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# GRÖBNER BASIS TECHNIQUES IN ALGEBRAIC COMBINATORICS

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Abstract. Gröbner basis techniques in the algebraic study of triangulations of convex polytopes as well as of the number of faces of simplicial complexes will be discussed. Of these two traditional topics in combinatorics, the first will be studied by using initial ideals of toric ideals and the second will be studied by using generic initial ideals of monomial ideals.

## Introduction

It turns out that, in the current trends on the study of convex polytopes and simplicial complexes, the algebraic techniques based on the theory of Gröbner bases could play fundamental roles. Especially, Gröbner basis techniques exert a great influence on the developments of the traditional topics in combinatorics such as triangulations of convex polytopes and the number of faces of simplicial complexes.

The present article will provide the reader with the way how to use Gröbner bases in algebraic combinatorics. No special knowledge on commutative algebra and algebraic combinatorics will be required.

First of all, in Section 1 we will present fundamental materials on Gröbner bases as quickly as possible. Our discussion will open with Dickson's lemma in classical combinatorics on monomials. In the language of commutative algebra, Dickson's lemma is equivalent to saying that every monomial ideal of the polynomial ring is finitely generated. Based on Dickson's lemma, the notion of initial ideals together with Gröbner bases will be introduced. In one word, a Gröbner basis of an ideal $I$ of the polynomial ring is a finite set of polynomials belonging to $I$ which enjoys certain distinguished algebraic properties. In the language of Gröbner bases, the Hilbert basis theorem is an immediate consequence of the fact that every Gröbner basis of an ideal $I$ is a system of generators of $I$. On the other hand, the Buchberger criterion gives an explicit answer to the problem when a system of generators of an ideal $I$ can be a Gröbner basis of $I$. Moreover, the Buchberger criterion supplies an algorithm to compute a Gröbner basis of an ideal $I$ starting from a system of generators of $I$.

The theory of toric ideals is one of the fascinating topics lying between commutative algebra and combinatorics (Sturmfels [24]). Section 2 will be devoted to the basic idea of the study of triangulations of convex polytopes by using initial ideals

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of toric ideals. Unimodular triangulations together with flag triangulations will explain the reason why the initial ideal generated by squarefree quadratic monomials would be of interest in combinatorics. Typical examples of initial ideals generated by squarefree quadratic monomials arise from finite distributive lattices [11], classical root systems [21], finite bipartite graphs [20], and decomposable models in algebraic statistics [6].

Algebraic shifting, introduced by Gil Kalai in 1988, is a powerful tool for the study of the number of faces of simplicial complexes. Kalai defined algebraic shifting in the framework of combinatorics. However, in Section 3, we will introduce algebraic shifting by using the notion of generic initial ideals. The generic initial ideal is one of the indispensable and fundamental tools for the developments of computational commutative algebra. Among many pending problems (Kalai [14] and Herzog [9]) related with algebraic shifting, one of the most fundamental problems is to compute the algebraic shifted complexes of simplicial spheres. A simplicial sphere is, by definition, a simplicial complex whose geometric realization is homeomorphic to the sphere. In particular the boundary complex of a simplicial polytope is a simplicial sphere. In [13] Kalai introduced the squeezed sphere and showed that the class of squeezed spheres is much bigger than that of boundary complexes of simplicial polytopes. Even though Kalai introduced the squeezed sphere in the language of combinatorics, following Murai [15] we will define the squeezed sphere by using strongly stable monomial ideals. Murai [16] succeeded in computing the algebraic shifted complexes of squeezed spheres. The final goal of Section 3 is to state this nice result due to Murai.

## 1. Gröbner bases

Let $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in $n$ variables over a field $K$ with $\deg x_i = 1$ for $i = 1, 2, \ldots, n$, and let

$$\mathrm{Mon}(S) = \{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} : a_i \in \mathbb{Z}_+, i = 1, 2, \ldots, n\},$$

be the set of monomials of $S$, where $\mathbb{Z}_+$ is the set of nonnegative integers. In particular $1 \in \mathrm{Mon}(S)$. For monomials $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $\mathbf{x}^{\mathbf{b}} = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ of $S$, we say that $\mathbf{x}^{\mathbf{b}}$ *divides* $\mathbf{x}^{\mathbf{a}}$ if $b_i \leq a_i$ for $i = 1, 2, \ldots, n$. We write $\mathbf{x}^{\mathbf{b}} \,|\, \mathbf{x}^{\mathbf{a}}$ if $\mathbf{x}^{\mathbf{b}}$ divides $\mathbf{x}^{\mathbf{a}}$. Let $\mathcal{M}$ be a nonempty subset of $\mathrm{Mon}(S)$. A monomial $\mathbf{x}^{\mathbf{a}} \in \mathcal{M}$ is said to be a *minimal element* of $\mathcal{M}$ with respect to divisibility if whenever $\mathbf{x}^{\mathbf{b}} \,|\, \mathbf{x}^{\mathbf{a}}$ with $\mathbf{x}^{\mathbf{b}} \in \mathcal{M}$, then $\mathbf{x}^{\mathbf{b}} = \mathbf{x}^{\mathbf{a}}$. Let $\mathcal{M}^{\min}$ denote the set of minimal elements of $\mathcal{M}$.

**Theorem 1.1** (Dickson's lemma)**.** *Let $\mathcal{M}$ be a nonempty subset of $\mathrm{Mon}(S)$. Then $\mathcal{M}^{\min}$ is a finite set.*

*Proof.* We prove Dickson's lemma by using induction on $n$, the number of variables of $S = K[x_1, x_2, \ldots, x_n]$. Let $n = 1$. If $d$ is the smallest integer for which $x_1^d \in \mathcal{M}$, then $\mathcal{M}^{\min} = \{x_1^d\}$. Thus $\mathcal{M}^{\min}$ is a finite set.

Let $n \geq 2$ and $B = K[\mathbf{x}] = K[x_1, x_2, \ldots, x_{n-1}]$. We use the notation $y$ instead of $x_n$. Thus $S = K[x_1, x_2, \ldots, x_{n-1}, y]$. Let $\mathcal{M}$ be a nonempty subset of $\mathrm{Mon}(S)$. Write $\mathcal{N}$ for the subset of $\mathrm{Mon}(B)$ which consists of those monomials $\mathbf{x}^{\mathbf{a}}$, where $\mathbf{a} \in \mathbb{Z}_+^{n-1}$, such that $\mathbf{x}^{\mathbf{a}} y^b \in \mathcal{M}$ for some $b \geq 0$. Our induction hypothesis says that

$\mathcal{N}^{\min}$ is a finite set. Let $\mathcal{N}^{\min} = \{u_1, u_2, \ldots, u_s\}$. By the definition of $\mathcal{N}$, for each $1 \leq i \leq s$, there is $b_i \geq 0$ with $u_i y^{b_i} \in \mathcal{M}$. Let $b = \max\{b_1, b_2, \ldots, b_s\}$. Now, for each $0 \leq \xi < b$, define the subset $\mathcal{N}_\xi$ of $\mathcal{N}$ to be

$$\mathcal{N}_\xi = \{\mathbf{x}^\mathbf{a} \in \mathcal{N} : \mathbf{x}^\mathbf{a} y^\xi \in \mathcal{M}\}.$$

Again, our induction hypothesis says that, for each $0 \leq \xi < b$, the set ${\mathcal{N}_\xi}^{\min}$ is finite. Let ${\mathcal{N}_\xi}^{\min} = \{u_1^{(\xi)}, u_2^{(\xi)}, \ldots, u_{s_\xi}^{(\xi)}\}$. We now show that each monomial belonging to $\mathcal{M}$ is divisible by one of the monomials which appear in the following list:

$$\begin{gathered} u_1 y^{b_1}, u_2 y^{b_2}, \ldots, u_s y^{b_s}, \\ u_1^{(0)}, u_2^{(0)}, \ldots, u_{s_0}^{(0)}, \\ u_1^{(1)} y, u_2^{(1)} y, \ldots, u_{s_1}^{(1)} y, \\ \cdots\cdots \\ u_1^{(b-1)} y^{b-1}, u_2^{(b-1)} y^{b-1}, \ldots, u_{s_{b-1}}^{(b-1)} y^{b-1}. \end{gathered}$$

In fact, since, for each monomial $w = \mathbf{x}^\mathbf{a} y^\gamma \in \mathcal{M}$ with $\mathbf{x}^\mathbf{a} \in \mathrm{Mon}(B)$, one has $\mathbf{x}^\mathbf{a} \in \mathcal{N}$, it follows that if $\gamma \geq b$, then $w$ is divisible by one of the monomials $u_1 y^{b_1}, u_2 y^{b_2}, \ldots, u_s y^{b_s}$, and that if $0 \leq \gamma < b$, then $w$ is divisible by one of the monomials $u_1^{(\gamma)} y^\gamma, u_2^{(\gamma)} y^\gamma, \ldots, u_{s_\gamma}^{(\gamma)} y^\gamma$. Clearly, the monomials listed above are in $\mathcal{M}$. Hence $\mathcal{M}^{\min}$ is a subset of the set of monomials listed above. Thus $\mathcal{M}^{\min}$ is finite, as desired. $\square$

A *monomial order* on $S$ is a total order $<$ on $\mathrm{Mon}(S)$ such that

- $1 < u$ for all $1 \neq u \in \mathrm{Mon}(S)$;
- if $u, v \in \mathrm{Mon}(S)$ and $u < v$, then $uw < vw$ for all $w \in \mathrm{Mon}(S)$.

**Example 1.2.** (a) Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be vectors belonging to $\mathbb{Z}_+^n$. We define the total order $<_{\text{lex}}$ on $\mathrm{Mon}(S)$ by setting $\mathbf{x}^\mathbf{a} <_{\text{lex}} \mathbf{x}^\mathbf{b}$ if either (i) $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$, or (ii) $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and the left-most nonzero component of the vector $\mathbf{a} - \mathbf{b}$ is negative. It follows that $<_{\text{lex}}$ is a monomial order on $S$, which is called the *lexicographic order* on $S$ induced by the ordering $x_1 > x_2 > \cdots > x_n$.

(b) Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be vectors belonging to $\mathbb{Z}_+^n$. We define the total order $<_{\text{rev}}$ on $\mathrm{Mon}(S)$ by setting $\mathbf{x}^\mathbf{a} <_{\text{rev}} \mathbf{x}^\mathbf{b}$ if either (i) $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$, or (ii) $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and the right-most nonzero component of the vector $\mathbf{a} - \mathbf{b}$ is positive. It follows that $<_{\text{rev}}$ is a monomial order on $S$, which is called the *reverse lexicographic order* on $S$ induced by the ordering $x_1 > x_2 > \cdots > x_n$.

For example, $x_2 x_3 <_{\text{lex}} x_1 x_4$ and $x_1 x_4 <_{\text{rev}} x_2 x_3$ in $K[x_1, x_2, x_3, x_4]$. Among the monomials of degree 2 of $K[x_1, x_2, x_3]$, one has

$$x_3^2 <_{\text{lex}} x_2 x_3 <_{\text{lex}} x_2^2 <_{\text{lex}} x_1 x_3 <_{\text{lex}} x_1 x_2 <_{\text{lex}} x_1^2$$

and

$$x_3^2 <_{\text{rev}} x_2 x_3 <_{\text{rev}} x_1 x_3 <_{\text{rev}} x_2^2 <_{\text{rev}} x_1 x_2 <_{\text{rev}} x_1^2.$$

**Exercise 1.3.** List the 10 monomials of degree 3 of $K[x_1, x_2, x_3]$ with respect to each of $<_{\text{lex}}$ and $<_{\text{rev}}$.

**Lemma 1.4.** *Let $<$ be a monomial order on $S$. Let $u, v \in \mathrm{Mon}(S)$ with $u \neq v$ and suppose that $u$ divides $v$. Then $u < v$.*

*Proof.* Write $v = uw$ with $w \in \mathrm{Mon}(S)$. Since $w \neq 1$, one has $1 < w$. Thus $1 \cdot u < w \cdot u$. Hence $u < v$, as desired. □

We will work with a fixed monomial order $<$ on $S$. Let $f = \sum_{u \in \mathrm{Mon}(S)} a_u u$ be a nonzero polynomial of $S$ with each $a_u \in K$. The *support* of $f$ is the finite set

$$\mathrm{supp}(f) = \{u \in \mathrm{Mon}(S) : a_u \neq 0\}.$$

The *initial monomial* of $f$ with respect to $<$ is the biggest monomial with respect to $<$ among the monomials belonging to $\mathrm{supp}(f)$.

Recall that an ideal of $S$ is a nonempty subset $I$ of $S$ such that

- if $f, g \in I$, then $f \pm g \in I$;
- if $f \in I$ and $h \in S$, then $fh \in I$.

Given a subset $\{f_\lambda\}_{\lambda \in \Lambda}$ of $S$, we write $(\{f_\lambda\}_{\lambda \in \Lambda})$ for the set of polynomials of the form $\sum_{\lambda \in \Lambda} h_\lambda f_\lambda$, where $\{\lambda \in \Lambda : h_\lambda \neq 0\}$ is finite. Then $(\{f_\lambda\}_{\lambda \in \Lambda})$ is an ideal of $S$, which is called the ideal of $S$ *generated by* $\{f_\lambda\}_{\lambda \in \Lambda}$. When $\Lambda$ is finite, say, $\Lambda = \{1, 2, \ldots, s\}$, we write $(f_1, f_2, \ldots, f_s)$ instead of $(\{f_1, f_2, \ldots, f_s\})$. Conversely, given an ideal $I$ of $S$, there exists a subset $(\{f_\lambda\}_{\lambda \in \Lambda})$ of $S$ with $I = (\{f_\lambda\}_{\lambda \in \Lambda})$. We call $\{f_\lambda\}_{\lambda \in \Lambda}$ *a system of generators* of $I$. We say that an ideal $I$ of $S$ is *finitely generated* if $I$ possesses a system of generators consisting of a finite number of polynomials. Later, we will see that every ideal of $S$ is finitely generated (Corollary 1.9).

A *monomial ideal* is an ideal which is generated by a set of monomials. Let $I \subset S$ be a monomial ideal. It follows that $I$ is generated by a subset $\mathcal{N} \subset \mathrm{Mon}(S)$ if and only if $(I \cap \mathrm{Mon}(S))^{\min} \subset \mathcal{N}$. Hence $(I \cap \mathrm{Mon}(S))^{\min}$ is a *unique* minimal system of monomial generators of $I$. Dickson's lemma guarantees that $(I \cap \mathrm{Mon}(S))^{\min}$ is finite. Thus in particular every monomial ideal is finitely generated.

Let $I$ be a nonzero ideal of $S$. The *initial ideal* of $I$ with respect to $<$ is the monomial ideal of $S$ which is generated by $\{\mathrm{in}_<(f) : 0 \neq f \in I\}$. We write $\mathrm{in}_<(I)$ for the initial ideal of $I$. Thus

$$\mathrm{in}_<(I) = (\{\mathrm{in}_<(f) : 0 \neq f \in I\}).$$

Since $(\mathrm{in}_<(I) \cap \mathrm{Mon}(S))^{\min}$ is a minimal system of monomial generators of $\mathrm{in}_<(I)$, and since $\mathrm{in}_<(I) \cap \mathrm{Mon}(S) = (\{\mathrm{in}_<(f) : 0 \neq f \in I\})$, there exists a finite number of nonzero polynomials $g_1, g_2, \ldots, g_s$ belonging to $I$ such that $\mathrm{in}_<(I)$ is generated by the set $\{\mathrm{in}_<(g_1), \mathrm{in}_<(g_2), \ldots, \mathrm{in}_<(g_s)\}$ of their initial monomials.

**Definition 1.5.** Let $I$ be a nonzero ideal of $S$. A finite set $\{g_1, g_2, \ldots, g_s\}$ of nonzero polynomials with each $g_i \in I$ is said to be a *Gröbner basis* of $I$ with respect to $<$ if the initial ideal $\mathrm{in}_<(I)$ of $I$ is generated by the set $\{\mathrm{in}_<(g_1), \mathrm{in}_<(g_2), \ldots, \mathrm{in}_<(g_s)\}$ of their initial monomials.

A Gröbner basis of $I$ with respect to $<$ exists. If $\mathcal{G}$ is a Gröbner basis of $I$ with respect to $<$, then every finite set $\mathcal{G}'$ with $\mathcal{G} \subset \mathcal{G}' \subset I$ is also a Gröbner basis of $I$ with respect to $<$. If $\mathcal{G} = \{g_1, \ldots, g_s\}$ is a Gröbner basis of $I$ with respect to $<$

and if $f_1, \ldots, f_s$ are nonzero polynomials belonging to $I$ with each $\mathrm{in}_<(f_i) = \mathrm{in}_<(g_i)$, then $\{f_1, \ldots, f_s\}$ is also a Gröbner basis of $I$ with respect to $<$.

**Example 1.6.** Let $S = K[x_1, x_2, \ldots, x_7]$ and $I = (f, g)$, where $f = x_1x_4 - x_2x_3$ and $g = x_4x_7 - x_5x_6$. Let $<_{\mathrm{lex}}$ the lexicographic order on $S$ induced by $x_1 > x_2 > \cdots > x_7$. One has $\mathrm{in}_{<_{\mathrm{lex}}}(f) = x_1x_4$ and $\mathrm{in}_{<_{\mathrm{lex}}}(g) = x_4x_7$. We claim that $\{f, g\}$ is not a Gröbner basis of $I$ with respect to $<_{\mathrm{lex}}$. In fact, the polynomial $h = x_7f - x_1g = x_1x_5x_6 - x_2x_3x_7$ belongs to $I$, but its initial monomial $\mathrm{in}_{<_{\mathrm{lex}}}(h) = x_1x_5x_6$ can be divided by neither $\mathrm{in}_{<_{\mathrm{lex}}}(f)$ nor $\mathrm{in}_{<_{\mathrm{lex}}}(g)$. Hence $\mathrm{in}_{<_{\mathrm{lex}}}(h) \notin (\mathrm{in}_{<_{\mathrm{lex}}}(f), \mathrm{in}_{<_{\mathrm{lex}}}(g))$. Thus $\mathrm{in}_{<_{\mathrm{lex}}}(I) \neq (\mathrm{in}_{<_{\mathrm{lex}}}(f), \mathrm{in}_{<_{\mathrm{lex}}}(g))$. In other words, $\{f, g\}$ is not a Gröbner basis of $I$ with respect to $<_{\mathrm{lex}}$. Later, we will show that $\{f, g, h\}$ is a Gröbner basis of $I$ with respect to $<_{\mathrm{lex}}$ (Example 1.16).

**Lemma 1.7.** *Let $<$ be a monomial order on $S = K[x_1, \ldots, x_n]$. Then, for any monomial $u$ of $S$, there is no infinite descending sequence of the form*

$$u = u_0 > u_1 > u_2 > \cdots . \tag{1}$$

*Proof.* Suppose, on the contrary, that one has an infinite descending sequence (1) and write $\mathcal{M}$ for the set of monomials $\{u_0, u_1, u_2, \ldots\}$. It follows from Dickson's lemma that $\mathcal{M}^{\min}$ is a finite set, say $\mathcal{M}^{\min} = \{u_{i_1}, u_{i_2}, \ldots, u_{i_s}\}$ with $i_1 < i_2 < \cdots < i_s$. Then the monomial $u_{i_s+1}$ is divided by $u_{i_j}$ for some $1 \leq j \leq s$. Thus by Lemma 1.4 one has $u_{i_j} < u_{i_s+1}$, which contradicts $i_j < i_s + 1$. □

**Theorem 1.8.** *Let $I$ be a nonzero ideal of $S = K[x_1, \ldots, x_n]$ and $\mathcal{G} = \{g_1, \ldots, g_s\}$ a Gröbner basis of $I$ with respect to a monomial order $<$ on $S$. Then $I = (g_1, \ldots, g_s)$. In other words, every Gröbner basis of $I$ is a system of generators of $I$.*

*Proof.* (Gordan) Let $0 \neq f \in I$. Since $\mathrm{in}_<(f) \in \mathrm{in}_<(I)$ and since $\mathcal{G}$ is a Gröbner basis of $I$, i.e., $\mathrm{in}_<(I) = (\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s))$, it follows that there is $g_{i_0}$ such that $\mathrm{in}_<(g_{i_0})$ divides $\mathrm{in}_<(f)$. Let $\mathrm{in}_<(f) = w_0 \, \mathrm{in}_<(g_{i_0})$ with $w_0 \in \mathrm{Mon}(S)$. Let $h_0 = f - c_{i_0}^{-1}c_0w_0g_{i_0}$, where $c_0$ is the coefficient of $\mathrm{in}_<(f)$ in $f$ and where $c_{i_0}$ is the coefficient of $\mathrm{in}_<(g_{i_0})$ in $g_{i_0}$. Then $h_0 \in I$. Since $\mathrm{in}_<(w_0g_{i_0}) = w_0 \, \mathrm{in}_<(g_{i_0})$ it follows that $\mathrm{in}_<(h_0) < \mathrm{in}_<(f)$. If $h_0 = 0$, then $f \in (g_1, \ldots, g_s)$.

Let $h_0 \neq 0$. Then the same technique as we used for $f$ can be applied for $h_0$. Thus $h_1 = f - c_{i_1}^{-1}c_1w_1g_{i_1} - c_{i_0}^{-1}c_0w_0g_{i_0}$, where $c_1$ is the coefficient of $\mathrm{in}_<(h_0)$ in $h_0$ and where $c_{i_1}$ is the coefficient of $\mathrm{in}_<(g_{i_1})$ in $g_{i_1}$. Then $h_1 \in I$ and $\mathrm{in}_<(h_1) < \mathrm{in}_<(h_0)$. If $h_1 = 0$, then $f \in (g_1, \ldots, g_s)$.

If $h_1 \neq 0$, then we proceed as before. Lemma 1.7 guarantees that this procedure must terminate. Thus we obtain an expression of the form $f = \sum_{q=0}^{N} c_{i_q}^{-1}c_qw_qg_{i_q}$. In particular, $f$ belongs to $(g_1, g_2, \ldots, g_s)$. Thus $I = (g_1, g_2, \ldots, g_s)$, as desired. □

**Corollary 1.9** (Hilbert basis theorem)**.** *Every ideal of the polynomial ring $S = K[x_1, \ldots, x_n]$ is finitely generated.*

It is natural to ask if the converse of Theorem 1.8 is true or false. That is to say, if $I = (f_1, f_2, \ldots, f_s)$ is an ideal of $S = K[x_1, \ldots, x_n]$, then does there exist a monomial order $<$ on $S$ such that $\{f_1, f_2, \ldots, f_s\}$ is a Gröbner basis of $I$ with respect to $<$ ?

**Example 1.10** ([18])**.** Let $S = K[x_1, x_2, \ldots, x_{10}]$ and $I$ the ideal of $S$ generated by

$$f_1 = x_1x_8 - x_2x_6, \quad f_2 = x_2x_9 - x_3x_7, \quad f_3 = x_3x_{10} - x_4x_8,$$
$$f_4 = x_4x_6 - x_5x_9, \quad f_5 = x_5x_7 - x_1x_{10}.$$

We claim that there exists *no* monomial order $<$ on $S$ such that $\{f_1, \ldots, f_5\}$ is a Gröbner basis of $I$ with respect to $<$.

Suppose, on the contrary, that there exists a monomial order $<$ on $S$ such that $\mathcal{G} = \{f_1, \ldots, f_5\}$ is a Gröbner basis of $I$ with respect to $<$. First, note that each of the five polynomials

$$x_1x_8x_9 - x_3x_6x_7,\ x_2x_9x_{10} - x_4x_7x_8,\ x_2x_6x_{10} - x_5x_7x_8,$$
$$x_3x_6x_{10} - x_5x_8x_9,\ x_1x_9x_{10} - x_4x_6x_7$$

belongs to $I$. Let, say, $x_1x_8x_9 > x_3x_6x_7$. Since $x_1x_8x_9 \in \mathrm{in}_<(I)$, there is $g \in \mathcal{G}$ such that $\mathrm{in}_<(g)$ divides $x_1x_8x_9$. Such $g \in \mathcal{G}$ must be $f_1$. Hence $x_1x_8 > x_2x_6$. Thus $x_2x_6 \notin \mathrm{in}_<(I)$. Hence there exists no $g \in \mathcal{G}$ such that $\mathrm{in}_<(g)$ divides $x_2x_6x_{10}$. Hence $x_2x_6x_{10} < x_5x_7x_8$. Thus $x_5x_7 > x_1x_{10}$. Continuing these arguments, we obtain

$$x_1x_8x_9 > x_3x_6x_7,\ \ x_2x_9x_{10} > x_4x_7x_8,\ \ x_2x_6x_{10} < x_5x_7x_8,$$
$$x_3x_6x_{10} > x_5x_8x_9,\ \ x_1x_9x_{10} < x_4x_6x_7$$

and

$$x_1x_8 > x_2x_6,\ \ x_2x_9 > x_3x_7,\ \ x_3x_{10} > x_4x_8,$$
$$x_4x_6 > x_5x_9,\ \ x_5x_7 > x_1x_{10}.$$

Hence

$$(x_1x_8)(x_2x_9)(x_3x_{10})(x_4x_6)(x_5x_7) > (x_2x_6)(x_3x_7)(x_4x_8)(x_5x_9)(x_1x_{10}). \tag{2}$$

The opposite relation in (2) occurs in case of $x_1x_8x_9 < x_3x_6x_7$. However, both sides of the inequality (2) coincide with $x_1x_2\cdots x_{10}$.

In high school mathematics, we learn that, given polynomials $f$ and $g \neq 0$ in one variable $x$, there exist unique polynomials $q$ and $r$ such that $f = gq+r$, where either $r = 0$ or $\deg r < \deg g$. The division algorithm generalizes this well-known result.

**Theorem 1.11** (Division algorithm)**.** *Let $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in $n$ variables over a field $K$ and fix a monomial order $<$ on $S$. Let $g_1, g_2, \ldots, g_s$ be nonzero polynomials of $S$. Then, given a polynomial $0 \neq f \in S$, there exist polynomials $f_1, f_2, \ldots, f_s$ and $f'$ of $S$ with*

$$f = f_1g_1 + f_2g_2 + \cdots + f_sg_s + f' \tag{3}$$

*such that the following conditions are satisfied:*

(i) *if $f' \neq 0$ and if $u \in \mathrm{supp}(f')$, then none of $\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s)$ divides $u$, i.e., no $u \in \mathrm{supp}(f')$ belongs to $(\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s))$;*

(ii) *if $f_i \neq 0$, then*

$$\mathrm{in}_<(f_ig_i) \leq \mathrm{in}_<(f).$$

The right hand side of equation (3) is said to be a *standard expression* for $f$ with respect to $g_1, g_2, \ldots, g_s$, and the polynomial $f'$ is called a *remainder* of $f$ with respect to $g_1, g_2, \ldots, g_s$.

Instead of giving a detailed proof of Theorem 1.11, we discuss a typical example which clearly explains the procedure to obtain a standard expression.

**Example 1.12.** Let $<_{\text{lex}}$ denote the lexicographic order on $S = K[x, y, z]$ induced by $x > y > z$. Let $g_1 = x^2 - z, g_2 = xy - 1$ and $f = x^3 - x^2y - x^2 - 1$. Each of

$$\begin{aligned}
f &= x^3 - x^2y - x^2 - 1 = x(g_1 + z) - x^2y - x^2 - 1 \\
&= xg_1 - x^2y - x^2 + xz - 1 = xg_1 - (g_1 + z)y - x^2 + xz - 1 \\
&= xg_1 - yg_1 - x^2 + xz - yz - 1 = xg_1 - yg_1 - (g_1 + z) + xz - yz - 1 \\
&= (x - y - 1)g_1 + (xz - yz - z - 1)
\end{aligned}$$

and

$$\begin{aligned}
f &= x^3 - x^2y - x^2 - 1 = x(g_1 + z) - x^2y - x^2 - 1 \\
&= xg_1 - x^2y - x^2 + xz - 1 = xg_1 - x(g_2 + 1) - x^2 + xz - 1 \\
&= xg_1 - xg_2 - x^2 + xz - x - 1 = xg_1 - xg_2 - (g_1 + z) + xz - x - 1 \\
&= (x - 1)g_1 - xg_2 + (xz - x - z - 1)
\end{aligned}$$

is a standard expression of $f$ with respect to $g_1$ and $g_2$, and each of $xz - yz - z - 1$ and $xz - x - z - 1$ is a remainder of $f$.

Example 1.12 says that a remainder of a nonzero polynomial may not be unique. However, we have the following fact.

**Lemma 1.13.** *If $\mathcal{G} = \{g_1, \ldots, g_s\}$ is a Gröbner basis of $I = (g_1, \ldots, g_s)$, then for any nonzero polynomial $f$ of $S$, there is a unique remainder of $f$ with respect to $g_1, \ldots, g_s$.*

*Proof.* Suppose there exist remainders $f'$ and $f''$ with respect to $g_1, \ldots, g_s$ with $f' \neq f''$. Since $0 \neq f' - f'' \in I$, the initial monomial $w = \text{in}_<(f' - f'')$ must belong to $\text{in}_<(I)$. However, since $w \in \text{supp}(f') \cup \text{supp}(f'')$, none of the monomials $\text{in}_<(g_1), \ldots, \text{in}_<(g_s)$ divides $w$. Hence $\text{in}_<(I) \neq (\text{in}_<(g_1), \ldots, \text{in}_<(g_s))$. $\square$

Given nonzero polynomials $f$ and $g$ of $S$, the notation $\text{lcm}(\text{in}_<(f), \text{in}_<(g))$ stands for the least common multiple of $\text{in}_<(f)$ and $\text{in}_<(g)$. Let $c_f$ denote the coefficient of $\text{in}_<(f)$ in $f$ and $c_g$ the coefficient of $\text{in}_<(g)$ in $g$. The polynomial

$$S(f, g) = \frac{\text{lcm}(\text{in}_<(f), \text{in}_<(g))}{c_f \, \text{in}_<(f)} f - \frac{\text{lcm}(\text{in}_<(f), \text{in}_<(g))}{c_g \, \text{in}_<(g)} g$$

is called the *$S$-polynomial* of $f$ and $g$.

We say that $f$ *has remainder* 0 with respect to $g_1, g_2, \ldots, g_s$ if, in the division algorithm, there is a standard expression (3) of $f$ with respect to $g_1, g_2, \ldots, g_s$ with $f' = 0$.

**Lemma 1.14.** *Let $f$ and $g$ be nonzero polynomials and suppose that* $\text{in}_<(f)$ *and* $\text{in}_<(g)$ *are relatively prime, i.e.,* $\text{lcm}(\text{in}_<(f), \text{in}_<(g)) = \text{in}_<(f)\,\text{in}_<(g)$. *Then $S(f, g)$ has remainder* 0 *with respect to $f, g$.*

*Proof.* To simplify notation we will assume that each of the coefficients of $\text{in}_<(f)$ in $f$ and $\text{in}_<(g)$ in $g$ is equal to 1. Let $f = \text{in}_<(f) + f_1$ and $g = \text{in}_<(g) + g_1$. Since

$\text{in}_<(f)$ and $\text{in}_<(g)$ are relatively prime, it follows that

$$\begin{aligned} S(f,g) &= \text{in}_<(g)f - \text{in}_<(f)g \\ &= (g-g_1)f - (f-f_1)g \\ &= f_1 g - g_1 f. \end{aligned}$$

We claim $(\text{in}_<(f_1)\,\text{in}_<(g) =)\,\text{in}_<(f_1 g) \neq \text{in}_<(g_1 f)\ (= \text{in}_<(g_1)\,\text{in}_<(f))$. In fact, if $\text{in}_<(f_1)\,\text{in}_<(g) = \text{in}_<(g_1)\,\text{in}_<(f)$, then, since $\text{in}_<(f)$ and $\text{in}_<(g)$ are relatively prime, it follows that $\text{in}_<(f)$ must divide $\text{in}_<(f_1)$. However, since $\text{in}_<(f_1) < \text{in}_<(f)$, this is impossible. Let, say, $\text{in}_<(f_1)\,\text{in}_<(g) < \text{in}_<(g_1)\,\text{in}_<(f)$. Then $\text{in}_<(S(f,g)) = \text{in}_<(g_1 f)$ and $S(f,g) = f_1 g - g_1 f$ turns out to be a standard expression of $S(f,g)$ in terms of $f$ and $g$. Hence $S(f,g)$ has remainder 0 with respect to $f$ and $g$, and similarly for $\text{in}_<(g_1)\,\text{in}_<(f) < \text{in}_<(f_1)\,\text{in}_<(g)$. $\square$

We now come to the most fundamental theorem in the theory of Gröbner bases.

**Theorem 1.15** (Buchberger criterion)**.** *Let $I$ be a nonzero ideal of $S$ and $\mathcal{G} = \{g_1, g_2, \ldots, g_s\}$ a system of generators of $I$. Then $\mathcal{G}$ is a Gröbner basis of $I$ if and only if the following condition is satisfied:*

($*$) *For all $i \neq j$, $S(g_i, g_j)$ has remainder 0 with respect to $g_1, \ldots, g_s$.*

We refer the reader to a standard textbook on Gröbner bases, e.g., [1], [4] and [5] for a proof of the Buchberger criterion. However, for a (general) Gröbner basis "user," it may not be required to understand a detailed proof of the Buchberger criterion.

In Example 1.6, by using Lemma 1.14 together with the Buchberger criterion, it follows immediately that the set $\{f, g\}$ is a Gröbner basis of $I = (f, g)$ with respect to the reverse lexicographic order $<_{\text{rev}}$ induced by $x_1 > x_2 > \cdots > x_7$.

The Buchberger criterion supplies an algorithm to compute a Gröbner basis starting from a system of generators of an ideal.

Let $\{g_1, g_2, \ldots, g_s\}$ be a system of generators of a nonzero ideal $I$ of $S$ and suppose that $\{g_1, g_2, \ldots, g_s\}$ is *not* a Gröbner basis of $I$. The Buchberger criterion then guarantees that there is an $S$-polynomial $S(g_i, g_j)$ such that *no* remainder of $S(g_i, g_j)$ with respect to $g_1, g_2, \ldots, g_s$ is 0. Let $h_{ij} \in I$ be a remainder of a standard expression of $S(g_i, g_j)$ with respect to $g_1, g_2, \ldots, g_s$. Then $\text{in}_<(h_{ij})$ can be divided by none of the monomials $\text{in}_<(g_1), \text{in}_<(g_2), \ldots, \text{in}_<(g_s)$. In other words, the inclusion

$$(\text{in}_<(g_1), \text{in}_<(g_2), \ldots, \text{in}_<(g_s)) \subset (\text{in}_<(g_1), \text{in}_<(g_2), \ldots, \text{in}_<(g_s), \text{in}_<(h_{ij})).$$

is strict. With setting $g_{s+1} = h_{ij}$, suppose that $\{g_1, g_2, \ldots, g_s, g_{s+1}\}$ is not a Gröbner basis of $I$. Again, by using the Buchberger criterion, there is a $S$-polynomial $S(g_k, g_\ell)$ such that no remainder of $S(g_k, g_\ell)$ with respect to $g_1, g_2, \ldots, g_s, g_{s+1}$ is 0. Let $h_{k\ell} \in I$ be a remainder of $S(g_k, g_\ell)$ with respect to $g_1, g_2, \ldots, g_s, g_{s+1}$. Then the inclusion

$$\begin{aligned} &(\text{in}_<(g_1), \text{in}_<(g_2), \ldots, \text{in}_<(g_s), \text{in}_<(g_{s+1})) \\ &\qquad \subset (\text{in}_<(g_1), \text{in}_<(g_2), \ldots, \text{in}_<(g_s), \text{in}_<(g_{s+1}), \text{in}_<(h_{k\ell})). \end{aligned}$$

is strict. By virtue of Dickson's lemma, these procedures must terminate after a finite number of steps, and a Gröbner basis of $I$ can be obtained.

The above algorithm to find a Gröbner basis starting from a system of generators of an ideal is said to be the *Buchberger algorithm*.

**Example 1.16.** We continue Example 1.6. Let $S = K[x_1, x_2, \ldots, x_7]$ and $<_{\text{lex}}$ the lexicographic order on $S$ induced by $x_1 > x_2 > \cdots > x_7$. Let $f = x_1x_4 - x_2x_3$ and $g = x_4x_7 - x_5x_6$. Thus $\text{in}_{<_{\text{lex}}}(f) = x_1x_4$ and $\text{in}_{<_{\text{lex}}}(g) = x_4x_7$. Let $I = (f, g)$. Then $\{f, g\}$ is not a Gröbner basis of $I$ with respect to $<_{\text{lex}}$. Now, as a remainder of $S(f, g) = x_7f - x_1g = x_1x_5x_6 - x_2x_3x_7$ with respect to $f$ and $g$, we choose $S(f, g)$ itself. Let $h = x_1x_5x_6 - x_2x_3x_7$ with $\text{in}_{<_{\text{lex}}}(h) = x_1x_5x_6$. Then $\text{in}_{<_{\text{lex}}}(g)$ and $\text{in}_{<_{\text{lex}}}(h)$ are relatively prime. On the other hand, $S(f, h) = x_2x_3(x_4x_7 - x_5x_6)$ has remainder 0 with respect to $f, g, h$. It follows from the Buchberger criterion that $\{f, g, h\}$ is a Gröbner basis of $I$ with respect to $<_{\text{lex}}$.

## 2. Initial ideals and triangulations of convex polytopes

Let $A = (a_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq n}} \in \mathbb{Z}^{d \times n}$, i.e., $A$ is a $d \times n$ integer matrix, with $\mathbf{a}_1, \ldots, \mathbf{a}_n$ their column vectors. Such a matrix $A$ is called a *configuration* if there exists a vector $\mathbf{c} \in \mathbb{R}^d$ such that $\langle \mathbf{a}_j, \mathbf{c} \rangle = 1$ for each $1 \leq j \leq n$, where $\langle \mathbf{a}_j, \mathbf{c} \rangle$ is the usual inner product in $\mathbb{R}^d$.

Let, as before, $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in $n$ variables over a field $K$ with $\deg x_i = 1$ for $i = 1, 2, \ldots, n$. If $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{Z}^n$, then we define the binomial

$$f_{\mathbf{b}} = \prod_{b_i > 0} x_i^{b_i} - \prod_{b_i < 0} x_i^{-b_i}$$

of $S$. For example, if $n = 5$ and $\mathbf{b} = (-1, 3, 2, 0, -4)$, then $f_{\mathbf{b}} = x_2^3x_3^2 - x_1x_5^4$.

The *toric ideal* of a configuration $A \in \mathbb{Z}^{d \times n}$ is the ideal $I_A$ of $S$ which is generated by those binomials $f_{\mathbf{b}}$ with $A\,\mathbf{b}^\top = 0$, where $\mathbf{b} \in \mathbb{Z}^n$ and where $\mathbf{b}^\top$ stands for the transposed matrix of $\mathbf{b}$. Thus

$$I_A = (\{\, f_{\mathbf{b}} \,:\, \mathbf{b} \in \mathbb{Z}^n,\ A\,\mathbf{b}^\top = 0 \,\}).$$

The convex polytope arising from a configuration $A \in \mathbb{Z}^{d \times n}$ is the convex polytope $\mathcal{P}_A \subset \mathbb{R}^d$ which is the convex hull of the column vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of $A$.

Fix a monomial order $<$ on $S$ and write, as before, $\text{in}_<(I_A)$ for the initial ideal of $I_A$ with respect to $<$.

**Example 2.1.** The $4 \times 5$ matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

is a configuration, since $(0, 0, 0, 1)\,A = (1, 1, 1, 1)$. The toric ideal $I_A$ is generated by the binomial $x_1x_5^2 - x_2x_3x_4$ and the convex polytope $\mathcal{P}_A \subset \mathbb{R}^4$ is a bipyramid on a triangle. One has either $\text{in}_<(I_A) = (x_1x_5^2)$ or $\text{in}_<(I_A) = (x_2x_3x_4)$.

We now discuss the triangulations of the convex polytope $\mathcal{P}_A \subset \mathbb{R}^d$ arising from a configuration $A \in \mathbb{Z}^{d \times n}$. Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be the column vectors of $A$. To simplify the situation, we may assume that the additive group $\mathbb{Z}^d$ is generated by the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ of column vectors. In other words, every vector belonging to $\mathbb{Z}^d$ is of the form $c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n$ with each $c_j \in \mathbb{Z}$. In particular, since $A$ is a configuration, it follows that the dimension of $\mathcal{P}_A$ is $d-1$.

When $\mathbb{Z}^d$ cannot be generated by the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ of column vectors of $A$, write $\mathcal{L}$ for the subgroup of $\mathbb{Z}^d$ generated by the set of column vectors of $A$. Thus $\mathcal{L} \simeq \mathbb{Z}^e$ with $e < d$. Let $\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_e}\}$ with $1 \leq i_1 < \cdots < i_e \leq n$ be a $\mathbb{Z}$-basis of $\mathcal{L}$. Each vector $\mathbf{a} \in \mathcal{L}$ has a unique expression of the form $\mathbf{a} = \sum_{j=1}^{e} q_j \mathbf{a}_{i_j}$ with each $q_j \in \mathbb{Z}$. We then introduce the $\mathbb{Z}$-isomorphism $\psi : \mathcal{L} \to \mathbb{Z}^e$ by setting $\psi(\mathbf{a}) = (q_1, \ldots, q_e)^\top$. Let $A' = (a'_{ij})_{\substack{1\leq i\leq e\\1\leq j\leq n}}$ denote the $e \times n$ integer matrix with the column vectors $\varphi(\mathbf{a}_1), \ldots, \varphi(\mathbf{a}_n)$. Since $A$ is a configuration, it follows that $\sum_{i=1}^{e} a'_{ij} = 1$ for each $1 \leq j \leq n$. Thus $A' \in \mathbb{Z}^{e\times n}$ is a configuration and the set of columns of $A'$ generates $\mathbb{Z}^e$. Clearly discussing the toric ideal and the triangulations of $A$ is equivalent to discussing the toric ideal and the triangulations of $A'$.

**Example 2.2.** The $3 \times 5$ matrix

$$A = \begin{bmatrix} 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

is a configuration. The additive group $\mathcal{L}$ generated by the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_5\}$ of column vectors of $A$ is $\mathcal{L} = \{(a_1, a_2, a_3) \in \mathbb{Z}^3 : a_1 + a_2 \in 2 \cdot \mathbb{Z}\}$, where $2 \cdot \mathbb{Z}$ is the set of even integers, with $\{\mathbf{a}_1, \mathbf{a}_4, \mathbf{a}_5\}$ its $\mathbb{Z}$-basis. One has

$$A' = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix}.$$

Given a subset $F = \{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_r}\}$ of $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, the notation $\mathrm{conv}(F)$ stands for the convex hull of $\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_r}$ in $\mathbb{R}^d$. A subset $F$ of $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is called a *simplex belonging to* $A$ if $\mathrm{conv}(F)$ is an $(r-1)$-simplex in $\mathbb{R}^d$, where $r = |F|$. Every subset of a simplex belonging to $A$ is again a simplex belonging to $A$. A simplex $F = \{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_r}\}$ belonging to $A$ is called *unimodular* if $F$ is a subset of a $\mathbb{Z}$-basis of $\mathbb{Z}^d$. If $F$ is unimodular and if $F' \subset F$, then $F'$ is again unimodular.

A *triangulation* of $\mathcal{P}_A \subset \mathbb{R}^d$ is a collection $\Delta$ of simplices belonging to $A$ such that

- If $F$ belongs to $\Delta$ and $F' \subset F$, then $F'$ belongs to $\Delta$;
- If $F$ and $G$ belong to $\Delta$, then $\mathrm{conv}(F) \cap \mathrm{conv}(G) = \mathrm{conv}(F \cap G)$;
- $\mathcal{P}_A = \bigcup_{F\in\Delta} \mathrm{conv}(F)$.

Each simplex $F \in \Delta$ is called a *face* of $\Delta$. A *facet* of $\Delta$ is a face $F$ of $\Delta$ with $|F| = d$. A triangulation $\Delta$ of $\mathcal{P}_A$ is called *unimodular* if every $F \in \Delta$ is unimodular. A simplex $F$ belonging to $A$ is called a *nonface* of $\Delta$ if $F \notin \Delta$. A triangulation $\Delta$ of $\mathcal{P}_A$ is called *flag* if every minimal nonface of $\Delta$ is a 2-element subset of $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$.

Why is a unimodular triangulation of interest in combinatorics? Let $\mathbf{c} \in \mathbb{R}^d$ with $\langle \mathbf{a}_j, \mathbf{c} \rangle = 1$ for each $j$ and $\mathcal{H} \subset \mathbb{R}^d$ the hyperplane $\{\mathbf{a} \in \mathbb{R}^d : \langle \mathbf{a}, \mathbf{c} \rangle = 1\}$. One has $\mathcal{P}_A \subset \mathcal{H}$. Fix an invertible affine transformation $\varrho : \mathcal{H} \to \mathbb{R}^{d-1}$ with $\varrho(\mathbb{Z}^d \cap \mathcal{H}) = \mathbb{Z}^{d-1}$. The image $\varrho(\mathcal{P}_A)$ of $\mathcal{P}_A$ under $\varrho$ is a convex polytope in $\mathbb{R}^{d-1}$

of dimension $d-1$. Hence the volume (Lebesgue measure) $\mathrm{vol}(\varrho(\mathcal{P}_A))$ of $\varrho(\mathcal{P}_A)$ is positive and is independent of the particular choice of $\varrho$. We call the positive integer $(d-1)!\,\mathrm{vol}(\varrho(\mathcal{P}_A))$ the *normalized volume* of $\mathcal{P}_A$. If $\Delta$ is a triangulation of $\mathcal{P}_A$, then it follows that the number of the facets of $\Delta$ is at most the normalized volume of $\mathcal{P}_A$ and is equal to the normalized volume of $\mathcal{P}_A$ if and only if $\Delta$ is unimodular. Consequently, the existence of a unimodular triangulation of $\mathcal{P}_A$ makes it easy to compute the normalized volume of $\mathcal{P}_A$.

**Remark 2.3.** A combinatorial way to see the reason why $\mathrm{vol}(\varrho(\mathcal{P}_A))$ is independent of the particular choice of $\varrho$ is to show that the *Ehrhart polynomial* [10, p. 80] of $\mathcal{P}_A$ coincides with that of $\varrho(\mathcal{P}_A)$. It is well known, e.g., [23, Theorem 10.3, p. 45] and [10, Proposition (28.5)] that the leading coefficient of the Ehrhart polynomial of $\varrho(\mathcal{P}_A)$ is $\mathrm{vol}(\varrho(\mathcal{P}_A))$.

Why is a flag triangulation of interest in combinatorics? Given a triangulation $\Delta$ of $\mathcal{P}_A$, we write $G_\Delta$ for the finite graph on $[n]=\{1,\ldots,n\}$ whose edges, $E(G_\Delta)$, are those $\{i,j\}$, where $i\neq j$, with $\{\mathbf{a}_i,\mathbf{a}_j\}\in\Delta$. We call $G_\Delta$ the *skeleton* of $\Delta$, and we denote the set of edges by $E(G_\Delta)$, as usual. A *clique* of $G_\Delta$ is a subset $C\subset[n]$ such that $\{i,j\}\in E(G_\Delta)$ for all $i,j\in C$ with $i\neq j$. Let $\mathcal{C}(G_\Delta)$ denote the set of cliques of $G_\Delta$. It then follows that $\Delta\subset\{\{\mathbf{a}_i : i\in C\} : C\in\mathcal{C}(G_\Delta)\}$ and that $\Delta=\{\{\mathbf{a}_i : i\in C\} : C\in\mathcal{C}(G_\Delta)\}$ if and only if $\Delta$ is flag. Consequently, when $\Delta$ is flag, we can discuss the combinatorics on $\Delta$ (for example, the number of faces of $\Delta$) in terms of the combinatorics on the skeleton $G_\Delta$ of $\Delta$.

**Example 2.4.** Let $A$ be the configuration discussed in Example 2.1. The simplices belonging to $A$ are

$$F_1=\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4\},\quad F_2=\{\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4,\mathbf{a}_5\},$$
$$F_3=\{\mathbf{a}_1,\mathbf{a}_5,\mathbf{a}_2,\mathbf{a}_3\},\quad F_4=\{\mathbf{a}_1,\mathbf{a}_5,\mathbf{a}_2,\mathbf{a}_4\},\quad F_5=\{\mathbf{a}_1,\mathbf{a}_5,\mathbf{a}_3,\mathbf{a}_4\},$$

together with their subsets. Since $\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4\}$ cannot be a $\mathbb{Z}$-basis of $\mathbb{Z}^4$, it follows that $F_1$ is not unimodular. Each of the simplices $F_2,F_3,F_4$ and $F_5$ is unimodular.

Let $\Delta_1$ be the triangulation of $\mathcal{P}_A$ with the facets $F_1,F_2$, and let $\Delta_2$ be the triangulation of $\mathcal{P}_A$ with the facets $F_3,F_4,F_5$. Then $\Delta_1$ is flag and $\Delta_2$ is unimodular. Since $F_1\in\Delta_1$, it follows that $\Delta_1$ is not unimodular. Since $\{\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4\}$ is a minimal nonface of $\Delta_2$, it follows that $\Delta_2$ is not flag.

Let $A\in\mathbb{Z}^{d\times n}$ be a configuration with $\mathbf{a}_1,\ldots,\mathbf{a}_n$ its columns and $\mathcal{P}_A\subset\mathbb{R}^d$ the convex polytope arising from $A$. Let $I_A\subset S$ be the toric ideal of $A$ and $\mathrm{in}_<(I_A)$ the initial ideal of $I_A$ with respect to a monomial order $<$ on $S$. For a monomial $u=x_1^{a_1}\cdots x_n^{a_n}$, the notation $\sqrt{u}$ stands for the squarefree monomial $\prod_{i\,:\,a_i>0}x_i$. For example, if $u=x_1x_3^5x_4^2$, then $\sqrt{u}=x_1x_3x_4$. Moreover, when $\mathrm{in}_<(I_A)$ is generated by monomials $u_1,\ldots,u_s$, the notation $\sqrt{\mathrm{in}_<(I_A)}$ stands for the ideal of $S$ which is generated by the squarefree monomials $\sqrt{u_1},\ldots,\sqrt{u_s}$. Since a monomial ideal possesses a unique minimal system of monomial generators, it follows that $\sqrt{\mathrm{in}_<(I_A)}$ is independent of the particular choice of a system of monomial generators of $I_A$.

**Lemma 2.5.** *Let $F$ be a subset of $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ and suppose that*

$$\prod_{\mathbf{a}_i \in F} x_i \notin \sqrt{\mathrm{in}_<(I_A)}. \tag{4}$$

*Then $F$ is a simplex belonging to $A$.*

**Theorem 2.6** (Sturmfels [24])**.** *Write $\Delta(\mathrm{in}_<(I_A))$ for the set of those simplices $F$ of $A$ satisfying* (4)*. Then $\Delta(\mathrm{in}_<(I_A))$ is a triangulation of $\mathcal{P}_A$.*

Such a triangulation $\Delta(\mathrm{in}_<(I_A))$ of $\mathcal{P}_A$ is called a *regular* triangulation of $\mathcal{P}_A$. A typical example of a *nonregular* triangulation appears in, say, [24]. Now, it is natural to ask when a regular triangulation $\Delta(\mathrm{in}_<(I_A))$ is unimodular or flag.

**Theorem 2.7** ([24])**.** *A regular triangulation $\Delta(\mathrm{in}_<(I_A))$ of $\mathcal{P}_A$ is unimodular if and only if $\mathrm{in}_<(I_A)$ is generated by squarefree monomials, i.e., $\mathrm{in}_<(I_A) = \sqrt{\mathrm{in}_<(I_A)}$.*

**Lemma 2.8.** *A regular triangulation $\Delta(\mathrm{in}_<(I_A))$ of $\mathcal{P}_A$ is flag if and only if $\sqrt{\mathrm{in}_<(I_A)}$ is generated by quadratic monomials.*

**Corollary 2.9.** *A regular triangulation $\Delta(\mathrm{in}_<(I_A))$ of $\mathcal{P}_A$ is unimodular and flag if and only if $\mathrm{in}_<(I_A)$ is generated by squarefree quadratic monomials.*

**Example 2.10.** Let $\Delta_1$ and $\Delta_2$ be the triangulations of $\mathcal{P}_A$ discussed in Example 2.4. Recall from Example 2.1 that one has either $\mathrm{in}_<(I_A) = (x_1x_5^2)$ or $\mathrm{in}_<(I_A) = (x_2x_3x_4)$. When $\mathrm{in}_<(I_A) = (x_1x_5^2)$, its regular triangulation $\Delta(\mathrm{in}_<(I_A))$ coincides with the flag nonunimodular triangulation $\Delta_1$. When $\mathrm{in}_<(I_A) = (x_2x_3x_4)$, then its regular triangulation $\Delta(\mathrm{in}_<(I_A))$ coincides with the unimodular nonflag triangulation $\Delta_2$.

In [19] we discovered an example of a configuration $A$ for which the toric ideal $I_A$ possesses no initial ideal generated by squarefree monomials, but the convex polytope $\mathcal{P}_A$ possesses a (nonregular) unimodular triangulation.

Corollary 2.9 does explain the reason why the existence of an initial ideal generated by squarefree quadratic monomials of a toric ideal is important in algebraic combinatorics on convex polytopes. In commutative algebra, such an existence is useful to show that the *toric ring* of a configuration is normal and Koszul.

We now discuss initial ideals generated by squarefree quadratic monomials arising in combinatorics. We refer the reader to [22] for the fundamental techniques behind the proof of each of Theorems 2.12, 2.14 and 2.15.

### (a) Finite distributive lattices

Let $L = \{\alpha_1, \ldots, \alpha_n\}$ be a finite distributive lattice [10, p. 118], where $i < j$ if $\alpha_i < \alpha_j$ in $L$, and $\beta_1, \ldots, \beta_d$ the join-irreducible elements [10, p. 119] of $L$. In the literature on combinatorics, the unique minimal element of $L$ may not be regarded as a join-irreducible element of $L$. However, we will regard the unique minimal element of $L$ as a join-irreducible element of $L$. We then introduce the configuration $A(L) = (a_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq n}} \in \mathbb{Z}^{d \times n}$ by setting $a_{ij} = 1$ if $\beta_i \leq \alpha_j$ and $a_{ij} = 0$ if $\beta_i \not\leq \alpha_j$. Since we will regard the unique minimal element of $L$ as a join-irreducible element of $L$, it turns out that $A(L)$ is, in fact, a configuration.

**Example 2.11.** Let $L$ be the finite distribute lattice drawn below.

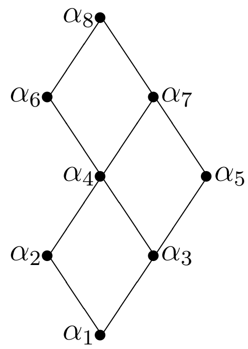


**Figure 2.1.** A finite distributive lattice

Its join-irreducible elements are $\alpha_1, \alpha_2, \alpha_3, \alpha_5$ and $\alpha_6$. The configuration $A_L$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Let $S = K[x_1, \ldots, x_n]$ and $I_{A(L)} \subset S$ the toric ideal of $A(L)$. For each pair $(\alpha_i, \alpha_j)$ with $i < j$ such that $\alpha_i$ and $\alpha_j$ are incomparable in $L$, we introduce the binomial $f_{i,j} = x_i x_j - x_k x_\ell$, where $\alpha_k = \alpha_i \wedge \alpha_j$ and $\alpha_\ell = \alpha_i \vee \alpha_j$. One has $f_{i,j} \in I_{A(L)}$. Write $\mathcal{G}$ for the finite set of those binomials $f_{i,j}$ of $I_{A(L)}$. Let $<_{\text{rev}}$ denote the reverse lexicographic order on $S$ induced by the ordering $x_1 > \cdots > x_n$ of the variables.

**Theorem 2.12** ([11]). *Working with the same notation as above, $\mathcal{G}$ is a Gröbner basis of $I_{A(L)}$ with respect to $<_{\text{rev}}$.*

For example, in the finite distributive lattice of Figure 2.1, the Gröbner basis $\mathcal{G}$ consists of the quadratic binomials

$$x_2x_3 - x_1x_4, \quad x_2x_5 - x_1x_7, \quad x_4x_5 - x_3x_7,$$
$$x_5x_6 - x_3x_8, \quad x_6x_7 - x_4x_8.$$

## (b) Classical root systems

Let $\Phi \subset \mathbb{Z}^n$ be one of the classical irreducible root systems $\mathbf{A}_{n-1}, \mathbf{B}_n, \mathbf{C}_n$ and $\mathbf{D_n}$ ([12, pp. 64–65]) and write $\Phi^{(+)}$ for the matrix whose column vectors are all positive roots of $\Phi$ together with the origin of $\mathbb{R}^n$. Let $\tilde{\Phi}^{(+)}$ be the configuration obtained by adding the row vector $(1, 1, , \ldots, 1)$ to $\Phi^{(+)}$. For example,

$$\tilde{\mathbf{A}}_2^{(+)} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \tilde{\mathbf{D}}_3^{(+)} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\tilde{\mathbf{B}}_3^{(+)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\tilde{\mathbf{C}}_3^{(+)} = \begin{bmatrix} 0 & 2 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

**Theorem 2.13** ([7]). *The toric ideal of the configurations $\tilde{\mathbf{A}}_{n-1}^{(+)}$ possesses an initial ideal generated by squarefree quadratic monomials.*

**Theorem 2.14** ([21]). *The toric ideal of each of the configurations $\tilde{\mathbf{B}}_n^{(+)}, \tilde{\mathbf{C}}_n^{(+)}$ and $\tilde{\mathbf{D}}_n^{(+)}$ possesses an initial ideal generated by squarefree quadratic monomials.*

### (c) Finite bipartite graphs

Let $G$ be a finite graph on the vertex set $[d] = \{1, \ldots, d\}$ and $e_1, \ldots, e_n$ its edges. The *incidence matrix* of $G$ is the configuration $A_G = \{a_{ij}\}_{\substack{1 \leq i \leq d \\ 1 \leq j \leq n}}$, where $a_{ij} = 1$ if $i \in e_j$ and $a_{ij} = 0$ if $i \notin e_j$. For example, if $G$ is the finite graph on [6] with edges $e_1 = \{1,2\}, e_2 = \{2,3\}, e_3 = \{1,3\}, e_4 = \{3,4\}, e_5 = \{4,5\}, e_6 = \{5,6\}$ and $e_7 = \{4,6\}$, then the configuration $A_G$ is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

**Theorem 2.15** ([20]). *Let $G$ be a finite bipartite graph and $I_{A_G}$ the toric ideal of the configuration $A_G$ arising from $G$. Then the following conditions are equivalent:*

(i) *$I_{A_G}$ is generated by quadratic binomials;*
(ii) *$I_{A_G}$ possesses an initial ideal generated by squarefree quadratic monomials;*
(iii) *Every cycle of $G$ of length $> 4$ has a chord.*

Finally, we refer the reader to, e.g., [2] for initial ideals generated by squarefree quadratic monomials arising in algebraic statistics.

## 3. Generic initial ideals and algebraic shifting

Let $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in $n$ variables over an *infinite* field $K$ with $\deg x_i = 1$ for $i = 1, 2, \ldots, n$. Write $\mathrm{GL}_n(K)$ for the group of all invertible $n \times n$ matrices with entries in $K$. For each $\varphi = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \in \mathrm{GL}_n(K)$ and for each $f = f(x_1, \ldots, x_n) \in S$, one defines

$$\varphi(f) = f\left(\sum_{i=1}^{n} a_{i1}x_i, \ldots, \sum_{i=1}^{n} a_{in}x_i\right).$$

If $I$ is a homogeneous ideal of $S$, i.e., an ideal of $S$ generated by homogeneous polynomials, and if $\varphi = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \in \mathrm{GL}_n(K)$, then $\varphi(I) = \{\varphi(f) : f \in I\}$ is again a homogeneous ideal of $S$.

**Theorem 3.1** (Galligo, Bayer–Stillman)**.** *Work with the reverse lexicographic order $<_{\mathrm{rev}}$ on $S$ induced by $x_1 > \cdots > x_n$. Let $I$ be a homogeneous ideal of $S$. If $\varphi, \varphi' \in \mathrm{GL}_n(K)$ are "generic," then $\mathrm{in}_{<_{\mathrm{rev}}}(\varphi(I)) = \mathrm{in}_{<_{\mathrm{rev}}}(\varphi'(I))$.*

To understand the precise definition that $\varphi, \varphi' \in \mathrm{GL}_n(K)$ are "generic," the language of algebraic geometry will be required. What Theorem 3.1 guarantees is that, whenever we choose the entries of $\varphi$ and $\varphi'$ "randomly," one has $\mathrm{in}_{<_{\mathrm{rev}}}(\varphi(I)) = \mathrm{in}_{<_{\mathrm{rev}}}(\varphi'(I))$. We write

$$\mathrm{Gin}(I) = \mathrm{in}_{<_{\mathrm{rev}}}(\varphi(I)),$$

where $\varphi \in \mathrm{GL}_n(K)$ is generic, and call $\mathrm{Gin}(I)$ the *generic initial ideal* of $I$. We refer the reader who is familiar with commutative algebra and algebraic geometry to [8] for the foundations of generic initial ideals.

**Example 3.2.** Let $n = 2$ and write $x$ and $y$ instead of $x_1$ and $x_2$ respectively. Let $I = (x^2, y^2) \subset K[x, y]$ and

$$\varphi = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Thus $\varphi(I) = ((ax + by)^2, (cx + dy)^2$. Suppose that $ad - bc \neq 0$ and $ac \neq 0$. We compute $\mathrm{in}_{<_{\mathrm{rev}}}(\varphi(I))$. Let $f = (ax + by)^2$ and $g = (cx + dy)^2)$.

- Since $a \neq 0$, one has $\mathrm{in}_{<_{\mathrm{rev}}}(f) = x^2 \in \mathrm{in}_{<_{\mathrm{rev}}}(\varphi(I))$. Since $c^2 f - a^2 g \in I$ and $ad - bc \neq 0$, it follows that $h_0 = 2acxy + (ad + bc)y^2 \in I$. Since $ac \neq 0$, one has $\mathrm{in}_{<_{\mathrm{rev}}}(h_0) = xy \in \mathrm{in}_{<_{\mathrm{rev}}}(\varphi(I))$.
- One has $y^2 \notin \mathrm{in}_{<_{\mathrm{rev}}}(\varphi(I))$. In fact, if $y^2 \in \mathrm{in}_{<_{\mathrm{rev}}}(\varphi(I))$, then $y^2 \in I$. Thus $Cy^2 = Af + Bg$, where $A, B, C \in K$ with $C \neq 0$. Hence $a^2 A + c^2 B = 0$ and $abA + cdB = 0$. However, since $ad - bc \neq 0$ and $ac \neq 0$, one has $A = B = 0$.
- Let $h_1 = 2cyf - axh_0 = a(3bc - ad)xy^2 + 2b^2cy^3 \in I$. Since
$$2ch_1 - (3bc - ad)yh_0 = (ad - bc)^2 y^3 \in I$$
and $ad - bc \neq 0$, one has $y^3 \in I$. Thus $y^3 \in \mathrm{in}_{<_{\mathrm{rev}}}(\varphi(I))$.
- Consequently, since $(x^2, xy, y^3) \subset \mathrm{in}_{<_{\mathrm{rev}}}(\varphi(I))$, it follows that all monomials in $K[x, y]$ except for $x, y$ and $y^2$ must belong to $\mathrm{in}_{<_{\mathrm{rev}}}(\varphi(I))$. Since $\deg f = \deg g = 2$, it is clear that none of $x$ and $y$ belongs to $\mathrm{in}_{<_{\mathrm{rev}}}(\varphi(I))$. Finally, since $y^2 \notin \mathrm{in}_{<_{\mathrm{rev}}}(\varphi(I))$, it follows that $\mathrm{in}_{<_{\mathrm{rev}}}(\varphi(I)) = (x^2, xy, y^3)$.

If we choose the entries $a, b, c$ and $d$ of $\varphi$ "randomly," then we could expect $ad - bc \neq 0$ and $ac \neq 0$. Hence if $\varphi$ is "generic," then one has $\mathrm{in}_{<_{\mathrm{rev}}}(\varphi(I)) = (x^2, xy, y^3)$. Thus the generic initial ideal of $I = (x^2, y^2)$ is $\mathrm{Gin}(I) = (x^2, xy, y^3)$.

Let $\mathcal{M}_n$ denote the set of monomials in the variables $x_1, \ldots, x_n$, and $\mathcal{M}_n^{(sq)}$ the subset of $\mathcal{M}_n$ consisting of all squarefree monomials belonging to $\mathcal{M}_n$. Let $\mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{M}_n$ and $\mathcal{M}^{(sq)} = \bigcup_{n=1}^{\infty} \mathcal{M}_n^{(sq)}$. Given a monomial

$$u = x_{i_1} x_{i_2} \cdots x_{i_j} \cdots x_{i_r}$$

of $\mathcal{M}$, where

$$1 \leq i_1 \leq i_2 \leq \cdots \leq i_j \leq \cdots \leq i_r,$$

we introduce the squarefree monomial $u^\sigma \in \mathcal{M}^{(sq)}$ by setting

$$u^\sigma = x_{i_1} x_{i_2+1} \cdots x_{i_j+(j-1)} \cdots x_{i_r+(r-1)}.$$

For example,

$$(x_1^3 x_2 x_3^2 x_5)^\sigma = (x_1 x_1 x_1 x_2 x_3 x_3 x_5)^\sigma = x_1 x_2 x_3 x_5 x_7 x_8 x_{11}.$$

The operator $\sigma : \mathcal{M} \to \mathcal{M}^{(sq)}$ is called the *squarefree operator* on $\mathcal{M}$.

**Example 3.3.** Let $\mathcal{M}_n(d)$ denote the set of monomials of degree $d$ belonging to $\mathcal{M}_n$ and $\mathcal{M}_n^{(sq)}(d)$ the set of squarefree monomials of degree $d$ belonging to $\mathcal{M}_n^{(sq)}$. Since $(x_1^d)^\sigma = x_1 x_2 \cdots x_d$ and $(x_n^d)^\sigma = x_n x_{n+1} \cdots x_{n+d-1}$, it follows that $\sigma$ induces a bijection between $\mathcal{M}_n(d)$ and $\mathcal{M}_{n+d-1}^{(sq)}(d)$.

We summarize fundamental materials on simplicial complexes from, e.g., [10] and [23]. Let $[n] = \{1, 2, \ldots, n\}$ be the vertex set and $\Delta$ a simplicial complex on $[n]$. Thus $\Delta$ is a collection of subsets of $[n]$ satisfying that (i) $\{i\} \in \Delta$ for all $i \in [n]$ and (ii) if $F \in \Delta$ and $F' \subset [n]$ with $F' \subset F$, then $F' \in \Delta$. Each element $F \in \Delta$ is called a *face* of $\Delta$. The dimension of a face $F$ is $|F| - 1$, where the notation $|F|$ stands for the cardinality of $F$. The *dimension* of $\Delta$ is $\dim \Delta = d - 1$, where $d = \max\{|F| : F \in \Delta\}$. A *facet* of $\Delta$ is a maximal face of $\Delta$ (with respect to inclusion). We say that $\Delta$ is *pure* if all facets of $\Delta$ have the same cardinality.

For each $i = 0, 1, \ldots, d-1$, we write $f_i = f_i(\Delta)$ for the number of faces of $\Delta$ of dimension $i$. The sequence $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ is called the *$f$-vector* of $\Delta$. The *$h$-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ is defined by the formula

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}$$

with $f_{-1} = 1$. In particular $h_0 = 1, h_1 = n - d$ and $h_0 + h_1 + \cdots + h_d = f_{d-1}$.

In order to visualize a simplicial complex $\Delta$ we often identify $\Delta$ with its *geometric realization* $|\Delta|$. For example, Figure 3.1 illustrates the simplicial complex $\Delta$ on $[5]$ of dimension 2 whose facets are $\{1, 2, 4\}, \{1, 2, 5\}, \{2, 3\}$ and $\{3, 4\}$. Its $f$-vector is $f(\Delta) = (5, 7, 2)$ and its $h$-vector is $h(\Delta) = (1, 2, 0, -1)$.

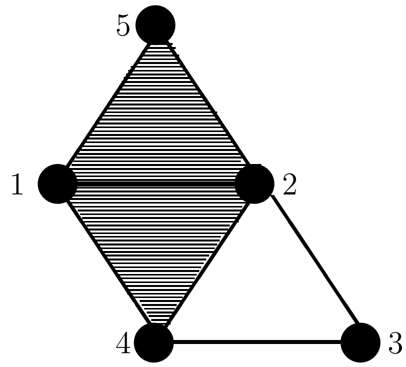


**Figure 3.1.** A geometric realization of $\Delta$

We associate each subset $F \subset [n]$ with the squarefree monomial $x_F = \prod_{i \in F} x_i$ of $S$. A *squarefree ideal* of $S$ is an ideal of $S$ which is generated by squarefree monomials. Given a simplicial complex $\Delta$ on $[n]$, we define its *Stanley–Reisner ideal* to be the squarefree ideal

$$I_\Delta = (\{x_F : F \subset [n], F \not\in \Delta\})$$

of $S$. Conversely, it follows easily that, given a squarefree ideal $I$ of $S$ with $x_i \not\in I$ for each $1 \leq i \leq n$, there exists a unique simplicial complex $\Delta$ on $[n]$ with $I = I_\Delta$.

**Example 3.4.** Let $\Delta$ be the simplicial complex of Figure 3.1. Its Stanley–Reisner ideal is $I_\Delta = (x_1x_3, x_3x_5, x_4x_5, x_2x_3x_4)$.

Algebraic shifting, introduced by Gil Kalai in 1984, turns out to be one of the powerful techniques in algebraic combinatorics on convex polytopes. In [14] the symmetric algebraic shifted complex is defined in a combinatorial way. However, our definition given here is algebraic and will require the generic initial ideal.

Let $\Delta$ be a simplicial complex on $[n]$ and $I_\Delta \subset S = K[x_1, \ldots, x_n]$ its Stanley–Reisner ideal. Let $\mathrm{Gin}(I_\Delta)$ denote the generic initial ideal of $I_\Delta$ and $\{u_1, \ldots, u_q\}$ the (unique) minimal system of monomial generators of $\mathrm{Gin}(I_\Delta)$.

**Lemma 3.5** ([9])**.** *Each squarefree monomial $u_i^\sigma$ belongs to $S$. In other words, $(u_1^\sigma, \ldots, u_q^\sigma)$ is a squarefree ideal of $S$.*

**Definition 3.6** ([9])**.** The *symmetric algebraic shifted complex* of $\Delta$ is the simplicial complex $\Delta^s$ on $[n]$ with $I_{\Delta^s} = (u_1^\sigma, \ldots, u_q^\sigma)$.

The operator $\Delta \to \Delta^s$ is called *symmetric algebraic shifting.* In [14] exterior algebraic shifting as well as symmetric algebraic shifting is introduced.

A simplicial complex $\Delta$ on $[n]$ is called *shifted* if $F \in \Delta$, $j \in F$ and $i > j$ imply $(F \setminus \{j\}) \cup \{i\} \in \Delta$. The shifted complexes played an important role in classical combinatorics on finite sets.

**Lemma 3.7** ([9] and [14])**.** *Let $\Delta$ and $\Delta'$ be simplicial complexes.*

- *$\Delta^s$ is shifted;*
- *$f(\Delta^s) = f(\Delta)$;*

- *If $\Delta$ is shifted, then $\Delta^s = \Delta$;*
- *If $\Delta \subset \Delta'$, then $\Delta^s \subset (\Delta')^s$.*

Let $\mathcal{P} \subset \mathbb{R}^N$ be a simplicial polytope [10, p. 9] of dimension $d$ with $n$ vertices and $\Delta(\mathcal{P})$ its boundary complex [10, p. 10]. It follows that $\Delta(\mathcal{P})$ is a simplicial complex on $[n]$ of dimension $d-1$ whose geometric realization is homeomorphic to the $(d-1)$-sphere $\mathbb{S}^{d-1}$.

A *simplicial* $(d-1)$*-sphere* is a simplicial complex of dimension $d-1$ whose geometric realization is homeomorphic to the $(d-1)$-sphere $\mathbb{S}^{d-1}$. Every boundary complex of a simplicial polytope of dimension $d$ is a simplicial $(d-1)$-sphere. The $h$-vector $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of a simplicial $(d-1)$-sphere is nonnegative ([10, pp. 44–45]) and symmetric ([10, p. 24]), i.e., each $h_i \geq 0$ and $h_i = h_{d-i}$ for $0 \leq i \leq d$.

One of the outstanding conjectures in algebraic and enumerative combinatorics is the "$g$-conjecture," which gives a complete characterization of the $h$-vectors of simplicial spheres. We recall what the $g$-conjecture is.

Let $\mathcal{M}$ denote the set of monomials in the variables $x_1, x_2, \ldots$ with $\deg x_i = 1$ for $i = 1, 2, \ldots$. A finite subset $\mathcal{B}$ of $\mathcal{M}$ is called an *order ideal of monomials* if whenever $u$ and $v$ are monomials with $u \in \mathcal{B}$ and if $v$ divides $u$, then $v \in \mathcal{B}$. A finite sequence $(h_0, h_1, \ldots, h_s)$ of integers is called an $M$*-vector* if there exists an order ideal of monomials $\mathcal{B}$ such that $|\{u \in \mathcal{B} : \deg u = i\}| = h_i$ for $0 \leq i \leq s$.

**Theorem 3.8** (Billera–Lee, Stanley)**.** *Given a finite sequence $(h_0, h_1, \ldots, h_d)$ of positive integers, there exists a simplicial polytope $\mathcal{P}$ of dimension $d$ with $h(\Delta(\mathcal{P}) = (h_0, h_1, \ldots, h_d)$ if and only if the following conditions are satisfied:*

(i) $h_0 = 1$;
(ii) $h_i = h_{d-i}$ *for* $0 \leq i \leq d$;
(iii) $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{[d/2]} - h_{[d/2]-1})$ *is an $M$-vector.*

In particular, the $h$-vector $h(\Delta(\mathcal{P}) = (h_0, h_1, \ldots, h_d)$ of the boundary complex of a simplicial polytope of dimension $d$ satisfies

$$h_0 \leq h_1 \leq \cdots \leq h_{[d/2]}.$$

**Conjecture 3.9** ($g$-conjecture)**.** *Let $\Delta$ be a simplicial $(d-1)$-sphere with $h(\Delta) = (h_0, h_1, \ldots, h_d)$. Then $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{[d/2]} - h_{[d/2]-1})$ is an $M$-vector.*

Let $n > d$. We write $C(n, d)$ for the convex hull of $n$ points $a(t_1), \ldots, a(t_n)$, where $t_1 < \cdots < t_n$, in the moment curve $\{a(t) = (t, t^2, \ldots, t^d) \in \mathbb{R}^d : t \in \mathbb{R}\}$. It is known ([10, p. 26]) that $C(n, d)$ is a simplicial polytope of dimension $d$ with $\{a(t_1), \ldots, a(t_n)\}$ its vertex set. We say that $C(n, d)$ is the *cyclic polytope of type* $(n, d)$. The combinatorial type [10, p. 27] of the boundary complex $\Delta(C(n, d))$ is independent of the particular choice of real numbers $t_1 < \cdots < t_n$. Thus in particular the $f$-vector (and $h$-vector) of $\Delta(C(n, d))$ is independent of the particular choice of real numbers $t_1 < \cdots < t_n$.

Kalai computed the symmetric algebraic shifted complex of $\Delta(C(n, d))$. The following conjecture due to Kalai and Sarkaria is called the "Shifting Theoretic Upper Bound Conjecture."

**Conjecture 3.10.** *Every simplicial* $(d-1)$*-sphere* $\Delta$ *on* $[n]$ *satisfies*

$$\Delta^s \subset \Delta(C(n,d))^s.$$

Conjecture 3.10 is much stronger than Conjecture 3.9. In fact, the conclusion in the above conjecture implies the conclusion in Conjecture 3.9.

**Lemma 3.11** (Kalai)**.** *Let* $\Delta$ *be a simplicial* $(d-1)$*-sphere with* $h(\Delta) = (h_0, h_1, \ldots, h_d)$*. Suppose that* $\Delta^s \subset \Delta(C(n,d))^s$*. Then* $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{[d/2]} - h_{[d/2]-1})$ *is an* $M$*-vector.*

From a computational viewpoint of commutative algebra, it is strongly desirable to characterize the symmetric shifted complexes of simplicial $(d-1)$-spheres.

Let $\mathcal{A}_n^d$ denote the set of simplicial complexes $\Sigma$ on $[n]$ of dimension $d-1$ such that

(i) $\Sigma$ is pure;
(ii) $\Sigma$ is shifted;
(iii) the $h$-vector of $\Sigma$ is symmetric;
(iv) $\Sigma \subset \Delta(C(n,d))^s$.

**Conjecture 3.12.** (a) *Every simplicial* $(d-1)$*-sphere* $\Delta$ *on* $[n]$ *satisfies* $\Delta^s \in \mathcal{A}_n^d$*.*

(b) *Conversely, for each* $\Sigma \in \mathcal{A}_n^d$*, there exists a simplicial* $(d-1)$*-sphere* $\Delta$ *on* $[n]$ *with* $\Delta^s = \Sigma$*.*

Later, Conjecture 3.12 (b) will be proved in Theorem 3.18 (b).

In 1988 Gil Kalai [14] introduced the squeezed sphere. The original definition given by Kalai is purely combinatorial. We will follow [16] and introduce the squeezed sphere in the language of monomial ideals.

A monomial ideal $I$ of $S = K[x_1, \ldots, x_n]$ is called *strongly stable* if, for each monomial $u \in I$ and for each $i \in [n]$ such that $x_i$ divides $u$, one has $x_j(u/x_i) \in I$ for all $i < j$.

**Example 3.13.** Each of the monomial ideals $(x_1^3, x_1^2x_2, x_1x_2^2, x_2^3)$ and $(x_1^3, x_1^2x_2, x_1x_2^2)$ of $K[x_1, x_2]$ is strongly stable. However, the monomial ideal $(x_1^3, x_1x_2^2, x_3^3)$ of $K[x_1, x_2]$ is not strongly stable.

Fix $p \geq 0$. Let $T_p = K[x_1, \ldots, x_p]$ denote the polynomial ring in $p$ variables with $\deg x_i = 1$ for $i = 1, 2, \ldots, p$ and $\mathfrak{m} = (x_1, \ldots, x_p)$. An $\mathfrak{m}$*-primary ideal* of $T_p$ is an ideal $I$ of $T_p$ for which $\mathfrak{m}^N \subset I$ for $N \gg 0$. Here $\mathfrak{m}^N$ is the ideal of $T_p$ generated by all monomials of $T$ of degree $N$.

Given a strongly stable $\mathfrak{m}$-primary ideal $I$ of $T_p$ with $I \subset \mathfrak{m}^2$, we construct the squeezed sphere arising from $I$ as follows:

- Let $\delta = \delta(I)$ denote the largest number $i \geq 1$ for which $\mathfrak{m}^i \not\subset I$.
- Fix an integer $n \geq p + 2\delta$. Let $d = n - p - 1$.
- Let $\{u_1, \ldots, u_q\}$ denote the (unique) minimal system of monomial generators of $I$.
- Since $n \geq p + 2\delta$, it follows that $(u_i^\sigma)^\sigma$ belongs to $S = K[x_1, \ldots, x_n]$ for each $1 \leq i \leq q$.
- Let $(I^\sigma)^\sigma$ denote the squarefree ideal $((u_1^\sigma)^\sigma, \ldots, (u_q^\sigma)^\sigma)$ of $S$.

- Write $B_d(I)$ for the simplicial complex on $[n]$ whose Stanley–Reisner ideal coincides with $(I^\sigma)^\sigma$.

**Lemma 3.14** ([13]). *The geometric realization of the simplicial complex $B_d(I)$ is homeomorphic to the $d$-ball $\mathbb{B}^d$.*

We say that $B_d(I)$ is the *squeezed ball* arising from $I$. Let $S_{d-1}(I)$ denote the boundary complex of $B_d(I)$. It follows that the geometric realization of $S_{d-1}(I)$ is homeomorphic to the $(d-1)$-sphere $\mathbb{S}^{d-1}$. We say that $S_{d-1}(I)$ is the *squeezed sphere* arising from $I$.

In the above algebraic definition of the squeezed ball, since the maximal ideal $I = \mathfrak{m}$ is excluded, the simplex cannot be a squeezed ball. However, for the sake of convenience, we will regard the simplex as a squeezed ball.

**Example 3.15.** Let $p = 2$ and $I = \mathfrak{m}^3 = (x_1^3, x_1^2x_2, x_1x_2^2, x_2^3)$. Thus $\delta = 2$. Let $n = p + 2\delta = 6$. Hence $d = n - p - 1 = 3$. Since $((x_1^3)^\sigma)^\sigma = x_1x_3x_5, (x_1^2x_2)^\sigma)^\sigma = x_1x_3x_6, (x_1x_2^2)^\sigma)^\sigma = x_1x_4x_6$ and $(x_2^3)^\sigma)^\sigma = x_2x_4x_6$, one has

$$(I^\sigma)^\sigma = (x_1x_3x_5, x_1x_3x_6, x_1x_4x_6, x_2x_4x_6).$$

The facets of the squeezed sphere $S_2(I)$ arising from $I$ are $\{1,2,3\}$, $\{1,3,4\}$, $\{1,4,5\}$, $\{1,5,6\}$, $\{1,2,6\}$, $\{2,3,6\}$, $\{3,4,6\}$, $\{4,5,6\}$. It follows that $S_2(I)$ is the unique simplicial 2-sphere, up to isomorphism, on $[6]$ with a vertex of degree 5.

**Example 3.16.** The boundary complex of an octahedron cannot be a squeezed sphere.

The $h$-vector of the squeezed sphere arising from $I$ can be computed easily. We write $h_i$, $0 \leq i \leq \delta$, for the number of monomials $u$ of $S_p$ of degree $i$ with $u \notin I$. Thus in particular $h_0 = 1$, $h_1 = p = n - (d+1)$ and $h_\delta \neq 0$. It follows that such a sequence $(h_0, h_1, \ldots, h_\delta)$ is an $M$-vector. Lemma 3.17 below then says immediately that every squeezed sphere satisfies the $g$-conjecture.

**Lemma 3.17.** (a) *The $h$-vector of the squeezed ball $B_d(I)$ is $(h_0, h_1, \ldots, h_\delta, 0, \ldots, 0)$.*
(b) *Let $\delta' = \min\{\delta, [d-1/2]\}$. Then the $h$-vector of the squeezed sphere $S_d(I)$ is*

$$(h_0, h_0 + h_1, \ldots, \sum_{i=0}^{\delta'-1} h_i, \sum_{i=0}^{\delta'} h_i, \ldots, \sum_{i=0}^{\delta'} h_i, \sum_{i=0}^{\delta'-1} h_i, \ldots, h_0 + h_1, h_0).$$

Murai [16] succeeded in giving a complete characterization of the symmetric algebraic shifted complexes of squeezed spheres.

**Theorem 3.18** (Murai [16]). (a) *Every squeezed $(d-1)$-sphere $\Delta$ on $[n]$ satisfies $\Delta^s \in \mathcal{A}_n^d$.*
(b) *Conversely, for each $\Sigma \in \mathcal{A}_n^d$, there exists a squeezed $(d-1)$-sphere $\Delta$ on $[n]$ with $\Delta^s = \Sigma$.*

Here is a brief outline of Murai's proof of Theorem 3.18.

- Each $\Sigma \in \mathcal{A}_n^d$ is strong Lefschetz. Thus in particular $\Sigma \in \mathcal{A}_n^d$ is determined by its faces of dimension $\leq \lfloor d/2 \rfloor - 1$.

- Kalai [13] constructed for the $(\lfloor d/2 \rfloor - 1)$-skeleton $\Sigma_{\lfloor d/2 \rfloor -1}$ of each $\Sigma \in \mathcal{A}_n^d$ a squeezed sphere $S_\Sigma$ and conjectured that $(S_\Sigma^s)_{\lfloor d/2 \rfloor -1} = \Sigma_{\lfloor d/2 \rfloor -1}$.
- The above conjecture by Kalai was proved by Murai [15]. Thus in order to complete a proof of Theorem 3.18 one must show that every squeezed sphere is strong Lefschetz.
- Murai [16] showed that every squeezed sphere is strongly edge decomposable, a notion introduced by Nevo [17], and that a strongly edge decomposable complex is strong Lefschetz. The last result was essentially proved in Babson and Nevo [3] independently.

Finally, the number of combinatorial types of squeezed spheres is much larger than that of boundary complexes of simplicial polytopes (Kalai [13]). However, it seems difficult to find a strongly stable $\mathfrak{m}$-primary ideal $I$ of $T_p$ for which the squeezed sphere $S_{d-1}(I)$ cannot be the boundary complex of a simplicial polytope.

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