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COUNTING DESCENT PAIRS WITH PRESCRIBED COLORS IN THE COLORED PERMUTATION GROUPS

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Abstract. We define new statistics, (c, d)-*descents*, on the colored permutation groups $\mathbb{Z}_r \wr S_n$ and compute the distribution of these statistics on the elements in these groups. We use some combinatorial approaches, recurrences, and generating functions manipulations to obtain our results.

1. Introduction

A permutation statistic on the symmetric group S_n, or more generally over a group, is a function from the group to the set of nonnegative integers. The study of permutation statistics started with Euler, who considered the number of descents of a permutation. Netto, at the beginning of the last century, considered the number of inversions, and Major Percy MacMahon [6, Vol. I, pp. 135, 186; Vol. II, p. viii], [7] considered the parameter called today major index and the excedance number.

On the symmetric group S_n, a *descent pair* of a permutation $\pi \in S_n$ is a pair $(i, i+1)$ such that $\pi_i > \pi_{i+1}$. The *descent set* of a permutation $\pi = \pi_1\pi_2 \cdots \pi_n \in S_n$, denoted by $\mathrm{Des}(\pi)$, is the set of indices i such that $(i, i+1)$ is a descent pair. The *number of descents* in a permutation π, denoted by $\mathrm{des}(\pi) = |\mathrm{Des}(\pi)|$, is a classical permutation statistic. This statistic was first studied by MacMahon [6] almost a century ago, and it still plays an important role in the study of permutation statistics.

Kitaev and Remmel [5] counted descents in S_n according to the parity of the first or second element of the descent pair. In [4], they generalize it to the case where the first or the second element in the descent pair is divisible by k for some $k \geq 2$. Some more research was done in the direction of studying the corresponding distribution for words (see [1, 2, 3]).

Let r, n be positive integers. A *colored permutation* on of n objects with r colors is a bijection π on the set

$$\Sigma_{r,n} = \{1, \ldots, n, 1^{[1]}, \ldots, n^{[1]}, \ldots, 1^{[r-1]}, \ldots, n^{[r-1]}\}$$

satisfying the following condition: if $\pi(i^{[\alpha]}) = j^{[\beta]}$, then $\pi(i^{[\alpha+1]}) = j^{[\beta+1]}$, where the exponents are taken modulo r. If $\pi_k = i^{[j]}$, we define $|\pi_k| = i$, where $\pi_k = \pi(k^{[0]})$ for each k. Moreover, the *color* of the position k, denoted by $c_k(\pi)$, is j. The set of all such colored permutations form a group, $G_{r,n}$, which is isomorphic to the wreath product $\mathbb{Z}_r \wr S_n = \mathbb{Z}_r^n \rtimes S_n$. The classical Weyl groups appear as special cases: the symmetric group $G_{1,n} = S_n$ and the hyperoctahedral group $G_{2,n} = B_n$.

We fix two different orders on the set $\Sigma_{r,n}$:

- The color order:

$$1^{[r-1]} < 2^{[r-1]} < \cdots < n^{[r-1]} < \cdots < 1^{[1]} < \cdots < n^{[1]} < 1^{[0]} < \cdots < n^{[0]}.$$

- The absolute order:

$$1^{[r-1]} < 1^{[r-2]} < \cdots < 1^{[0]} < 2^{[r-1]} < \cdots < 2^{[0]} < \cdots < n^{[r-1]} < \cdots < n^{[0]}.$$

Let $\prec$ be one of the orders defined above. Let $\pi \in G_{r,n}$ and let c, d be two nonnegative integers such that $c \leq d$. A (c,d)-*descent in position* i is a descent (with respect to $\prec$) in position i such that $c_i(\pi) = c$ and $c_{i+1}(\pi) = d$. For example, if $\pi = 6^{[1]}2^{[2]}4^{[0]}3^{[1]}5^{[2]}1^{[2]}$, then π has two $(1,2)$-descents with respect to the color order: $i = 1$ and $i = 4$, but only one $(1,2)$-descent with respect to the absolute order: $i = 1$. The number of (c,d)-descents in π with respect to the color order (respectively absolute order) is denoted by $\mathrm{des}^{\mathrm{Clr}}_{c,d}(\pi)$ (respectively $\mathrm{des}^{\mathrm{Abs}}_{c,d}(\pi)$).

It should be noted that actually, in the case of (c,d)-descents where $c < d$ with respect to the color order, one can forget about the colored permutations and deal with the underlying color words. For example, the underlying color word of $\pi = 6^{[1]}2^{[2]}4^{[0]}3^{[1]}5^{[2]}1^{[2]}$ is 120122, so our problem reduces to finding the number of words over $\{0, \ldots, r-1\}$ having a fixed number of places where the color d is subsequent to c. Hence, the enumeration of (c,d)-descents in the case of the color order can be done as a special case of the work of Hall and Remmel [2] where they compute the generating function of (X,Y)-descents (where X and Y are subsets of the alphabet). Nevertheless, since our enumeration covers both orders at once, we present here a simple and direct combinatorial proof for these enumerations.

In Section 2, we study the generating function for the number of colored permutations in $G_{r,n}$ having exactly m (c,d)-descents where $c < d$.

We get the following results.

Proposition 1.1. *The number of colored permutations in $G_{r,n}$ with exactly 0 (c,d)-descents $(0 \le c < d \le r-1)$, with respect to both orders is*

$$n!\sum_{k=0}^{n}(r-1)^k + n!\sum_{k=1}^{n-1}\sum_{b=1}^{k}\binom{k}{b}\binom{n-k}{b}(r-2)^b(r-1)^{n-k-b}.$$

The number of colored permutations in $G_{r,n}$ with exactly $m > 0$ (c,d)-descents $(0 \le c < d \le r-1)$ with respect to the color order, is given by

$$n!\sum_{k=1}^{n-1}\sum_{b=1}^{k}\binom{k}{b}\binom{n-k}{b}\binom{b}{m}(r-2)^{b-m}(r-1)^{n-k-b}.$$

The number of colored permutations in $G_{r,n}$ with exactly $m > 0$ (c,d)-descents $(0 \le c < d \le r-1)$ with respect to the absolute order, is given by

$$\frac{n!}{2^m}\sum_{k=1}^{n-1}\sum_{b=1}^{k}\binom{k}{b}\binom{n-k}{b}\binom{b}{m}(r-2)^{b-m}(r-1)^{n-k-b}.$$

Note that when $n = 2$, there will be no dependency on the value of r, as a simple verification shows.

The following is a special case for the hyperoctahedral group $B_n = G_{2,n}$.

Proposition 1.2. *The number of signed permutations in B_n with exactly m $(0,1)$-descents is given by $n!\binom{n+1}{n-2m}$.*

In Section 3, we study the generating function for the number of colored permutations in $G_{r,n}$ having exactly m (c,c)-descents. Here, we obtain our results by using recurrences and manipulations with exponential generating functions.

Proposition 1.3. *The number of colored permutations in $G_{r,n}$ with exactly m (c,c)-descents, $0 \le c \le r-1$, with respect to both orders is given by*

$$n!\sum_{j=0}^{n}\sum_{i=0}^{m}(-1)^i(r-1)^j\binom{j+m-i}{j}\binom{n+1}{i}\frac{(m-i+j+1)^{n-j}}{(n-j)!}.$$

2. Counting (c,d)-descents with $c < d$ by a direct combinatorial approach

Our approach for enumerating (c,d)-descents is based on a counting argument. We enumerate for both orders simultaneously.

Let $\pi = \pi_1^{[c_1]}\pi_2^{[c_2]}\cdots\pi_n^{[c_n]} \in G_{r,n}$. Define a c-block (respectively $\bar{c}$-block) in π to be a maximal subsequence $\pi_i^{[c_i]}\pi_{i+1}^{[c_{i+1}]}\cdots\pi_j^{[c_j]}$ of π such that for all $k \in [i,j]$, $c_k = c$ (respectively $c_k \neq c$). The cardinalities of the blocks form a composition of n. For example, in $G_{6,6}$, with $c = 2, d = 3$, if $\pi = 2^{[2]}4^{[2]}5^{[3]}1^{[4]}6^{[2]}3^{[1]}$, then the corresponding blocks of π are

$$\{2^{[2]}, 4^{[2]}\}, \{5^{[3]}, 1^{[4]}\}, \{6^{[2]}\}, \{3^{[1]}\},$$

while the corresponding composition is $(2,2,1,1)$.

Note that (c,d)-descents can appear in the transitions from a c-block to a $\bar{c}$-block, but not necessarily in all of these transitions.

Let $\mathrm{Comp}(n)$ be the set of compositions of n, and let $\mathrm{Comp}^{\mathrm{clr}}(n) = \mathrm{Comp}(n) \times \{c, \bar{c}\}$. For each $\varphi \in \mathrm{Comp}(n)$ and $x \in \{c, \bar{c}\}$, the element $(\varphi, x) \in \mathrm{Comp}^{\mathrm{clr}}(n)$ represents the composition φ where the first part is colored by x.

Now, let $\mu = (\varphi, x) \in \mathrm{Comp}^{\mathrm{clr}}(n)$. We define the following parameters:

- e_φ is the sum of parts in the even places of φ. For example, if $\varphi = (2,3,4,5)$ then $e_\varphi = 3 + 5 = 8$.
- $k = k_\mu = \begin{cases} n - e_\varphi & \text{if } x = c, \\ e_\varphi & \text{if } x = \bar{c}. \end{cases}$
- $b = b_\mu$ is the number of c-blocks and $\bar{b}_\mu$ is the number of $\bar{c}$-blocks. Explicitly,

$$b_\mu = \begin{cases} \frac{|\varphi|}{2} & \text{if } |\varphi| \text{ even}, \\ \frac{|\varphi|+1}{2} & \text{if } |\varphi| \text{ odd}, x = c, \\ \frac{|\varphi|-1}{2} & \text{if } |\varphi| \text{ odd}, x = \bar{c}, \end{cases}$$

  where $|\varphi|$ is the number of parts of φ, and $\bar{b}_\mu = |\varphi| - b_\mu$.
- t_μ is the number of transitions between a c-block to a $\bar{c}$-block.

Define

$$A_{r,n}^{\mathrm{Ord}}(q) = \sum_{\pi \in G_{r,n}} q^{\mathrm{des}_{c,d}^{\mathrm{Ord}}(\pi)},$$

where Ord stands for the orders Clr or Abs. For calculating $A_{r,n}^{\mathrm{Ord}}(q)$, we run over the elements of $\mathrm{Comp}^{\mathrm{clr}}(n)$, instead of running over the elements of $G_{r,n}$.

Lemma 2.1. *For all $n \geq 0$,*

$$A_{r,n}^{\mathrm{Clr}}(q) = n! \sum_{\mu \in \mathrm{Comp}^{\mathrm{clr}}(n)} (q + r - 2)^{t_\mu} (r-1)^{n - k_\mu - t_\mu}. \tag{2.1}$$

$$A^{\mathrm{Abs}}_{r,n}(q) = n! \sum_{\mu\in \mathrm{Comp}^{\mathrm{clr}}(n)} \left(\frac{q}{2}+r-2\right)^{t_\mu} (r-1)^{n-k_\mu-t_\mu}. \tag{2.2}$$

Proof. Each $\mu \in \mathrm{Comp}^{\mathrm{clr}}(n)$ gives rise to $n!(r-1)^{n-k_\mu}$ elements of $G_{r,n}$. $\pi \in G_{r,n}$ contributes a (c,d)-descent with respect to the color order in each transition from a c-block to a $\bar{c}$-block such that the first digit of the $\bar{c}$-block is colored by d. This proves Equation (2.1).

Only half of the (c,d)-descents with respect to the color order are also (c,d)-descents with respect to the absolute order (see an example in the introduction), so we have Equation (2.2) too. □

In order to get an explicit expression for the formulas appearing in Lemma 2.1, we treat separately $\mathrm{Comp}(n)\times\{c\}$ and $\mathrm{Comp}(n)\times\{\bar{c}\}$. We do this only for the color order. The other computation is similar.

Let $\mu = (\varphi, c) \in \mathrm{Comp}(n)\times\{c\}$.
We split the contribution into three cases according to $|\varphi|$.

(1) $|\varphi| = 1$: in this case we have $k_\mu = n$ and $t_\mu = 0$, thus its contribution is

$$n! \sum_{\substack{\mu=(\varphi,c)\in \mathrm{Comp}^{\mathrm{clr}}(n)\\ |\varphi|=1}} (q+r-2)^{t_\mu}(r-1)^{n-k_\mu-t_\mu} = n!.$$

(2) $|\varphi|$ is even: in this case, k_μ can vary in $\{1,\dots,n-1\}$ and hence $b_\mu \in \{1,\dots,k_\mu\}$. Moreover, $\bar{b}_\mu = t_\mu = b_\mu$. Then the contribution is

$$\begin{aligned} n! &\sum_{\substack{\mu=(\varphi,c)\in \mathrm{Comp}^{\mathrm{clr}}(n)\\ |\varphi| \text{ even}}} (q+r-2)^{t_\mu}(r-1)^{n-k_\mu-t_\mu} \\ &= n!\sum_{k=1}^{n-1}\sum_{b=1}^{k}\binom{k-1}{k-b}\binom{n-k-1}{n-k-b}(q+r-2)^b(r-1)^{n-k-b}. \end{aligned}$$

Note that the first binomial coefficient corresponds to the choice of dividing the k digits colored by c into b non-empty blocks, while the second binomial coefficient corresponds to the choice of dividing the remaining $n-k$ digits (which are not colored by c) into $\bar{b}$ non-empty blocks.

(3) $|\varphi| > 1$ odd: In this case, k_μ can vary in $\{2,\dots,n-1\}$ and hence $b_\mu \in \{2,\dots,k_\mu\}$. Moreover, $\bar{b}_\mu = b-1, t_\mu = b_\mu - 1$. Then

the contribution is

$$n! \sum_{\substack{\mu=(\varphi,c)\in\mathrm{Comp}^{\mathrm{clr}}(n)\\ |\varphi| \text{ odd}}} (q+r-2)^{t_\mu}(r-1)^{n-k_\mu-t_\mu}$$
$$= n!\sum_{k=2}^{n-1}\sum_{b=1}^{k}\binom{k-1}{k-b}\binom{n-k-1}{n-k-b+1}(q+r-2)^{b-1}(r-1)^{n-k-b+1}.$$

Now, let us treat the set $\mathrm{Comp}(n)\times\{\bar{c}\}$. Let $\mu=(\varphi,\bar{c})\in\mathrm{Comp}(n)\times\{\bar{c}\}$.

As in the case of $\mathrm{Comp}(n)\times\{c\}$, we split the computation into three cases according to $|\varphi|$.

(1) $|\varphi|=1$:

$$n! \sum_{\substack{\mu=(\varphi,\bar{c})\in\mathrm{Comp}^{\mathrm{clr}}(n)\\ |\varphi|=1}} (q+r-2)^{t_\mu}(r-1)^{n-k_\mu-t_\mu} = n!(r-1)^n.$$

(2) $|\varphi|$ is even: in this case, $k_\mu\in\{1,\dots,n-1\}$ and hence $b_\mu\in\{1,\dots,k_\mu\}$. Moreover, $\bar{b}_\mu=t_\mu=b_\mu-1$. Then we have

$$n! \sum_{\substack{\mu=(\varphi,\bar{c})\in\mathrm{Comp}^{\mathrm{clr}}(n)\\ |\varphi| \text{ even}}} (q+r-2)^{t_\mu}(r-1)^{n-k_\mu-t_\mu}$$
$$= n!\sum_{k=1}^{n-1}\sum_{b=1}^{k}\binom{k-1}{k-b}\binom{n-k-1}{n-k-b}(q+r-2)^{b-1}(r-1)^{n-k-b+1}.$$

(3) $|\varphi|>1$ odd: in this case, k_μ can vary in $\{1,\dots,n-2\}$ and hence $b_\mu\in\{2,\dots,k_\mu\}$. Moreover, $\bar{b}_\mu=b_\mu+1, t_\mu=b_\mu$. Then we have

$$n! \sum_{\substack{\mu=(\varphi,\bar{c})\in\mathrm{Comp}^{\mathrm{clr}}(n)\\ |\varphi| \text{ odd}}} (q+r-2)^{t_\mu}(r-1)^{n-k_\mu-t_\mu}$$
$$= n!\sum_{k=1}^{n-2}\sum_{b=1}^{k}\binom{k-1}{k-b}\binom{n-k-1}{n-k-b-1}(q+r-2)^{b}(r-1)^{n-k-b}.$$

Summing up all the parts, we get

$$\begin{aligned}A^{\mathrm{Clr}}_{r,n}(q) &= n!(1+(r-1)^n)\\ &\quad + n!\sum_{k=1}^{n-1}\sum_{b=1}^{k}\binom{k-1}{k-b}\binom{n-k}{n-k-b}(q+r-2)^b(r-1)^{n-k-b}\\ &\quad + n!\sum_{k=1}^{n-1}\sum_{b=1}^{k}\binom{k-1}{k-b}\binom{n-k}{n-k-b+1}\\ &\qquad\qquad \cdot (q+r-2)^{b-1}(r-1)^{n-k-b+1}\\ &= n!\sum_{k=0}^{n}(r-1)^k\\ &\quad + n!\sum_{k=1}^{n-1}\sum_{b=1}^{k}\binom{k}{b}\binom{n-k}{b}(q+r-2)^b(r-1)^{n-k-b}.\end{aligned}$$

Here is the analogous result for the absolute order.

Corollary 2.2.

$$\begin{aligned}A^{\mathrm{Abs}}_{r,n}(q) &= n!\sum_{k=0}^{n}(r-1)^k\\ &\quad + n!\sum_{k=1}^{n-1}\sum_{b=1}^{k}\binom{k}{b}\binom{n-k}{b}\left(\frac{q}{2}+r-2\right)^b(r-1)^{n-k-b}.\end{aligned}$$

Hence, we get the following result.

Proposition 2.3. *The number of colored permutations in $G_{r,n}$ with exactly* 0 *(c,d)-descents, $0 \le c < d \le r-1$, is*

$$n!\sum_{k=0}^{n}(r-1)^k + n!\sum_{k=1}^{n-1}\sum_{b=1}^{k}\binom{k}{b}\binom{n-k}{b}(r-2)^b(r-1)^{n-k-b}.$$

The number of colored permutations in $G_{r,n}$ with exactly $m > 0$ (c,d)-descents ($0 \le c < d \le r-1$) with respect to the color order, is given by

$$n!\sum_{k=1}^{n-1}\sum_{b=1}^{k}\binom{k}{b}\binom{n-k}{b}\binom{b}{m}(r-2)^{b-m}(r-1)^{n-k-b}.$$

The number of colored permutations in $G_{r,n}$ with exactly $m > 0$ (c,d)-descents ($0 \le c < d \le r-1$) with respect to the absolute order, *is given by*

$$\frac{n!}{2^m}\sum_{k=1}^{n-1}\sum_{b=1}^{k}\binom{k}{b}\binom{n-k}{b}\binom{b}{m}(r-2)^{b-m}(r-1)^{n-k-b}.$$

This proves Proposition 1.1.

3. (c,c)-descents

In this section, we calculate the number of permutations of $G_{r,n}$ having a fixed number of (c,c)-descents. Note that in this case, the enumeration is identical in the color order and the absolute order, since both orders satisfy

$$1^{[c]} < 2^{[c]} < \cdots < n^{[c]}.$$

Hence, we denote: $\mathrm{des}_c(\pi) = \mathrm{des}^{\mathrm{Clr}}_{c,c}(\pi) = \mathrm{des}^{\mathrm{Abs}}_{c,c}(\pi)$.

We use here a recursive approach. Let $\pi = i_1^{[j_1]}\cdots i_n^{[j_n]} \in G_{r,n}$. Using a map $f : G_{r,n} \to G_{r,n}$ which takes each colored digit $i^{[j]}$ to $i^{[(j+1) \pmod r)]}$, we are led to the following.

Observation 3.1. *For each color $c \in \{0,\dots,r-1\}$, the number of colored permutations with exactly m (c,c)-descents is equal to the number of colored permutations with exactly m $(0,0)$-descents.*

By the above observation, it suffices to find the generating function for the number of colored permutations in $G_{r,n}$ having $(0,0)$-descents.

Let

$$g_{r,n}(q) = \sum_{\pi\in G_{r,n}} q^{\mathrm{des}_0(\pi)}.$$

In order to find a recurrence for $g_{r,n}(q)$, we will use the following notations. Define

$$g^+_{r,n}(q) = \sum_{\pi\in G_{r,n},\ c_1(\pi)=0} q^{\mathrm{des}_0(\pi)},$$

$$g^-_{r,n}(q) = \sum_{\pi\in G_{r,n},\ c_1(\pi)\neq 0} q^{\mathrm{des}_0(\pi)}.$$

Note that $g_{r,n}(q) = g^+_{r,n}(q) + g^-_{r,n}(q)$.

Let $\pi \in G_{r,n}$. If $|\pi_n| = n$, denote $\pi = \pi'\pi_n$ and we have

$$\mathrm{des}_0(\pi) = \mathrm{des}_0(\pi').$$

Otherwise, let $j < n$ be such that $|\pi_j| = n$. Denote: $\pi = \pi'\pi_j\pi'' \in G_{r,n}$. If $\pi_j = n^{[c]}$ for $c \neq 0$, then $\mathrm{des}_0(\pi) = \mathrm{des}_0(\pi') + \mathrm{des}_0(\pi'')$. If

$\pi_j = n$, we have two cases depending on the color of π''_1 (the first element of π''):

$$\mathrm{des}_0(\pi) = \begin{cases} \mathrm{des}_0(\pi') + \mathrm{des}_0(\pi'') & \text{if } c_1(\pi'') \neq 0, \\ 1 + \mathrm{des}_0(\pi') + \mathrm{des}_0(\pi'') & \text{otherwise.} \end{cases}$$

Since there are no restrictions on the positions of the digits $1, 2, \ldots, n-1$, we have the following recurrences.

Lemma 3.2. *For all $n \geq 1$, we have*

$$\begin{aligned} g_{r,n}(q) = r g_{r,n-1}(q) &+ (r-1)\sum_{j=1}^{n-1}\binom{n-1}{j-1} g_{r,j-1}(q) g_{r,n-j}(q) \\ &+ \sum_{j=1}^{n-1}\binom{n-1}{j-1} g_{r,j-1}(q)(g^-_{r,n-j}(q) + q g^+_{r,n-j}(q)), \end{aligned}$$

and for $n \geq 2$, we have

$$\begin{aligned} g^+_{r,n}(q) = r g^+_{r,n-1}(q) &+ (r-1)\sum_{j=2}^{n-1}\binom{n-1}{j-1} g^+_{r,j-1}(q) g_{r,n-j}(q) \\ &+ \sum_{j=1}^{n-1}\binom{n-1}{j-1} g^+_{r,j-1}(q)(g^-_{r,n-j}(q) + q g^+_{r,n-j}(q)). \end{aligned}$$

Proof. In both expressions, the first summand corresponds to the case $|\pi_n| = n$, the second corresponds to the case $\pi_j = n^{[c]}$ ($c \neq 0$) for some j, $1 \leq j \leq n-1$, and the third corresponds to the case $\pi_j = n$ for some j, $1 \leq j \leq n-1$. □

In order to find an explicit formula for $g_{r,n}(q)$, we rewrite the above recurrences in terms of exponential generating functions. Define

$$G_r(x,q) = \sum_{n\geq 0} g_{r,n}(q)\frac{x^n}{n!}, \quad G^+_r(x,q) = \sum_{n\geq 0} g^+_{r,n}(q)\frac{x^n}{n!},$$

and

$$G^-_r(x,q) = \sum_{n\geq 0} g^-_{r,n}(q)\frac{x^n}{n!}.$$

Since $g_{r,n}(q) = g^+_{r,n}(q) + g^-_{r,n}(q)$, we have

$$G_r(x,q) = G^+_r(x,q) + G^-_r(x,q).$$

Moreover, for all $n \geq 1$, Lemma 3.2 yields

$$\begin{aligned}\frac{d}{dx}\left(\frac{g_{r,n}(q)}{n!}x^n\right) &= \frac{rg_{r,n-1}(q)}{(n-1)!}x^{n-1} \\ &\quad + (r-1)x^{n-1}\sum_{j=0}^{n-2}\frac{g_{r,j}(q)}{j!}\cdot\frac{g_{r,n-1-j}(q)}{(n-1-j)!} \\ &\quad + x^{n-1}\sum_{j=0}^{n-2}\frac{g_{r,j}(q)}{j!}\cdot\frac{g^-_{r,n-1-j}(q)+qg^+_{r,n-1-j}(q)}{(n-1-j)!},\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dx}\left(\frac{g^+_{r,n}(q)}{n!}x^n\right) &= r\frac{g^+_{r,n-1}(q)}{(n-1)!}x^{n-1} \\ &\quad + (r-1)x^{n-1}\sum_{j=1}^{n-2}\frac{g^+_{r,j}(q)}{j!}\cdot\frac{g_{r,n-1-j}(q)}{(n-1-j)!} \\ &\quad + x^{n-1}\sum_{j=0}^{n-2}\frac{g^+_{r,j}(q)}{j!}\cdot\frac{g^-_{r,n-1-j}(q)+qg^+_{r,n-1-j}(q)}{(n-1-j)!}.\end{aligned}$$

Summing over all $n \geq 1$ and using the initial conditions $g_{r,0}(q) = g^+_{r,0}(q) = g^+_{r,1}(q) = 1$, $g^-_{r,0}(q) = 0$ and $g^-_{r,1}(q) = r-1$, we get

$$\begin{aligned}G_r(x,q) &= G^+_r(x,q) + G^-_r(x,q), \\ \frac{d}{dx}\left(G_r(x,q)\right) &= (1-q)G_r(x,q) + (r-1)(G_r(x,q))^2 \\ &\quad + G_r(x,q)G^-_r(x,q) + qG_r(x,q)G^+_r(x,q), \\ \frac{d}{dx}\left(G^+_r(x,q)\right) &= (1-q)G^+_r(x,q) + (r-1)G^+_r(x,q)G_r(x,q) \\ &\quad + G^+_r(x,q)G^-_r(x,q) + q(G^+_r(x,q))^2.\end{aligned}$$

Using any computer algebra package, such as *Maple* or *Mathematica*, we obtain the following result.

Theorem 3.3. *The generating functions $G_r(x,q)$, $G^+_r(x,q)$ and $G^-_r(x,q)$ are given by*

$$\begin{aligned}G_r(x,q) &= \frac{1-q}{(r-1)x(q-1)-q+e^{(q-1)x}}, \\ G^+_r(x,q) &= \frac{(1-q)(1-(r-1)x)}{(r-1)x(q-1)-q+e^{(q-1)x}}, \\ G^-_r(x,q) &= \frac{x(r-1)(1-q)}{(r-1)x(q-1)-q+e^{(q-1)x}},\end{aligned}$$

respectively.

In order to obtain an explicit formula for the number of colored permutations in $G_{r,n}$ with exactly m $(0,0)$-descents, we find the coefficient of $x^n q^m$ in $G_r(x,q)$:

$$\begin{aligned}
G_r(x,q) &= \frac{1-q}{e^{(q-1)x}-q}\cdot\frac{1}{1-\frac{(r-1)x(1-q)}{e^{(q-1)x}-q}}\\
&= \sum_{j\geq 0}\frac{(r-1)^j x^j (1-q)^{j+1}}{(e^{(q-1)x}-q)^{j+1}}\\
&= \sum_{j,i,k\geq 0}(r-1)^j x^j (1-q)^{j+1}\binom{j+i}{j}q^i\frac{(1-q)^k x^k (i+j+1)^k}{k!}\\
&= \sum_{j,i,k\geq 0}\sum_{\ell=0}^{j+k+1}(-1)^\ell (r-1)^j\binom{j+i}{j}\binom{j+k+1}{\ell}\\
&\qquad\qquad\cdot q^{\ell+i}x^{j+k}\frac{(i+j+1)^k}{k!}.
\end{aligned}$$

Thus the coefficient of $x^n q^m$ in $G_r(x,q)$ is given by

$$\sum_{j=0}^{n}\sum_{i=0}^{m}(-1)^i(r-1)^j\binom{j+m-i}{j}\binom{n+1}{i}\frac{(m-i+j+1)^{n-j}}{(n-j)!}.$$

This implies that the number of colored permutations in $G_{r,n}$ with exactly m $(0,0)$-descents (or (c,c)-descents for any $0\leq c\leq r-1$) is

$$n!\sum_{j=0}^{n}\sum_{i=0}^{m}(-1)^i(r-1)^j\binom{j+m-i}{j}\binom{n+1}{i}\frac{(m-i+j+1)^{n-j}}{(n-j)!},$$

as stated in Proposition 1.3.

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