s}([n_1],[n_2])\big| = \begin{cases} 1+\big|(\bar F_s)_{\bar{\boldsymbol{\eta}}_s}([1],[1])\big|, & n_1=n_2=1,\\ \big|(\bar F_s)_{\bar{\boldsymbol{\eta}}_s}([n_1],[n_2])\big|, & \text{otherwise,}\end{cases}$$

and hence, by Equations (8.1) and (8.2),

$$\widetilde{\mathrm{GF}}_{\bar F_s}(z_1,z_2,y) = e^{-z_1z_2y}\,\widetilde{\mathrm{GF}}_{F_s}(z_1,z_2,y). \tag{8.5}$$

Similarly, we have

$$\widetilde{\mathrm{GF}}_{\bar F_s^*}(z,y) = e^{-y(z+z^2/2)}\widetilde{\mathrm{GF}}_{F_s^*}(z,y). \tag{8.6}$$

Indeed, for $n=1,2$, there exist indecomposable symmetric $s$-magic matrices on $([n],[n])$ containing an entry $s$:

$$(s)\ \text{ and }\ \begin{pmatrix}0&s\\ s&0\end{pmatrix}.$$

Now let $m$ be a symmetric $s$-magic matrix on $([n],[n])$ with $n\ge 3$, and suppose that $m$ contains an entry equal to $s$ in position $(i,j)$. Then, if $i=j$, we have $m=(s)\oplus m'$, where $m'$ has support $([n]-\{i\},[n]-\{i\})$. If, on the other hand, $i\ne j$, then, by symmetry, $m$ also contains $s$ in position $(j,i)$, and we find that $m$ splits as

$$m=\begin{pmatrix}0&s\\ s&0\end{pmatrix}\oplus m',$$

where $m'$ has support $([n]-\{i,j\},[n]-\{i,j\})$, and is non-empty since $n\ge 3$. Thus, in both cases, $m$ is in fact decomposable. Hence,

$$\big|(F_s^*)_{\boldsymbol{\eta}_s^*}([n])\big| = \begin{cases} 1+\big|(\bar F_s^*)_{\bar{\boldsymbol{\eta}}_s^*}([n])\big|, & n=1,2,\\ \big|(\bar F_s^*)_{\bar{\boldsymbol{\eta}}_s^*}([n])\big|, & \text{otherwise,}\end{cases}$$

and (8.6) follows from Equations (8.3) and (8.4).

The enumeration can be done exactly if $s=2$. For, according to Birkhoff's Theorem (cf. [5] or [1, Corollary 8.40]), a 2-magic matrix $m$ is the sum of two permutation matrices, say $p_1$ and $p_2$. If $m$ is indecomposable, then the pair $\{p_1, p_2\}$ is uniquely determined. Premultiplying by $p_1^{-1}$, we obtain a situation where $p_1$ is the identity; indecomposability forces $p_2$ to be the permutation matrix corresponding to a cyclic permutation. So there are $n!(n-1)!$ choices for $(p_1, p_2)$, and half this many choices for $m$ (assuming, as we may, that $n > 1$). Note that this formula gives half the correct number for $n = 1$. So we have

$$\big|(F_2)_{\boldsymbol{\eta}_2}([n_1],[n_2])\big| = \begin{cases} 1, & n_1 = n_2 = n = 1, \\ \frac{n!(n-1)!}{2}, & n_1 = n_2 = n > 1, \\ 0, & n_1 \neq n_2, \end{cases}$$

that is,

$$\mathrm{GF}_{(F_2)_{\boldsymbol{\eta}_2}}(z_1, z_2) = \frac{1}{2} z_1 z_2 - \frac{1}{2} \log(1 - z_1 z_2),$$

and therefore

$$\widetilde{\mathrm{GF}}_{F_2}(z_1, z_2, y) = (1 - z_1 z_2)^{-y/2}\, e^{z_1 z_2 y/2} \tag{8.7}$$

by Equation (8.1). Also,

$$\widetilde{\mathrm{GF}}_{\bar{F}_2}(z_1, z_2, y) = (1 - z_1 z_2)^{-y/2}\, e^{-z_1 z_2 y/2}, \tag{8.8}$$

making use of Equation (8.5) and the last result. Special cases of Identities (8.7) and (8.8) appear in [2, Sections 8.1 and 8.3] (see also [26, Eqs. (23) and (24) in Example 6.11]). For $s > 2$, enumeration is more difficult; see Stanley's paper [25] and Comtet [7, pp. 124–125] for comments in this direction; also Goulden and Jackson [13, Sections 3.4 and 3.5] for some variations on this counting problem.[8]

For symmetric matrices, it is again possible to count the indecomposables with $s = 2$. For $n > 2$, such a matrix can be represented as a graph in which every vertex has degree 2; loops are permitted, but contribute only one to the degree of a vertex. Indecomposability of the matrix is reflected in connectedness of the graph. So the graphs we must consider are paths (with a loop at each end) and cycles; and, for $n > 2$, their number is $\frac{1}{2}(n! + (n-1)!)$. Including the cases where $n \le 2$, we obtain

$$\mathrm{GF}_{(F_2^*)_{\boldsymbol{\eta}_2^*}}(z) = \frac{z^2}{4} + \frac{z}{2(1-z)} - \frac{1}{2}\log(1 - z),$$

and, hence

$$\widetilde{\mathrm{GF}}_{F_2^*}(z, y) = (1 - z)^{-y/2} \exp\Big(\frac{yz^2}{4} + \frac{yz}{2(1-z)}\Big), \tag{8.9}$$

as well as

$$\widetilde{\mathrm{GF}}_{\bar{F}_2^*}(z, y) = (1 - z)^{-y/2} \exp\Big(-\frac{yz^2}{4} - yz + \frac{yz}{2(1-z)}\Big). \tag{8.10}$$

Identity (8.9) generalises [14, Eq. (6.3)], whereas (8.10) generalises [14, Eq. (6.4)].

[8] Note however, that the formula given in [7] for $s = 3$ is erroneous.

## 9. Simple commutative monoids in species

In this final section, we address the natural question: *'Can one characterise all possible composition operators in $r$-sort species?'* In particular: how far can the composition operator $\boldsymbol{\eta}$ of our theory differ from the standard operation for the species of sets of combinatorial structures given by forming the disjoint union, of which Example 1 in Section 7 is a prototypical example? We do not have an answer in general. However, while addressing this question, we also clarify the relation of our work to the theory developed by Menni in [21].

Already Joyal pointed out in [16, Sec. 7.1] that ($r$-sort) species are endowed with the structure of a *symmetric monoidal category*. The main objects in Menni's theory [21] are *simple commutative monoids* in an *arbitrary* symmetric monoidal category. He defines the notion of a *decomposition* as a simple commutative monoid that satisfies a certain pullback condition (see [21, Def. 2.1]), and, in the case where the symmetric monoidal category that we start with is the category of ($r$-sort) species, he shows that a decomposition is equivalent with a composition operator as defined in Section 2. He defines an exponential principle in this general setting, and he proves this principle to hold for a large family of symmetric monoidal categories that includes $r$-sort species (see [21, Prop. 1.4, Ex. 3.2, and Ex. 3.5 with $I = 1 + 1 + \cdots + 1$, the summand 1 appearing $r$ times in the sum]). He then moves on to characterise decompositions in symmetric monoidal categories (see [21, Sec. 2.6]).

We shall next discuss the notions mentioned in the previous two paragraphs in some more detail, in order to provide a better feeling of what Menni's theory is about. Subsequently, we shall make Menni's characterisation of decompositions explicit for the case of $r$-sort species, that is, of composition operators in the sense of Section 2 that define the structure of a simple commutative monoid in $r$-sort species. We shall see that the latter is equivalent to the condition of the composition operator being "pointwise associative" and "pointwise commutative" (see (9.4) and (9.5)). For the sake of being self-contained, and for including the case of weighted species as well (weights not being addressed in [21], which however could be built in without too much effort), we provide an independent proof. We close this section by exhibiting a large family of examples of composition operators (see Theorem 23), generalising Example 3, that are *not* pointwise associative or commutative and, thus, do not define the structure of a commutative monoid. These examples are therefore not included in Menni's theory [21], which shows that our axiomatic set-up is wider than Menni's theory in the special case of $r$-sort species.

Let $\mathrm{Sp}_r$ denote the category of $r$-sort species. Given two species $F$ and $G$ in $\mathrm{Ob}(\mathrm{Sp}_r)$ and $\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$, we define their product $*$ (a special case of the more general concept of Day convolution; cf. [15] and [22, Sec. 3.1]) by

$$(F * G)[\boldsymbol{\Omega}] := \coprod_{\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2 = \boldsymbol{\Omega}} F[\boldsymbol{\Omega}_1] \times G[\boldsymbol{\Omega}_2],$$

with the obvious morphisms. In order to discuss monoids in species, we need to introduce the "unit" species $\mathbf{1}$ by

$$\mathbf{1}[\boldsymbol{\Omega}] := \begin{cases} \{1\} & \text{if } \boldsymbol{\Omega} = \emptyset, \\ \emptyset & \text{if } \boldsymbol{\Omega} \neq \emptyset. \end{cases}$$

Here, 1 denotes a "canonical" element, and the morphisms are the obvious ones. Then the triple $(\mathrm{Sp}_r, *, \mathbf{1})$ forms a *symmetric monoidal category* (see [16, Sec. 7.1]). We refer the reader to [17] (see also [19]) for the precise definition of a symmetric monoidal category. For our purposes, it suffices to say that, in the case of $r$-sort species, this involves the natural transformation(s)

$$\alpha = \alpha_{F,G,H} : F * (G * H) \to (F * G) * H$$

given by

$$\alpha : (x, (y, z)) \to ((x, y), z), \quad \text{for } x \in F[\mathbf{\Omega}_1],\ y \in G[\mathbf{\Omega}_2],\ z \in H[\mathbf{\Omega}_3],$$

where $F, G, H \in \mathrm{Ob}(\mathrm{Sp}_r)$, and where $\mathbf{\Omega}_1, \mathbf{\Omega}_2, \mathbf{\Omega}_3 \in \mathrm{Ob}(\mathbf{Set}^r)$ are pairwise disjoint, the natural transformation(s)

$$\beta = \beta_{F,G} : F * G \to G * F$$

given by

$$\beta : (x, y) \to (y, x), \quad \text{for } x \in F[\mathbf{\Omega}_1],\ y \in G[\mathbf{\Omega}_2],$$

the natural transformation(s)

$$\lambda = \lambda_F : \mathbf{1} * F \to F$$

given by

$$\lambda : (1, x) \to x, \quad \text{for } x \in F[\mathbf{\Omega}_1],$$

and the natural transformation(s)

$$\rho = \rho_F : F * \mathbf{1} \to F$$

given by

$$\rho = \rho_F : (x, 1) \to x, \quad \text{for } x \in F[\mathbf{\Omega}_1],$$

so that — roughly speaking — all association and commutation laws that one may think of are satisfied.

A *monoid* in the symmetric monoidal category $\mathrm{Sp}_r$ is by definition a triple $(F, *, \mathbf{1})$, where $F \in \mathrm{Ob}(\mathrm{Sp}_r)$, such that there are natural transformations $\mu : F * F \to F$ and $\nu : \mathbf{1} \to F$ such that the diagrams

$$\begin{tikzcd}
F * (F * F) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id} * \mu"'] & (F * F) * F \arrow[r, "\mu * \mathrm{id}"] & F * F \arrow[d, "\mu"] \\
F * F \arrow[rr, "\mu"] & & F
\end{tikzcd} \tag{9.1}$$

and

$$\begin{tikzcd}
\mathbf{1} * F \arrow[r, "\nu * \mathrm{id}"] \arrow[dr, "\lambda"'] & F * F \arrow[d, "\mu"] & F * \mathbf{1} \arrow[l, "\mathrm{id} * \nu"'] \arrow[dl, "\rho"] \\
& F &
\end{tikzcd} \tag{9.2}$$

commute. A monoid $(F, *, \mathbf{1})$ is *commutative* if the diagram

$$\begin{tikzcd} F * F \arrow[rr, "\beta"] \arrow[dr, "\mu"'] & & F * F \arrow[dl, "\mu"] \\ & F & \end{tikzcd} \tag{9.3}$$

commutes. It is called *simple* if the natural transformation $\nu : \mathbf{1} \to F$ is unique.

As Menni explains in [21, Example 3.2], a natural transformation $\boldsymbol{\eta}$ from $F \times F$ to $F \circ \amalg$ as in Section 2 induces a natural transformation $\mu : F * F \to F$ by

$$F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2] \overset{\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}}{\longrightarrow} F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2] \overset{\text{id}}{\longrightarrow} F[\boldsymbol{\Omega}]$$

(where, as usual, $\boldsymbol{\Omega} = \boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2$ is a partition of $\boldsymbol{\Omega}$) and by the universality property of coproducts. Conversely, a natural transformation $\mu : F * F \to F$ induces a natural transformation $\boldsymbol{\eta}$ from $F \times F$ to $F \circ \amalg$ by

$$F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2] \longrightarrow (F * F)[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2] \overset{\mu}{\longrightarrow} F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2]$$

In this precise sense, natural transformations $\mu : F*F \to F$ and natural transformations $\boldsymbol{\eta} : F \times F \to F \circ \amalg$ are equivalent notions. Under this equivalence, commutativity of the diagrams (9.1) and (9.3) is equivalent to the composition operator $\boldsymbol{\eta}$ being *pointwise associative* and *pointwise commutative*, respectively. Here, we say that $\boldsymbol{\eta}$ is pointwise associative if, for all pairwise disjoint $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \boldsymbol{\Omega}_3 \in \mathrm{Ob}(\mathbf{Set}^r)$, and all elements $x_1 \in F[\boldsymbol{\Omega}_1]$, $x_2 \in F[\boldsymbol{\Omega}_2]$, $x_3 \in F[\boldsymbol{\Omega}_3]$, we have

$$\eta_{(\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2, \boldsymbol{\Omega}_3)}\Big(\big(\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}\big((x_1,x_2)\big), x_3\big)\Big) = \eta_{(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_3)}\Big(\big(x_1, \eta_{(\boldsymbol{\Omega}_2,\boldsymbol{\Omega}_3)}\big((x_2,x_3)\big)\big)\Big), \tag{9.4}$$

and we say that $\boldsymbol{\eta}$ is pointwise commutative if, for all $(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_r)$, and all elements $x_1 \in F[\boldsymbol{\Omega}_1]$, $x_2 \in F[\boldsymbol{\Omega}_2]$, we have

$$\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}\big((x_1,x_2)\big) = \eta_{(\boldsymbol{\Omega}_2,\boldsymbol{\Omega}_1)}\big((x_2,x_1)\big). \tag{9.5}$$

The $\boldsymbol{\eta}$-maps in Examples 1 and 2 are instances of pointwise associative and commutative composition operators, while the composition operator in Example 3 is neither pointwise associative nor pointwise commutative. The notion of pointwise commutativity and associativity should *not* be confused with the commutativity and the 3-associativity proved in Lemmas 6 and 7, respectively, which are (in general) strictly weaker assertions. In particular, if $(F, *, \mathbf{1})$ is a monoid, then all "permutabilities" in Lemmas 6, 7, 9, 11, 15 come for free since they hold already on a functorial level, but, as Theorem 23 shows, the converse is not true; that is, these "permutabilities" do *not* guarantee that $(F, *, \mathbf{1})$ forms a monoid.

Finally, it is easy to see that there is a unique natural transformation $\nu : \mathbf{1} \to F$ if and only if $|F[\emptyset]| = 1$, a condition automatically satisfied by a composition operator (see Lemma 5), and it is also easy to see that the diagram (9.2) always commutes.

To summarise the above discussion: the notion of $(F, *, \mathbf{1})$ being a simple commutative monoid is equivalent to the corresponding composition operator $\boldsymbol{\eta}$ being pointwise associative and commutative.

Now that we have discussed the precise relationship between the theory in [21] (specialised to $r$-sort species) and our setting laid down in Section 2, we want to make Menni's characterisation of simple commutative monoids in the case of $r$-sort species, that is — in our language — of pointwise and associative composition operators, explicit. In order to do so, we need two preparatory results. Recall that a *species isomorphism* between two $r$-sort species $F_1$ and $F_2$ is a collection of maps $\boldsymbol{\varphi} = (\varphi_{\boldsymbol{\Omega}})_{\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)}$, where, for each $\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$,

$$\varphi_{\boldsymbol{\Omega}} : F_1[\boldsymbol{\Omega}] \to F_2[\boldsymbol{\Omega}]$$

is a bijection, with the property that, for every morphism $\boldsymbol{f} : \boldsymbol{\Omega} \to \widetilde{\boldsymbol{\Omega}}$ in the category $\mathbf{Set}^r$, the diagram

$$\begin{tikzcd} F_1[\boldsymbol{\Omega}] \arrow[r, "F_1[\boldsymbol{f}]"] \arrow[d, "\varphi_{\boldsymbol{\Omega}}"'] & F_1[\widetilde{\boldsymbol{\Omega}}] \arrow[d, "\varphi_{\widetilde{\boldsymbol{\Omega}}}"] \\ F_2[\boldsymbol{\Omega}] \arrow[r, "F_2[\boldsymbol{f}]"'] & F_2[\widetilde{\boldsymbol{\Omega}}] \end{tikzcd} \tag{9.6}$$

commutes. If $F_1$ carries a weak $\Lambda_1$-weight $\boldsymbol{w}_1$ and $F_2$ carries a weak $\Lambda_2$-weight $\boldsymbol{w}_2$, where, by "weak," we mean that $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ satisfy Axioms (W0) and (W1), but not necessarily (W2) (cf. Section 2), then an isomorphism $\boldsymbol{\varphi} : F_1 \to F_2$ is called *weight-preserving*, if there exists a ring homomorphism $\lambda : \Lambda_1 \to \Lambda_2$ such that the diagram

$$\begin{tikzcd} F_1[\boldsymbol{\Omega}] \arrow[r, "(w_1)_{\boldsymbol{\Omega}}"] \arrow[d, "\varphi_{\boldsymbol{\Omega}}"'] & \Lambda_1 \arrow[d, "\lambda"] \\ F_2[\boldsymbol{\Omega}] \arrow[r, "(w_2)_{\boldsymbol{\Omega}}"'] & \Lambda_2 \end{tikzcd} \tag{9.7}$$

commutes.

The lemma below tells us that, if $F_1$ and $F_2$ are two isomorphic $r$-sort species, where $F_1$ is decomposable with composition operator $\boldsymbol{\eta}_1$, then $\boldsymbol{\eta}_1$ can be lifted to a composition operator for $F_2$, demonstrating that $F_2$ is decomposable as well.

**Lemma 20.** *Let $F_1$ and $F_2$ be two isomorphic $r$-sort species, where $F_1$ is decomposable with composition operator $\boldsymbol{\eta}_1$. Furthermore, let $\boldsymbol{\varphi}$ be an isomorphism between $F_1$ and $F_2$. Then $F_2$ is decomposable, and the family of maps $\boldsymbol{\eta}_2 = ((\eta_2)_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)})_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_r)}$ defined by*

$$(\eta_2)_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}(x_1, x_2) := \varphi_{\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2}\Big((\eta_1)_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}\big((\varphi^{-1}_{\boldsymbol{\Omega}_1}(x_1), \varphi^{-1}_{\boldsymbol{\Omega}_2}(x_2))\big)\Big),$$
$$x_1 \in F_2[\boldsymbol{\Omega}_1],\ x_2 \in F_2[\boldsymbol{\Omega}_2],$$

*is a composition operator for $F_2$.*

*Proof.* We have to show that $\boldsymbol{\eta}_2$ is a natural transformation from $F_2 \times F_2$ to $F_2 \circ \amalg$, and that the pair $(F_2, \boldsymbol{\eta}_2)$ satisfies Axiom (D1). The former follows immediately from the corresponding property for $(F_1, \boldsymbol{\eta}_1)$ and the naturality condition (9.6). In order to verify (D1), we start with the left-hand side of (2.3) for the pair $(F_2, \boldsymbol{\eta}_2)$, suppressing the indices of $\eta_1, \eta_2, \varphi$ for better readability:

$$\begin{aligned}
&\eta_2\big(F_2[\mathbf{\Omega}_1] \times F_2[\mathbf{\Omega}_2]\big) \cap \eta_2\big(F_2[\widetilde{\mathbf{\Omega}}_1] \times F_2[\widetilde{\mathbf{\Omega}}_2]\big)\\
&\quad = \varphi\Big(\eta_1\big(\varphi^{-1}(F_2[\mathbf{\Omega}_1]) \times \varphi^{-1}(F_2[\mathbf{\Omega}_2])\big)\Big) \cap \varphi\Big(\eta_1\big(\varphi^{-1}(F_2[\widetilde{\mathbf{\Omega}}_1]) \times \varphi^{-1}(F_2[\widetilde{\mathbf{\Omega}}_2])\big)\Big)\\
&\quad = \varphi\Big(\eta_1\big(F_1[\mathbf{\Omega}_1] \times F_1[\mathbf{\Omega}_2]\big)\Big) \cap \varphi\Big(\eta_1\big(F_1[\widetilde{\mathbf{\Omega}}_1] \times F_1[\widetilde{\mathbf{\Omega}}_2]\big)\Big)\\
&\quad = \varphi\Big(\eta_1\big(F_1[\mathbf{\Omega}_1] \times F_1[\mathbf{\Omega}_2]\big) \cap \eta_1\big(F_1[\widetilde{\mathbf{\Omega}}_1] \times F_1[\widetilde{\mathbf{\Omega}}_2]\big)\Big).
\end{aligned}$$

Here we have used the injectivity of $\varphi$ to obtain the last line. Now we substitute the right-hand side of (2.3) for the pair $(F_1, \boldsymbol{\eta}_1)$, to obtain

$$\begin{aligned}
&\eta_2\big(F_2[\mathbf{\Omega}_1] \times F_2[\mathbf{\Omega}_2]\big) \cap \eta_2\big(F_2[\widetilde{\mathbf{\Omega}}_1] \times F_2[\widetilde{\mathbf{\Omega}}_2]\big)\\
&\qquad\qquad = \varphi\Big(\eta_1\big(\eta_1\big(F_1[\mathbf{\Omega}_{11}] \times F_1[\mathbf{\Omega}_{12}]\big) \times \eta_1\big(F_1[\mathbf{\Omega}_{21}] \times F_1[\mathbf{\Omega}_{22}]\big)\big)\Big),
\end{aligned}$$

where $\mathbf{\Omega}_{ij} := \mathbf{\Omega}_i \cap \widetilde{\mathbf{\Omega}}_j$ for $i, j \in \{1, 2\}$. Using $F_1[\mathbf{\Omega}_{ij}] = \varphi^{-1}(F_2[\mathbf{\Omega}_{ij}])$ at each possible place, and inserting $\mathrm{id} = \varphi^{-1} \circ \varphi$ at two places, we arrive at

$$\begin{aligned}
&\eta_2\big(F_2[\mathbf{\Omega}_1]\times F_2[\mathbf{\Omega}_2]\big) \cap \eta_2\big(F_2[\widetilde{\mathbf{\Omega}}_1] \times F_2[\widetilde{\mathbf{\Omega}}_2]\big)\\
&\qquad = \varphi\Big(\eta_1\big(\varphi^{-1}\big(\varphi\big(\eta_1\big(\varphi^{-1}(F_2[\mathbf{\Omega}_{11}]) \times \varphi^{-1}(F_2[\mathbf{\Omega}_{12}])\big)\big)\big)\\
&\qquad\qquad\qquad \times \varphi^{-1}\big(\varphi\big(\eta_1\big(\varphi^{-1}(F_2[\mathbf{\Omega}_{21}]) \times \varphi^{-1}(F_2[\mathbf{\Omega}_{22}])\big)\big)\big)\big)\Big)\\
&\qquad = \eta_2\big(\eta_2\big(F_2[\mathbf{\Omega}_{11}] \times F_2[\mathbf{\Omega}_{12}]\big) \times \eta_2\big(F_2[\mathbf{\Omega}_{21}] \times F_2[\mathbf{\Omega}_{22}]\big)\big),
\end{aligned}$$

which is exactly (2.3) for the pair $(F_2, \boldsymbol{\eta}_2)$. □

The second preparatory result, Proposition 21 below, states that, given a decomposable $r$-sort species $F$ with pointwise associative and commutative composition operator $\boldsymbol{\eta}$, $F$ is isomorphic to $E(F_{\boldsymbol{\eta}})$, where $E(F_{\boldsymbol{\eta}})$ denotes the species of sets of $F_{\boldsymbol{\eta}}$-structures (cf. [6, p. 8] for the definition of the species of sets, $E$, and [6, p. 41] for the definition of composition of species). In rigorous terms, for $\mathbf{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$, the set $E(F_{\boldsymbol{\eta}})[\mathbf{\Omega}]$ can be defined by

$$\begin{aligned}
E(F_{\boldsymbol{\eta}})[\mathbf{\Omega}] := \Big\{&\big\{(x_1, \mathbf{\Omega}_1), \dots, (x_k, \mathbf{\Omega}_k)\big\} : x_i \in F_{\boldsymbol{\eta}}[\mathbf{\Omega}_i],\ i = 1, \dots, k,\\
&\text{for some } k \in \mathbb{N}_0 \text{ and } \mathbf{\Omega}_1 \amalg \cdots \amalg \mathbf{\Omega}_k = \mathbf{\Omega},\ \text{all } \mathbf{\Omega}_i\text{'s being non-empty}\Big\},
\end{aligned}$$

with the obvious notion of induced morphisms. If $F$ carries a weak $\Lambda$-weight $\boldsymbol{w}$, then $\boldsymbol{w}$ can be lifted to a weak $\Lambda$-weight of $E(F_{\boldsymbol{\eta}})$ by setting

$$w_{\mathbf{\Omega}}\Big(\big\{(x_1, \mathbf{\Omega}_1), \dots, (x_k, \mathbf{\Omega}_k)\big\}\Big) := w_{\mathbf{\Omega}_1}(x_1) \cdots w_{\mathbf{\Omega}_k}(x_k).$$

**Proposition 21.** *Let $F$ be a decomposable weighted $r$-sort species with composition operator $\boldsymbol{\eta}$, where $\boldsymbol{\eta}$ is pointwise associative and commutative. Then there exists a weight-preserving isomorphism between $F$ and $E(F_{\boldsymbol{\eta}})$.*

*Proof.* The starting point is the combination of Lemmas 12 and 13. It says that, for each non-empty $\mathbf{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$ and every choice of base point $(\omega, \rho) \in \mathbf{\Omega}$, we have

$$F[\mathbf{\Omega}] = \coprod_{\substack{\mathbf{\Omega}_1 \in \mathrm{Ob}(\mathbf{Set}^r)\\ (\omega,\rho)\in\mathbf{\Omega}_1\subseteq\mathbf{\Omega}}} \eta_{(\mathbf{\Omega}_1,\mathbf{\Omega}-\mathbf{\Omega}_1)}\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times F[\mathbf{\Omega} - \mathbf{\Omega}_1]\big). \tag{9.8}$$

We are now going to construct bijective maps $\psi_{\mathbf{\Omega}} : F[\mathbf{\Omega}] \to E(F_{\boldsymbol{\eta}})[\mathbf{\Omega}]$ by induction on $\|\mathbf{\Omega}\|$, where $\|\,.\,\|$ has been defined in (2.6). For $\mathbf{\Omega} = \boldsymbol{\emptyset}$, we have $|F[\mathbf{\Omega}]| = |E(F_{\boldsymbol{\eta}})[\mathbf{\Omega}]| = 1$ by Lemma 5 respectively the definition of $E(F_{\boldsymbol{\eta}})$, whence the construction of $\psi_{\boldsymbol{\emptyset}}$ is trivial. Henceforth, we shall suppose that $\|\mathbf{\Omega}\| \geq 1$, and we assume that we have constructed maps $\psi_{\widetilde{\mathbf{\Omega}}}$ for all $\widetilde{\mathbf{\Omega}} \in \mathrm{Ob}(\mathbf{Set}^r)$ with $\|\widetilde{\mathbf{\Omega}}\| < N$.

Now let $\|\mathbf{\Omega}\| = N$. Choose a base point $(\omega, \rho) \in \mathbf{\Omega}$, and let $x \in F[\mathbf{\Omega}]$. By (9.8), there is a unique $\mathbf{\Omega}_1$ such that $x \in \eta_{(\mathbf{\Omega}_1,\mathbf{\Omega}-\mathbf{\Omega}_1)}\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times F[\mathbf{\Omega} - \mathbf{\Omega}_1]\big)$ and $(\omega, \rho) \in \mathbf{\Omega}_1$. Let $(y_1, x_1)$ be the uniquely determined pair with $y_1 \in F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1]$ and $x_1 \in F[\mathbf{\Omega} - \mathbf{\Omega}_1]$, such that

$$(y_1, x_1) := \eta^{-1}_{(\mathbf{\Omega}_1,\mathbf{\Omega}-\mathbf{\Omega}_1)}(x). \tag{9.9}$$

By the inductive hypothesis, there exist uniquely determined elements $y_2, \dots, y_k$, for some $k \in \mathbb{N}$ and $y_i \in F_{\boldsymbol{\eta}}[\mathbf{\Omega}_i]$, $i = 2, \dots, k$, with $\mathbf{\Omega}_2 \amalg \cdots \amalg \mathbf{\Omega}_k = \mathbf{\Omega} - \mathbf{\Omega}_1$, such that

$$\psi_{\mathbf{\Omega}-\mathbf{\Omega}_1}(x_1) = \big\{(y_2, \mathbf{\Omega}_2), \dots, (y_k, \mathbf{\Omega}_k)\big\}. \tag{9.10}$$

Define

$$\psi_{\mathbf{\Omega}}(x) := \big\{(y_1, \mathbf{\Omega}_1), (y_2, \mathbf{\Omega}_2), \dots, (y_k, \mathbf{\Omega}_k)\big\}.$$

We claim that this yields a well-defined bijection $\psi_{\mathbf{\Omega}} : F[\mathbf{\Omega}] \to E(F_{\boldsymbol{\eta}})[\mathbf{\Omega}]$. What needs to be checked here first of all is that different choices of base points would always lead to the same result. So, let us suppose, that, by choosing a different base point, we would have obtained

$$\bar{\psi}_{\mathbf{\Omega}}(x) := \big\{(\bar{y}_1, \bar{\mathbf{\Omega}}_1), (\bar{y}_2, \bar{\mathbf{\Omega}}_2), \dots, (\bar{y}_l, \bar{\mathbf{\Omega}}_l)\big\},$$

for some $l$, instead. Since we must have

$$\mathbf{\Omega} = \mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2 \amalg \cdots \mathbf{\Omega}_k = \bar{\mathbf{\Omega}}_1 \amalg \bar{\mathbf{\Omega}}_2 \amalg \cdots \bar{\mathbf{\Omega}}_l,$$

there is a $j$ such that $(w, \rho) \in \bar{\mathbf{\Omega}}_j$. By our inductive construction via (9.9) and (9.10), we have

$$x \in \eta\Big(F_{\boldsymbol{\eta}}[\bar{\mathbf{\Omega}}_1] \times \eta\big(F_{\boldsymbol{\eta}}[\bar{\mathbf{\Omega}}_2] \times \cdots \times \big(F_{\boldsymbol{\eta}}[\bar{\mathbf{\Omega}}_{l-1}] \times F_{\boldsymbol{\eta}}[\bar{\mathbf{\Omega}}_l]\big) \cdots\big)\Big).$$

By Lemma 15 ($m$-permutability for $(F_{\boldsymbol{\eta}}, \boldsymbol{\eta})$), this is equivalent to saying that

$$x \in \eta\Big(F_{\boldsymbol{\eta}}[\bar{\mathbf{\Omega}}_j] \times \eta\big(F_{\boldsymbol{\eta}}[\bar{\mathbf{\Omega}}_{\sigma(2)}] \times \cdots \times \big(F_{\boldsymbol{\eta}}[\bar{\mathbf{\Omega}}_{\sigma(l-1)}] \times F_{\boldsymbol{\eta}}[\bar{\mathbf{\Omega}}_{\sigma(l)}]\big) \cdots\big)\Big), \tag{9.11}$$

where $\sigma(2), \dots, \sigma(l-1), \sigma(l)$ is some permutation of $\{1, \dots, j-1, j+1, \dots, l\}$. If $\bar{\mathbf{\Omega}}_j \neq \mathbf{\Omega}_1$, then (9.9) and (9.11) would contradict the disjointness in (9.8). Hence, we must have $\bar{\mathbf{\Omega}}_j = \mathbf{\Omega}_1$, and, by our assumption that $\boldsymbol{\eta}$ be pointwise associative and commutative, we even must have $\bar{y}_j = y_1$. The inductive hypothesis applied to $\mathbf{\Omega} - \mathbf{\Omega}_1$ then guarantees that, moreover,

$$\{(y_2, \mathbf{\Omega}_1), \dots, (y_k, \mathbf{\Omega}_k)\} = \{(\bar{y}_1, \mathbf{\Omega}_1), \dots, (\bar{y}_{j-1}, \bar{\mathbf{\Omega}}_{j-1}), (\bar{y}_{j+1}, \bar{\mathbf{\Omega}}_{j+1}), \dots, (\bar{y}_l, \mathbf{\Omega}_l)\}.$$

This proves that $\psi_{\mathbf{\Omega}}$ is indeed well-defined.

The facts that each map $\psi_{\mathbf{\Omega}}$ is a bijection, and that the family $\boldsymbol{\psi} = (\psi_{\mathbf{\Omega}})_{\mathbf{\Omega}\in \mathrm{Ob}(\mathbf{Set}^r)}$ is an isomorphism between $F$ and $E(F_{\boldsymbol{\eta}})$, are not hard to verify. The fact that $\boldsymbol{\psi}$ is weight-preserving is obvious from the definition of $\boldsymbol{\psi}$ and Axiom (W2) for $(F, \boldsymbol{\eta}, \boldsymbol{w})$. This completes the proof of the proposition. $\square$

If we combine Lemma 20 and Proposition 21, then we can say exactly how a decomposable $r$-sort species $F$ with pointwise associative and commutative composition operator $\boldsymbol{\eta}$ arises from the composition of the species of sets with the species of $F_{\boldsymbol{\eta}}$-structures ("components"). We should point out here that, clearly, a natural composition operator for $E(F_{\boldsymbol{\eta}})$ is given by

$$(y_1, y_2) \mapsto y_1 \amalg y_2, \quad y_1 \in E(F_{\boldsymbol{\eta}})[\mathbf{\Omega}_1],\ y_2 \in E(F_{\boldsymbol{\eta}})[\mathbf{\Omega}_2],\ \mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2 = \mathbf{\Omega}. \tag{9.12}$$

**Theorem 22.** *Let $F$ be a decomposable $r$-sort species with composition operator $\boldsymbol{\eta}$, where $\boldsymbol{\eta}$ is pointwise associative and commutative. Then, for all $(\mathbf{\Omega}_1, \mathbf{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_r)$, the composition operator $\boldsymbol{\eta}$ can be expressed as follows:*

$$\eta_{(\mathbf{\Omega}_1,\mathbf{\Omega}_2)}(x_1, x_2) = \psi^{-1}_{\mathbf{\Omega}_1\amalg\mathbf{\Omega}_2}\Big(\psi_{\mathbf{\Omega}_1}(x_1) \amalg \psi_{\mathbf{\Omega}_2}(x_2)\Big), \quad x_1 \in F[\mathbf{\Omega}_1],\ x_2 \in F[\mathbf{\Omega}_2], \tag{9.13}$$

*where $\boldsymbol{\psi}$ is the isomorphism between $F$ and $E(F_{\boldsymbol{\eta}})$ constructed in the proof of Proposition 21.*

*Proof.* One combines Lemma 20 with Proposition 21, where the role of the isomorphism $\boldsymbol{\varphi}$ in Lemma 20 is played by the family of maps $\boldsymbol{\psi}^{-1}$ constructed in the proof of Proposition 21. $\square$

In summary, all decomposable $r$-sort species with pointwise associative and commutative composition operator can be constructed from $E(G)$, for some species $G$, equipped with the natural composition operator as given in (9.12) (in the case where $G = F_{\boldsymbol{\eta}}$), by applying a lift in the sense of Lemma 20 via a species isomorphism. In the more general setting of [21], the identification of the "indecomposable" objects $G$ (denoted by $\mathtt{L}(F, \zeta)$ there) is explained in [21, Sec. 2.6], while the isomorphism between $F$ and "composite objects built from $G$" (denoted by $\mathtt{E}G = \mathtt{E}\,\mathtt{L}(F, \zeta)$ there) is constructed in [21, Lemma 2.8].

In order to see how Example 2 in Section 7 fits into the setting of Theorem 22, recall that the isomorphism between $F$ and $E(F_{\boldsymbol{\eta}})$ in that example can be defined by mapping the bipartite graph $b \in F[\mathbf{\Omega}]$ to its complement $b^c$, identifying the connected components (in the classical sense of graph theory) of $b^c$, and forming the set of complements of these connected components (restricted to the set of vertices which a component involves). If this isomorphism is inserted in (9.13), the result is (7.5).

We conclude our paper by pointing out that the construction in Theorem 22 can be "twisted" to produce pointwise non-associative and non-commutative composition operators as well, thereby obtaining a large family of examples that still fit under our theory but not under Menni's.

**Theorem 23.** *Let $G$ be a weighted $r$-sort species, and let $\boldsymbol{g} : G \to G$ be a weight-preserving isomorphism. We extend $\boldsymbol{g}$ to $E(G)$ by setting*

$$g_{\mathbf{\Omega}\amalg\cdots\amalg\mathbf{\Omega}_k}\Big(\big\{(y_1, \mathbf{\Omega}_1), \dots, (y_k, \mathbf{\Omega}_k)\big\}\Big) = \big\{(g_{\mathbf{\Omega}_1}(y_1), \mathbf{\Omega}_1), \dots, (g_{\mathbf{\Omega}_k}(y_k), \mathbf{\Omega}_k)\big\}.$$

*Then the family* $\boldsymbol{\eta} = (\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)})_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)\in \mathrm{Ob}(\mathfrak{D}_r)}$ *of maps defined by*

$$\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}(x_1,x_2) = x_1 \amalg g_{\boldsymbol{\Omega}_2}(x_2), \quad x_1 \in E(G)[\boldsymbol{\Omega}_1],\ x_2 \in E(G)[\boldsymbol{\Omega}_2],$$

*where* $(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_r)$*, is a composition operator for the weighted species* $E(G)$*.*

It is obvious from the definition that the composition operator $\boldsymbol{\eta}$ of Theorem 23 will, in general, be neither pointwise associative nor pointwise commutative and, thus, not fit into the theory in [21]. Example 3 in Section 7 provides a typical example of the above construction, with $G$ given by

$$G[\boldsymbol{\Omega}] = \begin{cases} \{0_{\boldsymbol{\Omega}}, 1_{\boldsymbol{\Omega}}\}, & \text{if } |\boldsymbol{\Omega}| = 1, \\ \{\}, & \text{otherwise,} \end{cases}$$

where $0_{\boldsymbol{\Omega}}$ and $1_{\boldsymbol{\Omega}}$ are the constant functions on $\boldsymbol{\Omega}$ taking the value 0 and 1, respectively, and where the isomorphism $\boldsymbol{g}$ is given by $g_{\boldsymbol{\Omega}}(0_{\boldsymbol{\Omega}}) = 1_{\boldsymbol{\Omega}}$ and $g_{\boldsymbol{\Omega}}(1_{\boldsymbol{\Omega}}) = 0_{\boldsymbol{\Omega}}$ for $|\boldsymbol{\Omega}| = 1$. However, we expect that there are many composition operators $\boldsymbol{\eta}$ not obtainable in this way.

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