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ON SOME ANALOGUES OF DESCENT NUMBERS AND MAJOR INDEX FOR THE HYPEROCTAHEDRAL GROUP

RICCARDO BIAGIOLI AND JIANG ZENG

À la mémoire de Pierre Leroux

Abstract. We give a new description of the flag major index, introduced by Adin and Roichman, by using a major index defined by Reiner. This allows us to establish a connection between an identity of Reiner and some more recent results due to Chow and Gessel. Furthermore we generalize the main identity of Chow and Gessel by computing the four-variate generating series of descents, major index, length, and number of negative entries over Coxeter groups of type B and D.

1. Introduction

It is well known that enumerating permutations by number of descents "des" and major index "maj" yields a remarkable q-analogue for Eulerian polynomials, which satisfies the equation

$$\frac{\sum_{\sigma\in S_n} t^{\mathrm{des}(\sigma)}q^{\mathrm{maj}(\sigma)}}{\prod_{i=0}^{n}(1-tq^i)} = \sum_{k\geq 0}(1+q+\cdots+q^k)^n\, t^k, \tag{1.1}$$

where S_n denotes the group of permutations of $\{1,\dots,n\}$. Although this identity is usually attributed to Carlitz [8], it is actually a special case of a result of MacMahon [12, Volume 2, Chapter 4]. It is clear that (1.1) is equivalent to

$$\sum_{n\geq 0}\frac{\sum_{\sigma\in S_n} t^{\mathrm{des}(\sigma)}q^{\mathrm{maj}(\sigma)}}{\prod_{i=0}^{n}(1-tq^i)}\frac{u^n}{n!} = \sum_{k\geq 0} t^k \exp\,(1+q+\cdots+q^k)u. \tag{1.2}$$

At the end of 1970's, Gessel [11], and Garsia and Gessel [10] gave substantial extensions of (1.2). In a different direction, Reiner [14, 15] generalized Garsia and Gessel's work to the group of signed permutations of $\{1,\dots,n\}$, denoted by B_n. By using Coxeter group theory, he defined some B_n-analogues of descent number, major index, and inversions number, denoted by "d_R", "maj_R", "ℓ_R", respectively, and obtained the identity

$$\sum_{n\geq 0}\frac{\sum_{\beta\in B_n} t^{d_R(\beta)}q^{\mathrm{maj}_R(\beta)}p^{\ell_R(\beta)}a^{\mathrm{neg}(\beta)}}{\prod_{i=0}^{n}(1-tq^i)}\frac{u^n}{[\hat{n}]_{a,p}!} = \sum_{k\geq 0} t^k \hat{e}[u]_{a,p}e[qu]_p\cdots e[q^k u]_p, \tag{1.3}$$

where $\mathrm{neg}(\beta)$ denotes the number of negative entries in the window notation of $\beta\in B_n$. The precise definitions of the above statistics as well as all undefined notation will be given in Section 2.

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Motivated by their work on invariant algebras, Adin and Roichman introduced the flag major index "fmaj" in [1]. This statistic turned out to be the key ingredient of new B_n-analogues of Carlitz identity (see [2, Problem 1.1] on Foata's problem). Two such analogues were given by Adin, Brenti and Roichman [2, Theorem 4.2, Corollary 4.5]. More recently, a third one was proposed by Chow and Gessel [9, Theorem 3.7]. In its equivalent form, their result reads as follows [9, Theorem 3.8]:

$$\sum_{n\geq 0} \frac{\sum_{\beta\in B_n} t^{\operatorname{des}_B(\beta)} q^{\operatorname{fmaj}(\beta)}}{\prod_{i=0}^{n}(1-tq^{2i})} \frac{u^n}{n!} = \sum_{k\geq 0} t^k \exp(1+q+\cdots+q^{2k})u. \tag{1.4}$$

Here "des_B" denotes a B_n-analogue of the descent number, computed with respect to a generating set different from the aforementioned one of Reiner.

Our starting point is the observation that the substitution $q \leftarrow q^2$, $a \leftarrow q^{-1}$, $p \leftarrow 1$, and $u \leftarrow (1+q^{-1})u$ in Reiner's identity (1.3) yields

$$\sum_{n\geq 0} \frac{\sum_{\beta\in B_n} t^{d_R(\beta)} q^{2\operatorname{maj}_R(\beta)-\operatorname{neg}(\beta)}}{\prod_{i=0}^{n}(1-tq^{2i})} \frac{u^n}{n!} = \sum_{k\geq 0} t^k \exp(1+q+\cdots+q^{2k})u. \tag{1.5}$$

This implies immediately that the two pairs $(\mathrm{des_B}, \mathrm{fmaj})$ and $(d_R, 2\operatorname{maj}_R - \operatorname{neg})$ are equidistributed over B_n. Actually our first result (cf. Proposition 3.1) shows that these two pairs of statistics are equal over B_n. Therefore Equations (1.4) and (1.5) are the same. This prompted us to look for refinements of Chow and Gessel's identity by computing the generating function of the four statistics $(\operatorname{des}_B, \operatorname{maj}, \ell_B, \operatorname{neg})$ over B_n. It turns out (cf. Theorem 5.1) that this distribution is different from that of Reiner's four statistics in (1.3). In contrast to the method of Chow and Gessel [9], that uses q-difference calculus and recurrence relations, we adopt a more combinatorial approach inspired by that of Garsia and Gessel [10], and of Reiner [14]. The basic idea is to encode signed sequences by pairs made of a signed permutation and a partition.

Finally we consider the group of even-signed permutations of $\{1,\dots,n\}$, denoted D_n. For the Eulerian polynomials of type D, Brenti [7, Theorem 4.10] proved the identity

$$\frac{\sum_{\gamma\in D_n} t^{\operatorname{des}_D(\gamma)}}{(1-t)^{n+1}} = \sum_{k\geq 0} \{(2k+1)^n - n2^{n-1}[\mathcal{B}_n(k+1)-\mathcal{B}_n]\} t^k, \tag{1.6}$$

where $\mathcal{B}_n(x)$ denotes the nth Bernoulli polynomial, and $\mathcal{B}_n$ the nth Bernoulli number. Motivated by this identity and our results in type B, we compute the generating function of some D_n-analogues of descents, major index, length, and number of negative entries (cf. Theorem 6.3), by using the same encoding technique for the hyperoctahedral group.

2. Preliminaries and notation

In this section we give some definitions, notation and results that will be used in the rest of this work. Let P be a statement: the characteristic function χ of P is defined as $\chi(P) = 1$ if P is true, and $\chi(P) = 0$ otherwise. For $n \in \mathbb{N}$ we let $[n] := \{1, 2, \dots, n\}$ (where $[0] := \emptyset$). Given $n, m \in \mathbb{Z}$, $n \leq m$, we let $[n, m] := \{n, n+1, \dots, m\}$. The cardinality of a set A will be denoted either by $|A|$ or by $\#A$. For $n \in \mathbb{N}$, we let

$$(a;p)_n := \begin{cases} 1, & \text{if } n = 0; \\ (1-a)(1-ap)\cdots(1-ap^{n-1}), & \text{if } n \geq 1. \end{cases}$$

For our study we need notation for p-analogues of integers and factorials. These are defined by the expressions

$$\begin{aligned} [n]_p &:= 1 + p + p^2 + \cdots + p^{n-1}, \\ [n]_p! &:= [n]_p[n-1]_p \cdots [2]_p[1]_p, \\ [\hat{n}]_{a,p}! &:= (-ap;p)_n[n]_p!, \end{aligned}$$

where $[0]_p! = 1$. For $n = n_0 + n_1 + \cdots + n_k$ with $n_0, \ldots, n_k \geq 0$ we define the p-multinomial coefficient by

$$\begin{bmatrix} n \\ n_0, n_1, \ldots, n_k \end{bmatrix}_p := \frac{[n]_p!}{[n_0]_p!\,[n_1]_p! \cdots [n_k]_p!}.$$

Finally,

$$e[u]_p := \sum_{n \geq 0} \frac{u^n}{[n]_p!}, \quad \text{and} \quad \hat{e}[u]_{a,p} := \sum_{n \geq 0} \frac{u^n}{[\hat{n}]_{a,p}!}$$

are the two classical p-analogues of the exponential function.

Recall that the group of *signed permutations* on $[n]$, or the *hyperoctahedral group* of rank n, is denoted by B_n. This is the group of all bijections β of the set $[-n, n] \setminus \{0\}$ onto itself such that

$$\beta(-i) = -\beta(i)$$

for all $i \in [-n, n] \setminus \{0\}$, with composition as the group operation. If $\beta \in B_n$ then we write $\beta = [\beta(1), \ldots, \beta(n)]$ and we call this the *window* notation of β. We denote by $|\beta(i)|$ the absolute value of $\beta(i)$, and by $\mathrm{sgn}(\beta(i))$ its sign. For $\beta \in B_n$ we let

$$\begin{aligned} \mathrm{inv}(\beta) &:= |\{(i,j) \in [n] \times [n] \mid i < j,\ \beta(i) > \beta(j)\}|, \text{ and} \\ \mathrm{neg}(\beta) &:= |\{i \in [n] \mid \beta(i) < 0\}|. \end{aligned}$$

As set of generators for B_n we take $S_B := \{s_1^B, \ldots, s_{n-1}^B, s_0^B\}$ where for $i \in [n-1]$

$$s_i^B := [1, \ldots, i-1, i+1, i, i+2, \ldots, n] \text{ and } s_0^B := [-1, 2, \ldots, n].$$

It is well-known that (B_n, S_B) is a Coxeter system of type B (see e.g., [6, §8.1]). The following characterization of the *length* function $\ell_B(\beta) := \min\{r \mid \beta = s_{i_1}^B \cdots s_{i_r}^B, s_{i_i}^B \in S_B\}$ of B_n with respect to S_B is well-known [6, Proposition 8.1.1]:

$$\ell_B(\beta) = \mathrm{inv}(\beta) - \sum_{\beta(i) < 0} \beta(i).$$

The B*-descent set* of $\beta \in B_n$ is defined by

$$\mathrm{Des}_B(\beta) := \{i \in [0, n-1] \mid \beta(i) > \beta(i+1)\},$$

where $\beta(0) := 0$, and its cardinality is denoted by $\mathrm{des}_B(\beta)$. The inversion number inv and the B-descent set Des_B are computed by using the natural order

$$-n < \cdots < -1 < 0 < 1 < \cdots < n \tag{2.1}$$

on the set $[-n, n]$. As usual the *major index* is defined to be the sum of descents

$$\operatorname{maj}(\beta) = \sum_{i \in \operatorname{Des}_B(\beta)} i,$$

and the *flag-major index* [2, Theorem 4.1] by

$$\operatorname{fmaj}(\beta) := 2\operatorname{maj}(\beta) + \operatorname{neg}(\beta).$$

We should point out that in their first paper [1] Adin and Roichman defined the flag-major index by using a different order. Here we use their second definition which appears in the subsequent paper with Brenti [2], and that uses the natural order (2.1).

Reiner defined descents and length for B_n in the usual geometric way by using root systems [14, §2]. Given an element $\beta \in B_n$, let $\ell_R(\beta)$ denote this length of β and $\operatorname{Des}_R(\beta)$ the set of descents of β. Reiner also gave a combinatorial description of these statistics in another paper [17, §2]. Fix the following order on the set $[-n, n] \setminus \{0\}$, denoted by $<_R$:

$$1 <_R \cdots <_R n <_R -n <_R \cdots <_R -1. \tag{2.2}$$

Write $<$ instead of $<_R$ when they coincide. Denote by $\operatorname{inv}_R(\beta)$ the cardinality of the set of inversions of $\beta \in B_n$ computed with respect to the order (2.2), i.e.,

$$\operatorname{Inv}_R(\beta) := \{(i, j) \in [n] \times [n] \mid i < j,\ \beta(j) <_R \beta(i)\}.$$

Then the following result holds [17, §2].

Proposition 2.1. *For any $\beta \in B_n$ we have*

$$\ell_R(\beta) = \operatorname{inv}_R(\beta) + \sum_{\beta(i)<0} (n + 1 + \beta(i)), \tag{2.3}$$

$$\operatorname{Des}_R(\beta) = \{i \in [1, n] \mid \beta(i + 1) <_R \beta(i)\}, \tag{2.4}$$

where $\beta(n + 1) := n$.

The cardinality of $\operatorname{Des}_R(\beta)$ is denoted by $d_R(\beta)$. Note that Reiner used the notation d and *inv* instead of d_R and ℓ_R. To avoid confusion we denote the major index associated to Reiner's descent set by maj_R, i.e.,

$$\operatorname{maj}_R(\beta) = \sum_{i \in \operatorname{Des}_R(\beta)} i.$$

Remark 2.2. The main difference between the two descent sets is that 0 can be in Des_B but not in Des_R, while n can be in Des_R but not in Des_B. More precisely, $0 \in \operatorname{Des}_B(\beta)$ if and only if $\beta(1) < 0$, while $n \in \operatorname{Des}_R(\beta)$ if and only if $\beta(n) < 0$. Clearly, this difference reflects on the corresponding major indices. Besides, the two length statistics ℓ_B and ℓ_R are different since they are computed with respect to different sets of generators. However they are equidistributed over B_n, as will be explained in Remark 5.3.

Example 2.3. If $\beta = [-3, 1, 5, 2, -4, -6] \in B_5$ then

$$\operatorname{Des}_B(\beta) = \{0, 3, 4, 5\}, \quad \text{and} \quad \operatorname{Des}_R(\beta) = \{1, 3, 5, 6\}.$$

Hence $\operatorname{maj}(\beta) = 12$, while $\operatorname{maj}_R(\beta) = 15$. For the length functions we have that $\ell_B(\beta) = 10 + 13 = 23$, while $\ell_R(\beta) = 7 + 8 = 15$.

We denote by D_n the subgroup of B_n consisting of all the signed permutations having an even number of negative entries in their window notation, more precisely

$$D_n := \{\gamma \in B_n \mid \mathrm{neg}(\gamma) \equiv 0 \ (\mathrm{mod}\ 2)\}.$$

It is usually called the *even-signed permutation group.* As a set of generators for D_n we take $S_D := \{s_0^D, s_1^D, \ldots, s_{n-1}^D\}$ where for $i \in [n-1]$

$$s_i^D := s_i^B \ \text{ and } \ s_0^D := [-2, -1, 3, \ldots, n].$$

The following is a well-known combinatorial way to compute the length, and the descent set of $\gamma \in D_n$ (see, e.g., [6, §8.2]):

$$\begin{aligned} \ell_D(\gamma) &= \ell_B(\gamma) - \mathrm{neg}(\gamma), \ \text{ and} \\ \mathrm{Des}_D(\gamma) &= \{i \in [0, n-1] \mid \gamma(i) > \gamma(i+1)\}, \end{aligned}$$

where $\gamma(0) := -\gamma(2)$. The cardinality of $\mathrm{Des}_D(\gamma)$ will be denoted by $\mathrm{des}_D(\gamma)$.

3. Connection between Reiner's and Chow and Gessel's identities

In the introduction, from Reiner's and Chow and Gessel's identities (1.5) and (1.4), we observed the equidistribution of the two pairs of statistics $(\mathrm{des}_B, \mathrm{fmaj})$ and $(d_R, 2\,\mathrm{maj} - \mathrm{neg})$ over B_n. In this section we show that these two pairs are in fact identical. Therefore, the two identities are the same.

Proposition 3.1. *For any $\beta \in B_n$ we have*

$$d_R(\beta) = \mathrm{des}_B(\beta), \tag{3.1}$$

$$\mathrm{maj}_R(\beta) = \mathrm{maj}(\beta) + \mathrm{neg}(\beta). \tag{3.2}$$

Proof. We first prove that $d_R(\beta) = \mathrm{des}_B(\beta)$ for every $\beta \in B_n$. Clearly it suffices to restrict our attention to the case of one descent.

a) If $i \in \mathrm{Des}_R(\beta)$ is such that $\beta(i) > \beta(i+1) > 0$ or $0 > \beta(i) > \beta(i+1)$, then clearly $i \neq n$ and $i \in \mathrm{Des}_B(\beta)$.
b) If $i \in \mathrm{Des}_R(\beta)$ is such that $\beta(i) < 0$ and $\beta(i+1) > 0$, then there exists a k such that

$$0 \leq k \leq i-1, \quad 0 < \beta(1) < \cdots < \beta(k) \quad \text{and} \quad \beta(k+1) < \cdots < \beta(i) < 0.$$

If $k \geq 1$ then $k \in \mathrm{Des}_B(\beta)$; if $k = 0$ then $\beta(1) < 0$ and so $0 \in \mathrm{Des}_B(\beta)$.

Now we consider the major indices. It suffices to check the situation of one descent, since the general case follows easily by induction. In the first following three cases we let $i \in \mathrm{Des}_R(\beta)$ be the only descent of β in $[1, n-1]$.

1) Suppose that $\beta(i) > \beta(i+1) > 0$. Then $0 < \beta(1) < \cdots < \beta(i)$, and there are h negative entries $(0 \leq h \leq n-i-1)$ such that

$$0 < \beta(i+1) < \cdots < \beta(n-h) \quad \text{and} \quad \beta(n-h+1) < \cdots < \beta(n) < 0.$$

If $h = 0$, then $\mathrm{maj}_R(\beta) - \mathrm{neg}(\beta) = i = \mathrm{maj}(\beta)$. If $h > 0$, then $\mathrm{Des}_R(\beta) = \{i, n\}$ while $\mathrm{Des}_B(\beta) = \{i, n-h\}$. Thus $\mathrm{maj}_R(\beta) - \mathrm{neg}(\beta) = (i+n) - h = \mathrm{maj}(\beta)$.

2) Suppose that $0 > \beta(i) > \beta(i+1)$. Then $\beta(i+1) < \cdots < \beta(n) < 0$, and there are k positive entries ($0 \le k \le i-1$) such that
$$0 < \beta(1) < \cdots < \beta(k) \quad \text{and} \quad \beta(k+1) < \cdots < \beta(i) < 0.$$
In this case $\mathrm{maj}_R(\beta) - \mathrm{neg}(\beta) = (i+n) - (n-k) = k+i = \mathrm{maj}(\beta)$.
3) Suppose that $\beta(i) < 0$ and $\beta(i+1) > 0$. Then there exist $0 \le k \le i-1$ and $0 \le h \le n-i-2$ such that
$$0 < \beta(1) < \cdots < \beta(k) \quad \text{and} \quad \beta(k+1) < \cdots < \beta(i) < 0, \quad \text{and}$$
$$0 < \beta(i+1) < \cdots < \beta(n-h) \quad \text{and} \quad \beta(n-h+1) < \cdots < \beta(n) < 0.$$
If $h, k > 0$, we obtain $\mathrm{maj}_R(\beta) - \mathrm{neg}(\beta) = (i+n) - (h+i-k) = k+(n-h) = \mathrm{maj}(\beta)$. The cases $h = 0$ and $k = 0$ are similar.
4) If $n \in \mathrm{Des}_R(\beta)$ is the only descent then there exists a k ($0 \le k \le n-1$) such that
$$0 < \beta(1) < \cdots < \beta(k) \quad \text{and} \quad \beta(k+1) < \cdots < \beta(n) < 0.$$
Hence $\mathrm{maj}_R(\beta) - \mathrm{neg}(\beta) = n - (n-k) = k = \mathrm{maj}(\beta)$. □

Remark 3.2. The first equality $d_R(\beta) = \mathrm{des}_B(\beta)$ can also be derived by using their geometric interpretations.

The next new description of the flag major index easily follows from Proposition 3.1.

Corollary 3.3. *For every $\beta \in B_n$ we have*
$$\mathrm{fmaj}(\beta) = 2\,\mathrm{maj}_R(\beta) - \mathrm{neg}(\beta).$$

4. Encoding signed sequences

In this section we introduce a procedure encoding signed sequences by signed permutations and partitions, which will be used in the following sections. The basic idea can be found in Garsia and Gessel [10] and Reiner [14].

Let $\mathcal{P}_n$ be the set of non decreasing sequences of nonnegative integers $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, i.e., partitions of length less than or equal to n.

Definition 4.1. Given a sequence $f = (f_1, \ldots, f_n) \in \mathbb{Z}^n$ we define a pair $(\beta, \lambda) \in B_n \times \mathcal{P}_n$, where:

a) β is the unique signed permutation satisfying the following three conditions:
   1) $|f_{|\beta(1)|}| \le |f_{|\beta(2)|}| \le \cdots \le |f_{|\beta(n)|}|$,
   2) $\mathrm{sgn}(\beta(i)) := \mathrm{sgn}(f_{|\beta(i)|})$,
   3) If $|f_{|\beta(i)|}| = |f_{|\beta(i+1)|}|$, then $\beta(i) < \beta(i+1)$;

b) $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the partition with
$$\lambda_i := |f_{|\beta(i)|}| - |\{j \in \mathrm{Des}_B(\beta) \mid j \le i-1\}| \quad \text{for} \quad 1 \le i \le n.$$

Let $\pi(f) := \beta$ and $\lambda(f) := \lambda$.

Remark 4.2. The above sequence λ is clearly a partition since for all $i \in [n]$, $\lambda_i \ge 0$ and
$$\lambda_{i+1} - \lambda_i = |f_{|\beta(i+1)|}| - |f_{|\beta(i)|}| - \chi(i \in \mathrm{Des}_B(\pi)) \ge 0 \quad \text{for } i \in [n-1].$$
Moreover, note that if $i \in \mathrm{Des}_B(\beta)$ then $|f_{|\beta(i)|}| < |f_{|\beta(i+1)|}|$, and $0 \in \mathrm{Des}_B(\beta)$ if and only if $f_{|\beta(1)|} < 0$.

We introduce the following statistics on the set of signed sequences $\mathbb{Z}^n$.

Definition 4.3. For $f = (f_1, \dots, f_n) \in \mathbb{Z}^n$ we let

$$\max(f) := \max\{|f_i|\}, \qquad |f| := \sum_{i=1}^{n} |f_i|,$$
$$\ell_B(f) := \ell_B(\pi(f)), \qquad \operatorname{neg}(f) := |\{i \in [n] \mid f_i < 0\}|.$$

Proposition 4.4. *The map $\psi : \mathbb{Z}^n \to B_n \times \mathcal{P}_n$ defined by (see Definition 4.1)*

$$f \mapsto (\pi(f), \lambda(f))$$

is a bijection. Moreover, if we let $\beta := \pi(f)$ and $\lambda := \lambda(f)$ then

$$\max(f) = \max(\lambda) + \operatorname{des}_B(\beta), \tag{4.1}$$
$$|f| = |\lambda| + n \operatorname{des}_B(\beta) - \operatorname{maj}(\beta). \tag{4.2}$$

Proof. To see that ψ is a bijection we construct its inverse as follows. To each $(\beta, \lambda) \in B_n \times \mathcal{P}_n$ associate the partition $\mu = (\mu_1, \dots, \mu_n)$ where

$$\mu_i := \lambda_i + |\{j \in \operatorname{Des}_B(\beta) \mid j \le i - 1\}|,$$

and define the sequence $f = (f_1, \dots, f_n) \in \mathbb{Z}^n$ by letting

$$f_i := \operatorname{sgn}(\beta^{-1}(i))\, \mu_{|\beta^{-1}(i)|} \quad \text{for} \quad 1 \le i \le n.$$

It is easy to see that $\psi(f) = (\beta, \lambda)$. Since λ is a partition, Equations (4.1) and (4.2) follow from

$$\sum_{i=1}^{n} |\{j \in \operatorname{Des}_B(\beta) | j \le i - 1\}| = \sum_{j \in \operatorname{Des}_B(\pi)} (n - j) = n \operatorname{des}_B(\beta) - \operatorname{maj}(\beta). \qquad \square$$

Example 4.5. If $f = (-4, 4, 1, -3, 6, 3, -4)$ then $(|f_{|\beta(1)|}|, \dots, |f_{|\beta(7)|}|) = (1, 3, 3, 4, 4, 4, 6)$, so

$$\pi(f) := \beta = (3, -4, 6, -7, -1, 2, 5) \in B_7.$$

Moreover $\operatorname{Des}_B(\beta) = \{1, 3\}$, so $\lambda = (1, 2, 2, 2, 2, 2, 4)$. We obtain $\max(f) = 6$ and $\operatorname{des}_B(\beta) = 2$. Hence the identity (4.2) reads $25 = 15 + 14 - 4$. Conversely, given the pair

$$([5, -3, 1, 2, -4], (0, 2, 2, 3, 3)) \in B_5 \times \mathcal{P}_5,$$

we find $\mu = (0, 3, 3, 4, 5)$ and $f = (3, 4, -3, -5, 0)$.

5. Main identity

In this section we compute the generating function of $(\operatorname{des}_B, \operatorname{maj}, \ell_B, \operatorname{neg})$ over B_n. As special instances, we recover several known identities listed as remarks after the proof.

Theorem 5.1. *We have*

$$\sum_{n \ge 0} \frac{u^n}{(t;q)_{n+1}[\hat{n}]_{a,p}!} \sum_{\beta \in B_n} t^{\operatorname{des}_B(\beta)} q^{\operatorname{maj}(\beta)} p^{\ell_B(\beta)} a^{\operatorname{neg}(\beta)} = \sum_{k \ge 0} t^k \prod_{j=0}^{k-1} e[q^j u]_p \cdot \hat{e}[q^k u]_{a,p}. \tag{5.1}$$

Proof. The proof consists in computing in two different ways the series

$$\sum_{f\in\mathbb{Z}^n} t^{\max(f)} q^{\max(f)\cdot n-|f|} p^{\ell_B(f)} a^{\mathrm{neg}(f)}.$$

We need the following preliminary result that can be proved as in [14, Lemma 3.1]. For $\underline{n}=(n_0,n_1,\dots,n_k)\in\mathbb{N}^{k+1}$ a composition of n, i.e., $n=n_0+\cdots+n_k$, we let

$$\mathbb{Z}^n(\underline{n}) := \{f\in\mathbb{Z}^n \mid \#\{i:|f_i|=j\}=n_j\}.$$

Then

$$\sum_{f\in\mathbb{Z}^n(\underline{n})} p^{\ell_B(f)} a^{\mathrm{neg}(f)} = \begin{bmatrix} n \\ n_0,n_1,\dots,n_k\end{bmatrix}_p \frac{(-ap;p)_n}{(-ap;p)_{n_0}}. \tag{5.2}$$

A version of (5.2) for the more general case of wreath products is given in [5, Lemma 4.2].

We can now start our computation:

$$\begin{aligned}
\sum_{f\in\mathbb{Z}^n|\max(f)\le k} q^{k\cdot n-|f|} p^{\ell_B(f)} a^{\mathrm{neg}(f)} &= \sum_{n_0+\cdots+n_k=n} q^{k\sum_i n_i-\sum_i in_i} \sum_{f\in\mathbb{Z}^n(\underline{n})} p^{\ell_B(f)} a^{\mathrm{neg}(f)}\\
&= \sum_{n_0+\cdots+n_k=n} \begin{bmatrix} n \\ n_0,n_1,\dots,n_k\end{bmatrix}_p \frac{(-ap;p)_n}{(-ap;p)_{n_0}} q^{\sum_i n_i(k-i)}\\
&= [\hat{n}]_{a,p}!\cdot\langle u^n\rangle \left(e[u]_p e[qu]_p\cdots e[q^{k-1}u]_p \hat{e}[q^k u]_{a,p}\right),
\end{aligned}$$

where the notation $\langle u^n\rangle f(u)$ stands for the coefficient of u^n in $f(u)$.

Using the formula

$$(1-t)\sum_{k\ge0} a_{\le k}t^k = \sum_{k\ge0}(a_{\le k}-a_{\le k-1})t^k = \sum_{k\ge0} a_k t^k, \tag{5.3}$$

where $a_{\le k}=a_0+\cdots+a_k$, we derive immediately

$$\sum_{f\in\mathbb{Z}^n} t^{\max(f)} q^{\max(f)\cdot n-|f|} p^{\ell_B(f)} a^{\mathrm{neg}(f)} = (1-tq^n)\sum_{k\ge0} t^k \sum_{f\in\mathbb{Z}^n|\max(f)\le k} q^{kn-|f|} p^{\ell_B(f)} a^{\mathrm{neg}(f)}.$$

Therefore

$$\sum_{n\ge0} \frac{u^n}{[\hat{n}]_{a,p}!(1-tq^n)} \sum_{f\in\mathbb{Z}^n} t^{\max(f)} q^{\max(f)\cdot n-|f|} p^{\ell_B(f)} a^{\mathrm{neg}(f)} = \sum_{k\ge0} t^k \prod_{j=0}^{k-1} e[q^j u]_p \cdot \hat{e}[q^k u]_{a,p}. \tag{5.4}$$

On the other hand, from the bijection ψ, and Equations (4.1) and (4.2) in Proposition 4.4, it follows that

$$\begin{aligned}
\sum_{f\in\mathbb{Z}^n|\pi(f)=\beta} t^{\max(f)} q^{|f|} &= \sum_{\lambda} t^{\max(\lambda)+\mathrm{des}_B(\beta)} q^{|\lambda|+n\,\mathrm{des}_B(\beta)-\mathrm{maj}(\beta)}\\
&= \frac{t^{\mathrm{des}_B(\beta)} q^{\sum_{i\in \mathrm{Des}_B(\beta)}(n-i)}}{(tq;q)_n}.
\end{aligned} \tag{5.5}$$

Replacing q by q^{-1} and t by tq^n in (5.5) we get

$$\sum_{f\in\mathbb{Z}^n|\pi(f)=\beta} t^{\max(f)} q^{\max(f)\cdot n-|f|} = \frac{t^{\mathrm{des}_B(\beta)} q^{\mathrm{maj}(\beta)}}{(t;q)_n}. \tag{5.6}$$

Hence

$$\begin{aligned}\sum_{f\in\mathbb{Z}^n} t^{\max(f)} q^{\max(f)\cdot n-|f|} p^{\ell_B(f)} a^{\mathrm{neg}(f)} &= \sum_{\beta\in B_n} p^{\ell_B(\beta)} a^{\mathrm{neg}(\beta)} \sum_{f\in\mathbb{Z}^n|\pi(f)=\beta} t^{\max(f)} q^{\max(f)\cdot n-|f|}\\ &= \sum_{\beta\in B_n} \frac{t^{\mathrm{des}_B(\beta)} q^{\mathrm{maj}(\beta)} p^{\ell_B(\beta)} a^{\mathrm{neg}(\beta)}}{(t;q)_n}. \end{aligned} \tag{5.7}$$

By comparing (5.4) and (5.7) the result follows. □

Remark 5.2 (Chow and Gessel's identities)**.** By letting $p = 1$, and substituting $q \leftarrow q^2$, $a \leftarrow q$, and $u \leftarrow (1 + q)u$ in (5.1) we obtain Chow and Gessel's formula (1.4). Letting $p = 1$ and replacing u by $(1+a)u$ in (5.1), then extracting the coefficient of $u^n/n!$ yields another result of Chow and Gessel [9, Equation (26)]

$$\frac{\sum_{\beta\in B_n} t^{\mathrm{des}_B(\beta)} q^{\mathrm{maj}(\beta)} a^{\mathrm{neg}(\beta)}}{(t;q)_{n+1}} = \sum_{k\geq 0} ([k+1]_q + a[k]_q)^n t^k. \tag{5.8}$$

This result also follows from (1.3) and Proposition 3.1. Indeed, substituting $a \leftarrow aq^{-1}$, $p \leftarrow 1$ and $u \leftarrow (1 + aq^{-1})u$ in (1.3) we get

$$\sum_{n\geq 0} \frac{\sum_{\beta\in B_n} t^{d(\beta)} q^{\mathrm{maj}_R(\beta)-\mathrm{neg}(\beta)} a^{\mathrm{neg}(\beta)}}{(t;q)_{n+1}} \frac{u^n}{n!} = \sum_{k\geq 0} t^k \exp\left([k+1]_q + a[k]_q\right) u. \tag{5.9}$$

In view of Proposition 3.1, Equation (5.8) follows then by extracting the coefficient of $u^n/n!$ in (5.9).

Since the right-hand side of (5.8) is invariant under the substitution $q \to 1/q, a \to a/q$ and $t \to tq^n$ we obtain immediately the following result:

$$\sum_{\beta\in B_n} t^{\mathrm{des}_B(\beta)} q^{\mathrm{maj}(\beta)} a^{\mathrm{neg}(\beta)} = \sum_{\beta\in B_n} t^{\mathrm{des}_B(\beta)} q^{n\,\mathrm{des}_B(\beta)-\mathrm{maj}(\beta)-\mathrm{neg}(\beta)} a^{\mathrm{neg}(\beta)}.$$

Remark 5.3 (Reiner and Brenti's identities)**.** Comparing the right-hand sides of (1.3) and (5.1) we conclude that the distributions of the two quadruples of statistics are different. However, if we set $q = 1$ in (5.1) we obtain

$$\sum_{n\geq 0} \frac{u^n}{(1-t)^{n+1}} \frac{\sum_{\beta\in B_n} t^{\mathrm{des}_B(\beta)} p^{\ell_B(\beta)} a^{\mathrm{neg}(\beta)}}{[\hat{n}]_{a,p}!} = \frac{1}{1-te[u]_p} \cdot \hat{e}[u]_{a,p}, \tag{5.10}$$

and comparing with Reiner's equation (1.3) with $q = 1$ we see that $(d_R, \ell_R, \mathrm{neg})$ and $(\mathrm{des}_B, \ell_B, \mathrm{neg})$ are equidistributed over B_n. By letting $p = 1$ in (5.10) we recover a formula of Brenti [7, (14)]:

$$\sum_{n\geq 0}\sum_{\beta\in B_n} t^{\mathrm{des}_B(\beta)} a^{\mathrm{neg}(\beta)} \frac{u^n}{n!} = \frac{(1-t)e^{u(1-t)}}{1-te^{(1+a)(1-t)u}}.$$

Remark 5.4 (Gessel and Roselle's identity for B_n)**.** To compute the generating function of major index and length we proceed as follows. Setting $a=1$ in equation (5.1) yields

$$\sum_{n\geq 0}\frac{(1-p)^n u^n}{(t;q)_{n+1}(p^2;p^2)_n}\sum_{\beta\in B_n} t^{\mathrm{des}_B(\beta)}q^{\mathrm{maj}(\beta)}p^{\ell_B(\beta)} = \sum_{k\geq 0} t^k e[u]_p e[qu]_p\cdots e[q^{k-1}u]_p\hat{e}[q^k u]_{1,p}. \tag{5.11}$$

By multiplying both sides of (5.11) by $(1-t)$, and then by sending $t\to 1$ we obtain

$$\sum_{n\geq 0}\frac{((1-p)u)^n}{(q;q)_n(p^2;p^2)_n}\sum_{\beta\in B_n} q^{\mathrm{maj}(\beta)}p^{\ell_B(\beta)} = \prod_{i\geq 0} e[q^i u]_p.$$

Replacing u by $u/(1-p)$ and applying q-binomial formula $e[u/(1-p)]_p=\prod_{j\geq 0}\frac{1}{1-p^j u}$ we get the following B_n-analogue of an identity of Gessel and Roselle (see [11, Theorem 8.5] and the historical note after Theorem 4.3 in [3]):

$$\sum_{n\geq 0}\frac{\sum_{\beta\in B_n} q^{\mathrm{maj}(\beta)}p^{\ell_B(\beta)}}{(q;q)_n(p^2;p^2)_n}u^n = \prod_{i,j\geq 0}\frac{1}{1-up^iq^j}. \tag{5.12}$$

We refer the reader to [4, Proposition 5.4] for a different generalization of this identity.

6. The D_n-case.

The aim of this section is to obtain a generating series for the four-variate distribution of descents, major index, length and number of negative entries over the group D_n by using the encoding of Section 4. This time we are unable to get a nice identity as in the B_n-case; this is not so surprising as will be explained in Section 7.

Let $\mathbb{Z}^n_e$ be the subset of sequences in $\mathbb{Z}^n$ with an even number of negative entries. For $f\in\mathbb{Z}^n_e$ Definition 4.3 is still valid and we let

$$\ell_D(f) := \ell_D(\pi(f)),$$

where $\pi(f)\in D_n$ is the even signed permutation defined in Definition 4.1.

For $\underline{n}=(n_0,n_1,\ldots,n_k)$ a composition of n, we define

$$\mathbb{Z}^n_e(\underline{n}) = \{f\in\mathbb{Z}^n_e \mid \#\{i : |f_i|=j\}=n_j\}.$$

We have the following lemma.

Lemma 6.1. *Let $\underline{n}=(n_0,n_1,\ldots,n_k)$ be a composition of n. Then*

$$\sum_{f\in\mathbb{Z}^n_e(\underline{n})} p^{\ell_D(f)}a^{\mathrm{neg}(f)} = \frac{1}{2}\begin{bmatrix} n \\ n_0,n_1,\ldots,n_k\end{bmatrix}_p\cdot\left\{\frac{(a;p)_n}{(a;p)_{n_0}}+\frac{(-a;p)_n}{(-a;p)_{n_0}}\right\}.$$

Proof. It is clear that

$$\sum_{f\in\mathbb{Z}^n_e(\underline{n})} p^{\ell_D(f)}a^{\mathrm{neg}(f)} = \frac{1}{2}\left[\sum_{f\in\mathbb{Z}^n(\underline{n})} p^{\ell_D(f)}a^{\mathrm{neg}(f)} + \sum_{f\in\mathbb{Z}^n(\underline{n})} p^{\ell_D(f)}(-a)^{\mathrm{neg}(f)}\right].$$

By definition we have that $\ell_B(\gamma) = \ell_D(\gamma) + \mathrm{neg}(\gamma)$, so by (5.2) with $a \leftarrow a/p$ we get

$$\sum_{f \in \mathbb{Z}^n(\underline{n})} p^{\ell_D(f)} a^{\mathrm{neg}(f)} = \begin{bmatrix} n \\ n_0, n_1, \dots, n_k \end{bmatrix}_p \cdot \frac{(-a;p)_n}{(-a;p)_{n_0}}.$$

The result follows by replacing a by $-a$ in the above formula. □

In order to obtain our identity, we want to compute the following generating series in two different ways:

$$\sum_{f \in \mathbb{Z}_e^n} t^{\max(f)} q^{\max(f)\cdot n - |f|} p^{\ell_D(f)} a^{\mathrm{neg}(f)}.$$

First of all we have

$$\begin{aligned}
\sum_{f \in \mathbb{Z}_e^n \mid \max(f) \le k} q^{k\cdot n - |f|} p^{\ell_D(f)} a^{\mathrm{neg}(f)} &= \sum_{n_0+\cdots+n_k=n} q^{\sum_i n_i(k-i)} \sum_{f \in \mathbb{Z}_e^n} p^{\ell_D(f)} a^{\mathrm{neg}(f)} \\
= \frac{1}{2} \sum_{n_0+\cdots+n_k=n} \frac{[n]_p!}{[n_0]_p! \cdots [n_k]!_p} q^{\sum_i n_i(k-i)} &\left\{ \frac{(a;p)_n}{(a;p)_{n_0}} + \frac{(-a;p)_n}{(-a,p)_{n_0}} \right\} \\
= \frac{1}{2} [n]_p! \langle u^n \rangle \Big(e[u]_p e[uq]_p \cdots e[uq^{k-1}]_p &\left\{ (a;p)_n \hat{e}[uq^k]_{-a/p,p} + (-a;p)_n \hat{e}[uq^k]_{a/p,p} \right\} \Big).
\end{aligned}$$

Using the formula (5.3) we derive immediately

$$\sum_{f \in \mathbb{Z}_e^n} t^{\max(f)} q^{\max(f)\cdot n - |f|} p^{\ell_D(f)} a^{\mathrm{neg}(f)} = (1 - tq^n) \sum_{k \ge 0} t^k \sum_{f \in \mathbb{Z}_e^n \mid \max(f) \le k} q^{k\cdot n - |f|} p^{\ell_D(f)} a^{\mathrm{neg}(f)}.$$

Therefore we obtain

$$\begin{aligned}
&\frac{\sum_{f \in \mathbb{Z}_e^n} t^{\max(f)} q^{\max(f)\cdot n - |f|} p^{\ell_D(f)} a^{\mathrm{neg}(f)}}{(1 - tq^n)[n]_p!} \qquad (6.1) \\
&= \frac{1}{2} \sum_{k \ge 0} t^k \langle u^n \rangle \left\{ e[u]_p e[uq]_p \cdots e[uq^{k-1}]_p \Big((a;p)_n \hat{e}[uq^k]_{-a/p,p} + (-a;p)_n \hat{e}[uq^k]_{a/p,p} \Big) \right\}.
\end{aligned}$$

On the other hand, we decompose D_n as the disjoint union of the three sets

$$\begin{aligned}
D_n^+ &:= \{\gamma \in D_n \mid 0 \notin \mathrm{Des}_D(\gamma) \text{ and } \gamma(1) < 0\}, \\
D_n^- &:= \{\gamma \in D_n \mid 0 \in \mathrm{Des}_D(\gamma) \text{ and } \gamma(1) > 0\}, \\
D_n^0 &:= D_n \setminus (D_n^+ \cup D_n^-).
\end{aligned}$$

We have the following lemma.

Lemma 6.2. *For $\gamma \in D_n$ we have*

$$\sum_{f \in \mathbb{Z}_e^n \mid \pi(f) = \gamma} t^{\max(f)} q^{\max(f)\cdot n - |f|} = \frac{D_n^\gamma(t,q)}{(t;q)_n},$$

where

$$D_n^\gamma(t,q) := \begin{cases} t^{\mathrm{des}_D(\gamma)+1} q^{\mathrm{maj}(\gamma)}, & \text{if } \gamma \in D_n^+; \\ t^{\mathrm{des}_D(\gamma)-1} q^{\mathrm{maj}(\gamma)}, & \text{if } \gamma \in D_n^-; \\ t^{\mathrm{des}_D(\gamma)} q^{\mathrm{maj}(\gamma)}, & \text{if } \gamma \in D_n^0. \end{cases}$$

Proof. First of all, for any $\gamma \in D_n$ we have $\{f \in \mathbb{Z}_e^n \mid \pi(f) = \gamma\} = \{f \in \mathbb{Z}^n \mid \pi(f) = \gamma\}$. Suppose that $\gamma \in D_n^+$, then $0 \in \mathrm{Des}_B(\gamma)$ since $\gamma(1) < 0$, but $0 \notin \mathrm{Des}_D(\gamma)$. It follows from Equation (5.6) that

$$\sum_{f \in \mathbb{Z}^n \mid \pi(f)=\gamma} t^{\max(f)} q^{\max(f)\cdot n - |f|} = \frac{t^{\mathrm{des}_B(\gamma)} q^{\mathrm{maj}(\gamma)}}{(t;q)_n} = \frac{t^{\mathrm{des}_D(\gamma)+1} q^{\mathrm{maj}(\gamma)}}{(t;q)_n}.$$

The other cases are similar. □

For $\gamma \in D_n$ set

$$w(\gamma) := t^{\mathrm{des}_D(\gamma)} q^{\mathrm{maj}(\gamma)} p^{\ell_D(\gamma)} a^{\mathrm{neg}(\gamma)}.$$

It follows from Lemma 6.2 that

$$\begin{aligned}\sum_{f \in \mathbb{Z}_e^n} t^{\max(f)} q^{\max(f)\cdot n - |f|} p^{\ell_D(f)} a^{\mathrm{neg}(f)} &= \sum_{\gamma \in D_n} p^{\ell_D(\gamma)} a^{\mathrm{neg}(\gamma)} \sum_{f \in \mathbb{Z}_e^n \mid \pi(f)=\gamma} t^{\max(f)} q^{\max(f)\cdot n - |f|} \\ &= \frac{1}{(t;q)_n} \left(\sum_{\gamma \in D_n^0} w(\gamma) + \frac{1}{t} \sum_{\gamma \in D_n^-} w(\gamma) + t \sum_{\gamma \in D_n^+} w(\gamma) \right).\end{aligned}$$

By comparing the above equation with Equation (6.1) we obtain the following identity.

Theorem 6.3.

$$\begin{aligned}&\sum_{\gamma \in D_n^0} \frac{w(\gamma)}{(t;q)_n} + \frac{1}{t} \sum_{\gamma \in D_n^-} \frac{w(\gamma)}{(t;q)_n} + t \sum_{\gamma \in D_n^+} \frac{w(\gamma)}{(t;q)_n} \\ &= \frac{1}{2} \sum_{k \geq 0} t^k \langle u^n \rangle \left\{ e[u]_p e[uq]_p \cdots e[uq^{k-1}]_p \left((a;p)_n \hat{e}[uq^k]_{-a/p,p} + (-a;p)_n \hat{e}[uq^k]_{a/p,p)} \right) \right\}. \quad (6.2)\end{aligned}$$

If $a = 1$, letting $t \to 1$ in the above identity yields a D_n-analogue of Gessel and Roselle's identity (see (5.12)):

$$1 + \sum_{n \geq 1} u^n \frac{\sum_{\gamma \in D_n} q^{\mathrm{maj}(\gamma)} p^{\ell_D(\gamma)}}{(-p;p)_{n-1} (q;q)_{n-1}} = \prod_{i,j \geq 0} \frac{1}{1 - (1-p) u p^i q^j}. \quad (6.3)$$

7. Final Remarks

Note that Brenti [7, Proposition 4.3] computed the generating function of $(\mathrm{des}_D, \mathrm{neg})$, and Reiner [16, Theorem 7] that of (des_D, ℓ_D) over D_n. Their formulas are much more involved than the corresponding B_n-cases, even in the one-variable case, see Equation (1.6). Reiner gave also a method to compute the distribution of $(\mathrm{des}_D, \ell_D, \mathrm{neg})$ over D_n. However, it does not seem that his method allows to include the statistic major index in the computation.

The restriction of the bijection ψ (defined in Proposition 4.4) to $\mathbb{Z}_e^n$ is not a well defined map from $\mathbb{Z}_e^n$ to $D_n \times \mathcal{P}_n$. For example, consider $f = (0, -3, -4) \in \mathbb{Z}_e^3$. Then $\pi(f) = [1, -2, -3] \in D_3$, and $\lambda(f) = (-1, 1, 1)$ which is not a partition. Hence the following question naturally arises.

Question 7.1. Is there a parametrization of the elements of $\mathbb{Z}_e^n$ which reduces the left-hand side of Equation (6.2) to a single sum with a simpler right-hand side ?

Hopefully, the desired equation, for $p = a = 1$, should provide a nice q-analogue of Brenti's identity (1.6). This problem was also raised in [9, §5]. Recently Mendes and Remmel [13] computed some generating series over B_n and D_n, closely related to ours. Unfortunately, their computations do not answer the above question.

In [5] we study the distribution of several statistics over the wreath product of a symmetric group with a cyclic group. In particular, we give an extension of Theorem 5.1.

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Université de Lyon, Université Lyon 1, Institut Camille Jordan, UMR 5208 du CNRS, F-69622, Villeurbanne Cedex, France

E-mail address: `biagioli@math.univ-lyon1.fr, zeng@math.univ-lyon1.fr`