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NUMBER OF "*udu*"S OF A DYCK PATH AND *ad*-NILPOTENT IDEALS OF PARABOLIC SUBALGEBRAS OF $sl_{\ell+1}(\mathbb{C})$

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Abstract. For an ad-nilpotent ideal $\mathfrak{i}$ of a Borel subalgebra of $sl_{\ell+1}(\mathbb{C})$, we denote by $I_{\mathfrak{i}}$ the maximal subset I of the set of simple roots such that $\mathfrak{i}$ is an ad-nilpotent ideal of the standard parabolic subalgebra $\mathfrak{p}_I$. We use the bijection of Andrews, Krattenthaler, Orsina and Papi [*Trans. Amer. Math. Soc.* **354** (2002), 3835–3853] between the set of ad-nilpotent ideals of a Borel subalgebra in $sl_{\ell+1}(\mathbb{C})$ and the set of Dyck paths of length $2\ell+2$, to exhibit a bijection between ad-nilpotent ideals $\mathfrak{i}$ of the Borel subalgebra such that $\sharp I_{\mathfrak{i}} = r$ and the Dyck paths of length $2\ell+2$ having r occurrences of "*udu*". We obtain also a duality between antichains of cardinality p and $\ell - p$ in the set of positive roots.

1. Introduction

Let $M_{\ell+1}(\mathbb{C})$ be the set of $(\ell+1)$-by-$(\ell+1)$ matrices with coefficients in $\mathbb{C}$, and $\mathfrak{g}$ be the simple Lie algebra $sl_{\ell+1}(\mathbb{C})$ consisting of elements of $M_{\ell+1}(\mathbb{C})$ whose trace is equal to zero. Let $\mathfrak{h}$ be the maximal toral subalgebra of $\mathfrak{g}$ consisting of trace zero diagonal matrices. Let $(E_{i,j})$ be the canonical basis of $M_{\ell+1}(\mathbb{C})$ and $(E^*_{i,j})$ be its dual basis. For $1 \leqslant i \leqslant \ell+1$, set $\epsilon_i = E^*_{i,i}$. Then $\Delta = \{\epsilon_i - \epsilon_j; 1 \leqslant i, j \leqslant \ell+1, i \neq j\}$ is the root system associated to $(\mathfrak{g}, \mathfrak{h})$, and $\Delta^+ = \{\epsilon_i - \epsilon_j; 1 \leqslant i < j \leqslant \ell+1\}$ is a system of positive roots. Denote by $\alpha_i = \epsilon_i - \epsilon_{i+1}$, for $i = 1, \ldots, \ell$. Then $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ is the corresponding set of simple roots. For each $\alpha \in \Delta$, let $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}; [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$ be the root space of $\mathfrak{g}$ relative to α.

For $I \subset \Pi$, set $\Delta_I = \mathbb{Z}I \cap \Delta$. We fix the corresponding standard parabolic subalgebra,

$$\mathfrak{p}_I = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_I \cup \Delta^+} \mathfrak{g}_\alpha \right).$$

Note that $\mathfrak{p}_\emptyset$ is a Borel subalgebra $\mathfrak{b}$ associated to the choice of Δ^+.

An ideal $\mathfrak{i}$ of $\mathfrak{p}_I$ is ad-nilpotent if and only if for all $x \in \mathfrak{i}$, $ad_{\mathfrak{p}_I} x$ is nilpotent. Since any ideal of $\mathfrak{p}_I$ is $\mathfrak{h}$-stable, we can deduce easily that

an ideal is ad-nilpotent if and only if it is nilpotent. Moreover, we have $\mathfrak{i} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, for some subset $\Phi \subset \Delta^+ \setminus \Delta_I$.

A Dyck path of length $2n$ can be defined as a word of $2n$ letters u or d, having the same number of u and d, and such that there is always more u's than d's to the left of a letter.

Andrews, Krattenthaler, Orsina and Papi established in [AKOP] a bijection between the set of ad-nilpotent ideals of the Borel subalgebra $\mathfrak{p}_\emptyset$ and the set of Dyck paths of length $2\ell+2$ which allows them to enumerate ad-nilpotent ideals of a fixed class of nilpotence. The purpose of this paper is to explain some applications of this correspondence for the ad-nilpotent ideals of parabolic subalgebras.

More precisely, let $\mathfrak{i}$ be an ad-nilpotent ideal of the Borel subalgebra $\mathfrak{p}_\emptyset$. Denote by $I_\mathfrak{i}$ the maximal subset $I \subset \Pi$ such that $\mathfrak{i}$ is an ad-nilpotent ideal of $\mathfrak{p}_I$. The main result we prove here is the following theorem.

Theorem 1. *There is a bijection between the ad-nilpotent ideals $\mathfrak{i}$ of $\mathfrak{b}$ such that $\sharp I_\mathfrak{i} = r$ and the Dyck paths of length $2\ell+2$ having r occurrences of "udu".*

We can deduce a formula for the desired number of ideals since the number of Dyck paths having r occurrences of "udu" have been calculated in [Sun].

This paper is organized as follows: we first recall the natural bijection between ℓ-partitions and Dyck paths of length $2\ell+2$, as in [Pa]. In Section 3, we recall the iterative construction of the bijection of [AKOP]. Then, in Section 4, we explain how to calculate the number of occurrences of "udu" of a Dyck path obtained by the previous construction. In Section 5, we recall some facts of [R] and [CP] on ad-nilpotent ideals and we prove Theorem 1. Finally, in Section 6, we establish a duality between ad-nilpotent ideals of $\mathfrak{p}_\emptyset$. Such a duality has already been constructed by Panyushev in [Pa], however, it is not the same as the one we have here.

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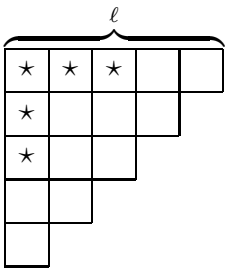
2. Partitions and Dyck paths

In this section, we shall see how to generate a Dyck path from a partition.

Recall that a partition is an ℓ-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathbb{N}^\ell$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell$. A partition will be called an ℓ-partition if $\lambda_i \leqslant i$ for $i = 1, \ldots, l$.

Partitions are usually represented by their Ferrers diagrams. Let T_ℓ be the Ferrers diagram of the ℓ-partition $(\ell, \ell-1, \ldots, 1)$. Then the Ferrers diagram F of any ℓ-partition λ can be viewed as a subdiagram of T_ℓ. For example, for $\ell = 5$, the Ferrers diagram of $\lambda = (3, 1, 1, 0, 0)$ is the subdiagram of T_ℓ, whose boxes are denoted by some $\star$:



Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an ℓ-partition and let F be its Ferrers diagram. We draw a dotted horizontal line from the top of the line $x + y = \ell + 1$ to F and a dotted vertical line from F to the bottom of the line $x + y = \ell + 1$. For example, when $\lambda = (5, 3, 1, 1, 1, 0, 0)$, we have:

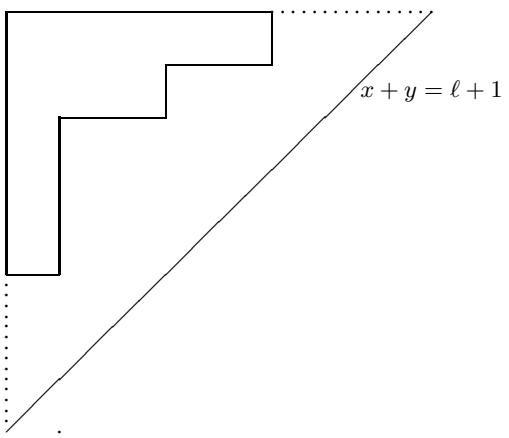
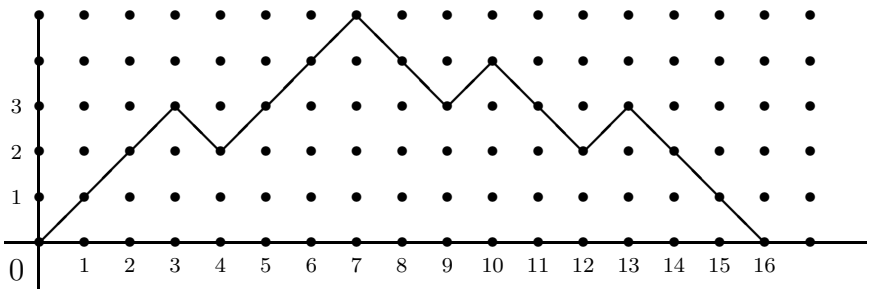


FIGURE 1

If we rotate the figure clockwise by 45 degrees, we can easily see that we obtain a Dyck path of length $2\ell + 2$ called $P(\lambda)$ as in [Pa]. This construction defines clearly a bijection $P : \lambda \mapsto P(\lambda)$ between ℓ-partitions and Dyck paths of length $2\ell + 2$. In the above example, the Dyck path $P(\lambda)$ is:



3. AKOP-bijection

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an ℓ-partition whose Ferrers diagram is F. We shall draw a dotted line associated to λ. We start at the top of the line $x + y = \ell + 1$. We go left until we meet F. Then, we continue downwards until we reach $x + y = \ell + 1$. Then we iterate the procedure until we reach the bottom. For example, for $\ell = 13$ and $\lambda = (10, 10, 9, 6, 5, 4, 4, 3, 1, 1, 1, 1, 0)$:

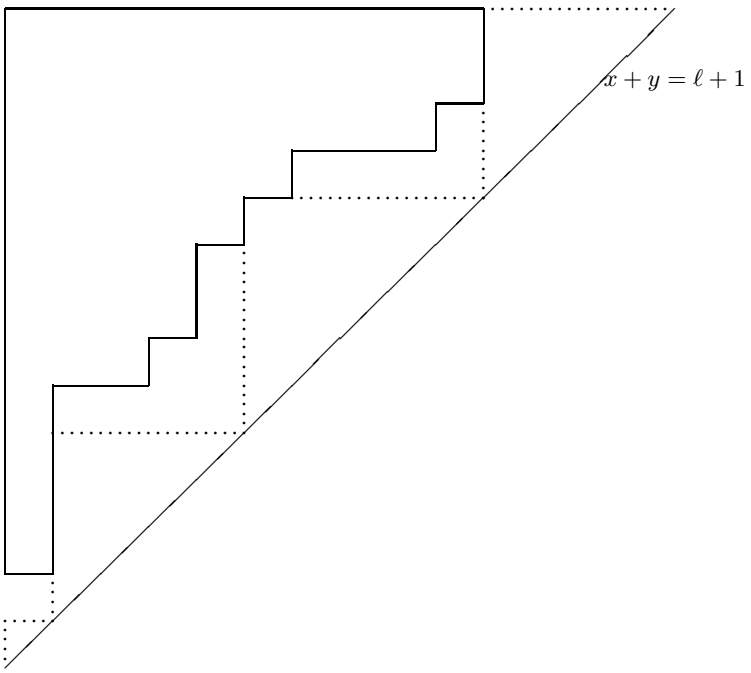


Figure 2

Let $n(\lambda)$ be the number of points of the dotted line on $x + y = \ell + 1$, which are not at the top or bottom. For example, we have

$n((0,\dots,0))=0$, and for the ℓ-partition λ of Figure 2, we have $n(\lambda)=3$.

We shall describe the construction of this line in a more formal way.

Let $k=n(\lambda)$. Set $i_n=\ell+1$ for all $n>k$, $i_k=\lambda_1$, $i_{k-1}=\lambda_{\ell-i_k+2}$, $i_{k-2}=\lambda_{\ell-i_{k-1}+2}$, $\dots$, $i_1=\lambda_{\ell-i_2+2}$ and $i_p=0$ for all $p\leqslant 0$. We have $0<i_1<\cdots<i_k<\ell+1$. The dotted line describes the shape of an ℓ-partition

$$\lambda^M=(i_k^{\ell-i_k+1},i_{k-1}^{i_k-i_{k-1}},\dots,i_1^{i_2-i_1},0^{i_1-1}). \tag{1}$$

Any ℓ-partition λ whose associated dotted line gives the partition λ^M must necessarily contain the cells

$$(1,i_k),(\ell-i_k+2,i_{k-1}),(\ell-i_{k-1}+2,i_{k-2}),\dots,(\ell-i_2+2,i_1).$$

The "minimal" ℓ-partition in the sense of inclusion of diagrams that contains these cells is

$$\lambda^m=(i_k,i_{k-1}^{\ell-i_k+1},i_{k-2}^{i_k-i_{k-1}},\dots,i_1^{i_3-i_2},0^{i_2-2}). \tag{2}$$

For example, take $\ell=13$ and $\lambda=(10,10,9,6,5,4,4,3,1,1,1,1,0)$, as above, we have $n(\lambda)=k=3$, $i_3=10, i_2=5, i_1=1$. The three distinguished cells above are

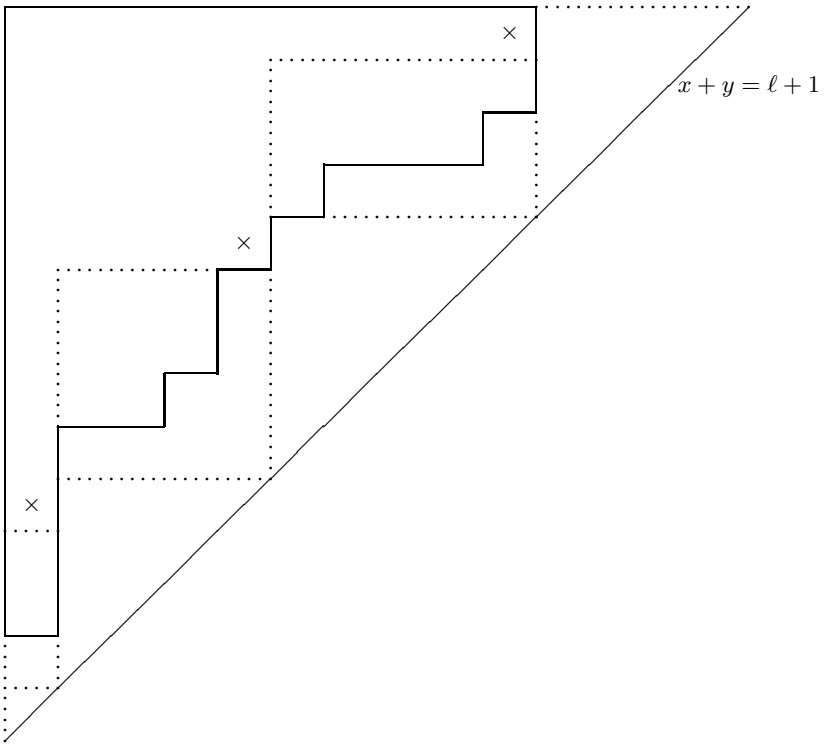
$$(1,10),(5,5),(10,1).$$

So we have

$$\lambda^M=(10,10,10,10,5,5,5,5,5,1,1,1,1),\text{ and}$$
$$\lambda^m=(10,5,5,5,5,1,1,1,1,1,0,0,0).$$

These partitions are illustrated in the figure below, where the distinguished cells are marked with $\times$, and λ^M is the partition corresponding to the dotted line outside λ, while λ^m is the one which corresponds to

the dotted line inside λ.



Observe that the difference $\lambda^M \backslash \lambda^m$ is a disjoint union of k rectangles, denoted by $R_k, \dots, R_1$ from the top to the bottom. More precisely,

$$R_j = \{(s,t); \ell - i_{p+1} + 2 < s < \ell - i_p + 2 \text{ and } i_{p-1} < t \leqslant i_p\}.$$

Inside each rectangle R_j, the shape of λ could be described by a word M_j, whose letters are d and l, where d indicates a down step and l indicates a left step.

Let h_j be the number of d's in M_j, which is at most the height of R_j and let l_j be the number of l's in M_j, which is the length of R_j. Then we have

$$h_j = i_{j+1} - i_j - 1 \text{ if } j \neq 1, \text{ and } h_j \leqslant i_{j+1} - i_j - 1 \text{ if } j = 1,$$
$$l_j = i_j - i_{j-1},$$

so $h_j \leqslant l_{j+1} - 1$ and the equality holds if $j \neq 1$. Furthermore the shape of M_j is $l^{a_{j,0}} d l^{a_{j,1}} d \dots d l^{a_{j,h_j}}$, where $a_{j,i} \in \mathbb{N}$, $0 \leqslant i \leqslant h_j$. We then have that

$$l_j = \sum_{i=0}^{h_j} a_{j,i}. \tag{3}$$

In the above example, we have $M_3 = dldl^3dl$, $M_2 = lddldl^2d$ and $M_1 = ddl$.

We shall now generate a Dyck path step by step from the M_j. We call a peak of a Dyck path, an occurrence of ud in the corresponding Dyck word.

First, let D_{k+1} be the Dyck path of length $2(\ell+1-i_k)$ containing $\ell+1-i_k$ peaks. Next, we have $M_k = l^{a_{k,0}}dl^{a_{k,1}}d\dots dl^{a_{k,h_k}}$. We insert $a_{k,0}$ peaks on the first peak of the already existing Dyck path D_{k+1}, then $a_{k,1}$ peaks on the second peak, and so on. We call D_k the new Dyck path obtained. Observed that the highest peaks of D_k are exactly those newly inserted, so there are exactly l_k. Since $h_{k-1} \leqslant l_k - 1$, the procedure can then be iterated by inserting peaks only on highest peaks. Each intermediate Dyck path obtained after using the word M_j is denoted by D_j. At the end, we obtain a Dyck path D_λ of length $2\ell+2$.

For example, let us consider $\ell = 7$ and $\lambda = (5,3,1,1,1,0,0)$:

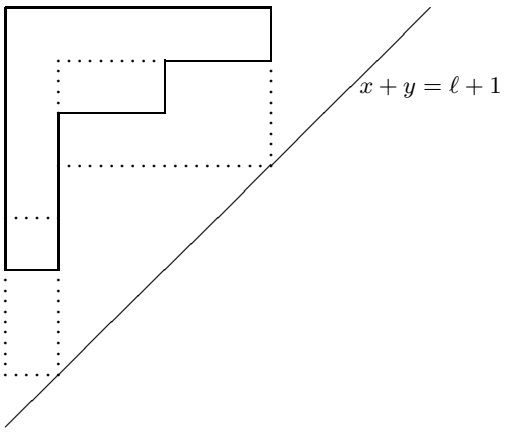
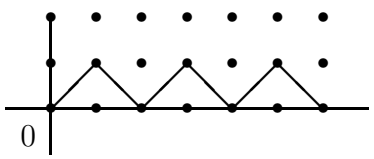
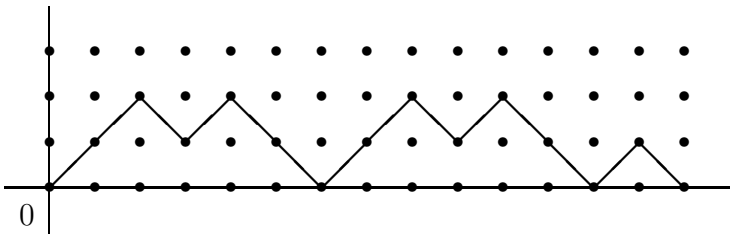


FIGURE 3

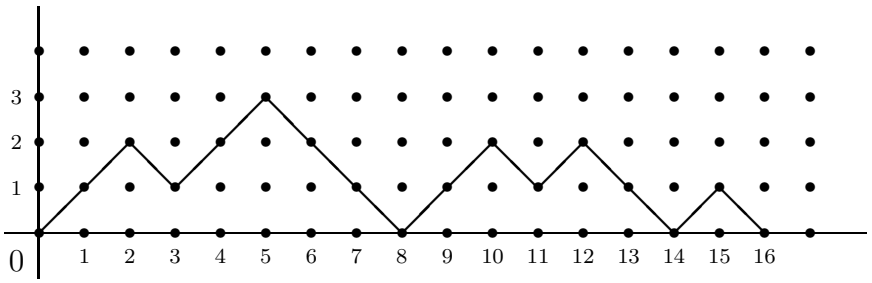
We have $n(\lambda) = k = 2$, $i_2 = 5$ and $i_1 = 1$. Then D_3 is the following Dyck path:



We have $M_2 = l^2dl^2d$, so we first insert 2 peaks on the first peak of D_3, then again two peaks on the second one. We obtain D_2:



Finally, $M_1 = dl$ so we insert $a_{1,0} = 0$ peak on the first highest peak of D_2 and $a_{1,1} = 1$ peak on the second highest peak. We obtain D_λ:



By [AKOP], we have the following proposition.

Proposition 3.1. *The map $D : \lambda \mapsto D_\lambda$ defines a bijection between the set of ℓ-partitions and the set of Dyck paths of length $2\ell + 2$.*

4. Dyck path and number of occurrences of "*udu*"

Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be an ℓ-partition such that $n(\lambda) = k$. Let D_λ be the Dyck path obtained from λ as described in Section 3. We shall see how to count the number of occurrences of "*udu*" contained in D_λ.

A peak could be followed by a "*u*", a "*d*" or nothing in the Dyck word. If it is followed by a "*u*", we call it a *u*-peak. Each *u*-peak will give an "*udu*" and vice versa.

Let $1 \leqslant j \leqslant k+1$. Let u_j be the number of *u*-peaks in the Dyck path D_j. For example, D_{k+1} contains $\ell - \lambda_1 + 1 = \ell - i_k + 1$ peaks, so it is easy to see that $u_{k+1} = \ell - \lambda_1$.

To construct D_{j-1} from D_j, we add some peaks on the highest peaks of D_j. Then, one must understand how the insertion of p peaks on a highest peak modifies the number of occurrences of "*udu*". Consider a peak P of maximal height on a Dyck path. If we add p peaks, the part of the Dyck word which corresponds to P (which was *ud*) becomes $uudud \dots udd$ (with p *ud*), so we obtain $p-1$ occurrences of *udu*. If P is a *u*-peak, then we also "destroy" the *udu* given by P. So at the end,

we only add $p-2$ occurrences of udu. For example, let us consider the following Dyck path which contains 2 occurrences of udu:

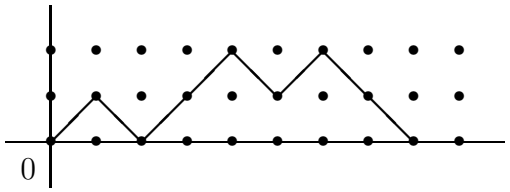
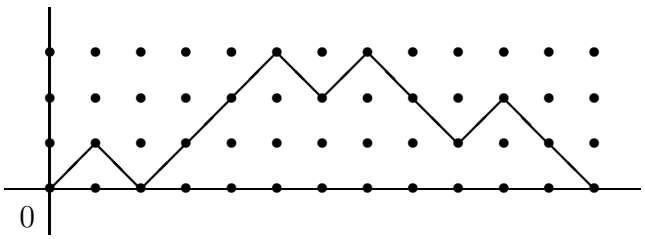
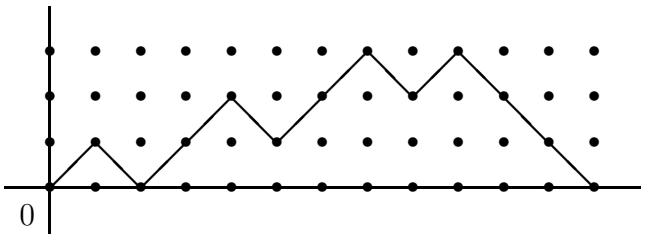


FIGURE 4

If we add 2 peaks on the first highest peak, we add $2-2=0$ occurrences of udu. So we obtain the following Dyck path with still 2 occurrences of udu:



If P is not a u-peak, then we do not "destroy" a udu, so we indeed add $p-1$ occurrences of "udu". For example, if we add 2 peaks on the second highest peak of Figure 4, we add $2-1=1$ occurrence of udu, so we obtain 3 occurrences of udu at the end:



Set $a_{k+1,0}=\ell-i_k+1$, $M_{k+1}=l^{a_{k+1,0}}$, and $h_{k+1}=0$. We have seen that each word M_j is in the form $l^{a_{j,0}}dl^{a_{j,1}}d\dots dl^{a_{j,h_j}}$. Let

$$\mathcal{A}_j=\{(j,t);t\in\{0,\dots,h_j\};a_{j,t}\neq 0\},$$

$$\mathcal{A}=\bigcup_{j=1}^{k}\mathcal{A}_j.$$

Recall from the construction that the number of highest peaks in D_j is

$$\sum_{t=0}^{h_j} a_{j,i} = l_j. \tag{4}$$

Observe that a highest peak is a u-peak if it is not the last one of a consecutive group of highest peaks. Hence, the q-th peak of D_j is not a u-peak if and only if there exists $r \in \{0, \dots, h_j\}$ such that $q = \sum_{s=0}^{r} a_{j,s}$. Set

$$\mathcal{L}_p = \left\{ (p,t); \text{there exists } \ 0 \leqslant r \leqslant h_{p+1}; t+1 = \sum_{q=0}^{r} a_{p+1,q} \right\},$$

$$\mathcal{U}_p = \mathcal{A}_p \setminus \mathcal{L}_p, \quad \mathcal{L} = \bigcup_{p=1}^{k} \mathcal{L}_p, \quad \mathcal{U} = \bigcup_{p=1}^{k} \mathcal{U}_p.$$

Thus $\mathcal{L}_j$ corresponds exactly to the set of highest peaks in D_j which are not u-peaks and where we insert new peaks. It follows that

$$u_{j-1} = u_j + \sum_{(j-1,t)\in\mathcal{U}_{j-1}} (a_{j-1,t} - 2) + \sum_{(j-1,t)\in\mathcal{L}_{j-1}} (a_{j-1,t} - 1).$$

At the end of the construction, the number of occurrences of "udu" in D_λ is u_1. By induction, we have

$$u_1 = \ell - \lambda_1 + \sum_{(j,t)\in\mathcal{U}} (a_{j,t} - 2) + \sum_{(j,t)\in\mathcal{L}} (a_{j,t} - 1).$$

Since $\sum_{(j,t)\in\mathcal{A}} a_{j,t} = \lambda_1$, we obtain the following proposition.

Proposition 4.1. *Let λ be an ℓ-partition. Then, the number of occurrences of "udu" in D_λ is $\ell - 2\sharp\mathcal{U} - \sharp\mathcal{L}$.*

To illustrate this, we could follow again the construction of the Dyck path which corresponds to $\lambda = (5,3,1,1,1,0,0)$. We first have the Dyck path D_3 in Section 3, with $n - \lambda_1 + 1 = 3$ peaks, and $u_3 = 2$. Then we use the word $M_2 = l^2dl^2d = l^{a_{2,0}}dl^{a_{2,1}}d$, where $a_{2,0}, a_{2,1} \in \mathcal{L}_2$, so we add $a_{2,0} - 2 + a_{2,1} - 2 = 0$ peak. So $u_2 = 2$. Then we use the word $M_1 = dl = l^{a_{1,0}}dl^{a_{1,1}}$, where $a_{1,1} \in \mathcal{U}_1$, so we add $a_{1,1} - 1 = 0$ peak. Hence, $u_1 = 2$.

5. Ad-nilpotent ideals of a parabolic subalgebra and Dyck paths

Let $I \subset \Pi$ and $\mathfrak{i}$ be an ad-nilpotent ideal of $\mathfrak{p}_I$. We set

$$\Phi_\mathfrak{i} = \{\alpha \in \Delta^+ \setminus \Delta_I; \ \mathfrak{g}_\alpha \subseteq \mathfrak{i}\}.$$

Then $\mathfrak{i} = \bigoplus_{\alpha\in\Phi_\mathfrak{i}} \mathfrak{g}_\alpha$ and if $\alpha \in \Phi_\mathfrak{i}$, $\beta \in \Delta^+ \cup \Delta_I$ are such that $\alpha+\beta \in \Delta^+$, then $\alpha+\beta \in \Phi_\mathfrak{i}$.

Conversely, set

$$\mathcal{F}_I = \{\Phi \subset \Delta^+\backslash\Delta_I; \text{if } \alpha \in \Phi, \beta \in \Delta^+\cup\Delta_I, \alpha+\beta \in \Delta^+, \text{then } \alpha+\beta \in \Phi\}.$$

Then for $\Phi \in \mathcal{F}_I$, $\mathfrak{i}_\Phi = \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ is an ad-nilpotent ideal of $\mathfrak{p}_I$.

We obtain therefore a bijection

$$\{\text{ad-nilpotent ideals of } \mathfrak{p}_I\} \to \mathcal{F}_I,\ \mathfrak{i} \mapsto \Phi_\mathfrak{i}.$$

Recall the following partial order on Δ^+: $\alpha < \beta$ if $\beta - \alpha$ is a sum of positive roots. Then it is easy to see that $\Phi \in \mathcal{F}_\emptyset$ if and only if for all $\alpha \in \Phi, \beta \in \Delta^+$, such that $\alpha < \beta$, then $\beta \in \Phi$.

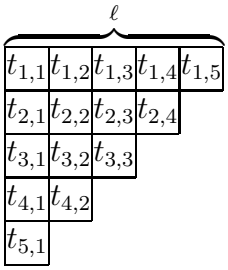
Let $\Phi \in \mathcal{F}_\emptyset$. Set

$$\Phi_{min} = \{\beta \in \Phi; \beta - \alpha \notin \Phi, \text{ for all } \alpha \in \Delta^+\}.$$

Then, Φ_{min} is an antichain of Δ^+ with respect to the above partial order. Conversely, if we consider an antichain Γ, then, the set of roots which are bigger than any one of the elements of Γ is an element of $\mathcal{F}_\emptyset$.

As in [CP], we display the positive roots Δ^+ in the Ferrers diagram T_ℓ of $(\ell, \ell-1, \dots, 1)$ as follows: we assign to each box in the i-th row and the j-th column, labelled (i,j) in T_ℓ, a positive root $t_{i,j} = \alpha_i + \cdots + \alpha_{\ell-j+1}$, $1 \leqslant i, j \leqslant \ell$.

For example, for $\ell = 5$, we have



Observe that given two positive roots α and β, α is bigger than or equal to β if the box corresponding to α is in the quadrant north-west of the box corresponding to β. It follows easily that the map which sends an element $\Phi \in \mathcal{F}_\emptyset$ to the subdiagram of T_ℓ consisting of the boxes corresponding to the roots of Φ defines a bijection between $\mathcal{F}_\emptyset$ and the set of northwest flushed subdiagrams of T_ℓ, i.e with the set of subdiagrams which contain the quadrant north-west of their boxes. Hence, by Section 2, we obtain a bijection σ from $\mathcal{F}_\emptyset$ to the set of ℓ-partitions.

By Proposition 3.1, $D \circ \sigma$ is a bijection from $\mathcal{F}_\emptyset$ to the set of Dyck paths of length $2\ell+2$.

For $\Phi \in \mathcal{F}_\emptyset$, set

$$I_\Phi = \{\alpha \in \Pi; \Phi \in \mathcal{F}_{\{\alpha\}}\}.$$

It is the maximal element of $\{I \subset \Pi; \Phi \in \mathcal{F}_I\}$ with respect to inclusion order. We shall see how to link the number of occurrences of "*udu*" of the Dyck path $(D \circ \sigma)(\Phi)$ and the cardinality of I_Φ.

Set $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j$, for all $1 \leqslant i \leqslant j \leqslant \ell$. We have easily the following lemma.

Lemma 5.1. *Let $I \subset \Pi$. An element $\Phi \in \mathcal{F}_\emptyset$ is an element of $\mathcal{F}_I$ if and only if for all $\alpha_{i,j} \in \Phi_{min}$, we have $\alpha_i, \alpha_j \notin I$.*

It follows from Lemma 5.1 that

$$I_\Phi = \Pi \setminus \{\alpha_i \in \Pi; \text{there exists } \alpha_{i,j} \text{ or } \alpha_{k,i} \in \Phi_{min}\}.$$

The problem is not to count the same root twice. For example, in A_7, for $\Phi_{min} = \{\alpha_{1,3}, \alpha_{2,5}, \alpha_{5,7}\}$, we have $\Pi \setminus I_\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_7\}$ but we find α_5 in the beginning or in the end of the support of two roots in Φ_{min}. So if we set

$$L = \{\alpha_{i,j} \in \Phi_{min};\ \text{there exists a root of shape } \alpha_{p,i} \in \Phi_{min}\},$$

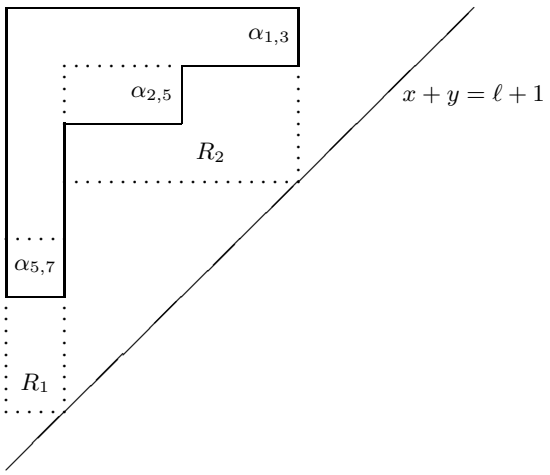
$$U = \Phi_{min} \setminus L,$$

we obtain that

$$\sharp I_\Phi = l - 2\sharp U - \sharp L. \tag{5}$$

Let $\lambda = \sigma(\Phi)$, F its Ferrers diagram and $D_\lambda = D(\lambda)$ be the Dyck path which corresponds to λ via the AKOP-bijection. Let $\alpha_{i,j} \in \Phi_{min}$. Then the cell $(i, \ell + 1 - j) = (i, \lambda_i)$ of $\alpha_{i,j}$ in F is a south-east corner of the diagram and two cases are possible: there exists a rectangle R_p such that $(i, \lambda_i) \in R_p$ or (i, λ_i) is not in any rectangle. If the latter case occurs, then $(i, \ell + 1 - j)$ is above a rectangle R_p. For example, if $\lambda = (5, 3, 1, 1, 1, 0, 0)$, we have that $\alpha_{2,5}, \alpha_{5,7}$ are in the first case and

$\alpha_{1,3}$ is in the second case.



If $\alpha_{i,j}$ is in the rectangle R_p, then the cell $(i,\lambda_i)=(i,\ell-j+1)$ which corresponds to $\alpha_{i,j}$ in F satisfies

$$\ell - i_{p+1} + 2 < i < \ell - i_p + 2, \tag{6}$$

$$i_{p-1} < \lambda_i \leqslant i_p, \tag{7}$$

and so we have

$$\ell - i_p + 1 \leqslant j < \ell - i_{p-1} + 1. \tag{8}$$

If $\alpha_{i,j}$ is above the rectangle R_p, then the cell $(i,\ell-j+1)$ which corresponds to $\alpha_{i,j}$ in F satisfies

$$(i,\ell-j+1) = (\ell - i_{p+1} + 2, i_p). \tag{9}$$

Define the map r from Φ_{min} to $\{1,\dots,k\}$ which associates to $\alpha_{i,j}$ the integer $r(\alpha_{i,j})=p$ such that $\alpha_{i,j}$ is in or immediately above the rectangle R_p.

Let $\alpha_{i,j}\in\Phi_{min}$ and $p=r(\alpha_{i,j})$. Since the cell $(i,\ell-j+1)$ which contains $\alpha_{i,j}$ in T_ℓ is a south-east corner, there is a horizontal line under this cell. If $c=(i,\ell-j+1)$ is in the rectangle R_p, then it is at the row $q=i-(\ell-i_{p+1}+2)$ of R_p and the line under c correspond to the part $l^{a_{p,q}}$ in M_p. Furthermore $(p,q)\in\mathcal{A}_p$.

If c is immediately above the rectangle R_p, then the line under c corresponds to $l^{a_{p,0}}$ in M_p and $(p,0)\in\mathcal{A}_p$. Since in this case, by (9) we have $(i,\ell-j+1)=(\ell-i_{p+1}+2,i_p)$, we obtain that $i-(\ell-i_{p+1}+2)=0$. We can define in any case the map s from Φ_{min} to $\mathbb{N}$ by

$$s(\alpha_{i,j}) = i - (\ell - i_{r(\alpha_{i,j})+1} + 2). \tag{10}$$

Furthermore, in both cases, the line under the cell which contains $\alpha_{i,j}$ is the part $l^{a_{r(\alpha_{i,j}),s(\alpha_{i,j})}}$ in $M_{r(\alpha_{i,j})}$ and $(r(\alpha_{i,j}),s(\alpha_{i,j}))\in\mathcal{A}_{r(\alpha_{i,j})}$.

Conversely, let $(p,q) \in \mathcal{A}_p$. Then, there is a horizontal line under the row $i = q - \ell - i_{p+1} + 2$ of F which is under a south-east corner of F. This south-east corner is a cell (i, λ_i) which corresponds to a root $\alpha_{i,j}$, where $\ell - j + 1 = \lambda_i$. So we have a bijection

$$\begin{array}{rccc} \Psi & : & \Phi_{min} & \to & \mathcal{A} \\ & & \alpha_{i,j} & \mapsto & (r(\alpha_{i,j}), s(\alpha_{i,j})). \end{array}$$

Lemma 5.2. *We have $\Psi(U) = \mathcal{U}$ and $\Psi(L) = \mathcal{L}$.*

Proof. Since $L = \Phi_{min} \setminus U$ and $\mathcal{L} = \mathcal{A} \setminus \mathcal{U}$, it suffices to prove that $\Psi(L) = \mathcal{L}$.

Let $\alpha_{i,j} \in L$. Set $p = r(\alpha_{i,j})$, $q = s(\alpha_{i,j})$ and let $c = (i, \lambda_i)$ be the cell which corresponds to $\alpha_{i,j}$ in F.

First assume that $i = j$. Then, we have $c = (i, \ell - i + 1)$. If $c \in R_p$, then by (6) and (8), we have

$$i = \ell - i_p + 1,$$

so by (10), we have that $q = i_{p+1} - i_p - 1$ so by (3), $a_{p,q} \in \mathcal{L}_p$.

If c is above R_p, then by (9), we have $c = (i, \ell - i + 1) = (\ell - i_{p+1} + 2, i_p)$, so $q = 0$ and $i_{p+1} - i_p = 1$, hence by (3) we also have $a_{p,q} \in \mathcal{L}_p$.

Now assume that $i \neq j$ and there exists a root of shape $\alpha_{m,i} \in \Phi_{min}$. Set $t = r(\alpha_{m,i})$. Let $(m, \lambda_m) = (m, \ell - i + 1)$ be the cell which corresponds to $\alpha_{m,i}$ in λ. If $c \in R_p$, then by (6), we have

$$i_p \leqslant \lambda_m \leqslant i_{p+1} - 2.$$

So either $(m, \lambda_m) \in R_{p+1}$ or $(m, \lambda_m) = (\ell - i_{p+1} + 2, i_p)$.

If $(m, \lambda_m) \in R_{p+1}$, then between the columns i_{p+1} and $\lambda_m = \ell - i + 1$, we have $i_{p+1} - (\ell - i + 1)$ columns, so there exists n such that $\sum_{u=0}^{n} a_{p+1,u} = i_{p+1} - (\ell - i + 1)$. Furthermore, by (10), we have $q = i - (\ell - i_{p+1} + 2)$, hence $a_{p,q} \in \mathcal{L}_p$.

If $(m, \lambda_m) = (\ell - i_{p+1} + 2, i_p)$, then $i = \ell - i_p + 1$ and by (10), we have that

$$q = (\ell - i_p + 1) - (\ell - i_{p+1} + 2) = i_{p+1} - i_p - 1.$$

Hence, by (3), we have $a_{p,q} \in \mathcal{L}_p$.

Conversely, let $a_{p,q} \in \mathcal{L}_p$, then there exists $0 \leqslant t \leqslant h_{p+1}$ such that $q + 1 = \sum_{f=0}^{t} a_{p+1,f}$. There also exists $\alpha_{i,j} \in \Phi_{min}$ such that $r(\alpha_{i,j}) = p$ and $s(\alpha_{i,j}) = q$. By (10), we have that

$$q = i - (\ell - i_{p+1} + 2).$$

Observe that for all $0 \leqslant j \leqslant h_{p+1}$, there exists a south-east corner (n_j, λ_{n_j}) in or above the rectangle R_{p+1} such that

$$\lambda_{n_j} = i_{p+1} - \sum_{f=0}^{j} a_{p+1,f}.$$

So there exists a south-east corner (n_j, λ_{n_j}) such that

$$\lambda_{n_j} = i_{p+1} - (q+1) = \ell - i + 1.$$

The element of Φ_{min} which corresponds to the cell (n_j, λ_{n_j}) is $\alpha_{n_j,i}$, so we have $\alpha_{i,j} \in L$. □

It follows by Proposition 4.1 and Equation (5) that we have the following theorem.

Theorem 5.3. *There is a bijection between the elements $\Phi \in \mathcal{F}_\emptyset$ such that $\sharp I_\Phi = r$ and the Dyck paths of length $2\ell+2$ having r occurrences of "udu".*

Since the number of Dyck paths having a fixed number of occurrences of *udu* is calculated in Theorem 2.1 of [Sun], we have the following corollary.

Corollary 5.4. *The number of elements of $\Phi \in \mathcal{F}_\emptyset$ such that $\sharp I_\Phi = r$ is*

$$\begin{pmatrix} \ell \\ r \end{pmatrix} \sum_{k=0}^{[\ell - r/2]} \begin{pmatrix} \ell - r \\ 2k \end{pmatrix} \mathcal{C}_k$$

where $\mathcal{C}_k$ denotes the k-th Catalan number.

Example 5.5. *Let N_r^ℓ be the number of elements $\Phi \in \mathcal{F}_\emptyset$ such that $\sharp I_\Phi = r$. We have by Corollary 5.4:*

r	N_r^1	N_r^2	N_r^3	N_r^4	N_r^5
0	1	2	4	9	21
1	1	2	6	16	45
2		1	3	12	40
3			1	4	20
4				1	5
5					1

6. Duality

We shall construct a duality between the elements of $\mathcal{F}_\emptyset$ such that $\sharp\Phi_{min} = p$ and those such that $\sharp\Phi_{min} = \ell - p$.

Proposition 6.1. *Let $\Phi \in \mathcal{F}_\emptyset$. Let N be the number of peaks in $(D \circ \sigma)(\Phi)$, then we have*

$$\sharp\Phi_{min} = \ell - (N-1).$$

Proof. Let $\lambda = \sigma(\Phi)$ be the corresponding ℓ-partition. Recall that the construction of $D(\lambda)$ is iterative. At each step, when we add $a_{p,q}$ peaks to a highest peak, for $(p,q) \in \mathcal{A}_p$, we also "destroy" the initial highest peak. So, we add only $a_{p,q} - 1$ peaks. At the end of the construction we have

$$\ell - \lambda_1 + 1 + \sum_{p=1}^{k} \sum_{(p,q)\in\mathcal{A}_p} (a_{p,q} - 1)$$

peaks. Since $\sum_{p=1}^{k} \sum_{(p,q)\in\mathcal{A}_p} a_{p,q} = \sum_{(p,q)\in\mathcal{A}} a_{p,q} = \lambda_1$ and $\mathcal{A}$ is in bijection with Φ_{min} by Section 5, we obtain the result. □

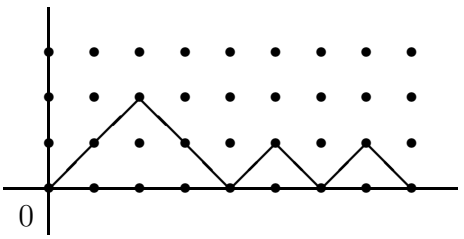
Proposition 6.2. *Let $\Phi \in \mathcal{F}_\emptyset$ and p be the number of peaks in $(P \circ \sigma)(\Phi)$, then we have*

$$\sharp\Phi_{min} = p - 1.$$

Proof. The result is clear by the construction of $(P \circ \sigma)(\Phi)$ defined in Section 2. □

Theorem 6.3. *The map $\sigma^{-1} \circ P^{-1} \circ D \circ \sigma$ induces a bijection from $\mathcal{F}_\emptyset$ to $\mathcal{F}_\emptyset$ which sends $\Phi \in \mathcal{F}_\emptyset$ such that $\sharp\Phi_{min} = p$ to $\Psi \in \mathcal{F}_\emptyset$ such that $\sharp\Psi_{min} = \ell - p$.*

For example, in $sl_4(\mathbb{C})$, the element $\Phi = \{\theta\} \in \mathcal{F}_\emptyset$ corresponds to the partition $\lambda = (1,0,0)$, and the Dyck path D_λ is:



Then, $P^{-1}(D_\lambda) = (3,2,0)$ which is the partition which corresponds to Ψ such that $\Psi_{min} = \{\alpha_1, \alpha_2\}$.

Remark 6.4. *It was proved in* [Pa] *that when $\mathfrak{g}$ is a simple Lie algebra of type A or C, the number of elements $\Phi \in \mathcal{F}_\emptyset$ such that $\sharp\Phi_{min} = p$ is the same as the number of elements $\Phi \in \mathcal{F}_\emptyset$ such that $\sharp\Phi_{min} = \ell - p$. But the duality of* [Pa] *is not the same as the one defined above. For example, in $sl_4(\mathbb{C})$, if we consider $\Phi = \{\theta\}$ like above, the dual ideal defined by* [Pa] *is Ψ where $\Psi_{min} = \{\alpha_1 + \alpha_2, \alpha_3\}$.*

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