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KOSTKA–FOULKES POLYNOMIALS FOR SYMMETRIZABLE KAC–MOODY ALGEBRAS

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Abstract. We introduce a generalization of the classical Hall–Littlewood and Kostka–Foulkes polynomials to all symmetrizable Kac–Moody algebras. We prove that these Kostka–Foulkes polynomials coincide with the natural generalization of Lusztig's t-analog of weight multiplicities, thereby extending a theorem of Kato. For $\mathfrak{g}$ an affine Kac–Moody algebra, we define t-analogs of string functions and use Cherednik's constant term identities to derive explicit product expressions for them.

1. Introduction

The theory of Hall–Littlewood polynomials and Kostka–Foulkes polynomials associated to a finite dimensional simple Lie algebra $\mathfrak{g}$ is a classical subject with numerous connections to representation theory, combinatorics, geometry and mathematical physics. We refer to [15] and [17] for nice surveys. Let W be the Weyl group of $\mathfrak{g}$, P its weight lattice and P^+ the set of dominant weights. The Hall–Littlewood polynomials $P_\lambda(t)$ and the Weyl characters χ_μ ($\lambda, \mu \in P^+$) each form a $\mathbb{C}[t]$-basis of the ring $\mathbb{C}[t][P]^W$. The Kostka–Foulkes polynomials $K_{\lambda\mu}(t)$ are the entries of the transition matrix between these two bases.

In this article, we introduce a generalization of the Hall–Littlewood polynomials to all *symmetrizable Kac–Moody algebras* $\mathfrak{g}$. This is defined by

$$P_\lambda(t) := \frac{1}{W_\lambda(t)} \frac{\sum_{w\in W} (-1)^{\ell(w)} w \left(e^{\lambda+\rho} \prod_{\alpha\in\Delta_+} (1-te^{-\alpha})^{m_\alpha}\right)}{e^\rho \prod_{\alpha\in\Delta_+} (1-e^{-\alpha})^{m_\alpha}}, \tag{1.1}$$

where W is the Weyl group and Δ_+ the set of positive roots of $\mathfrak{g}$ (see §2 for complete notation). This is a straightforward generalization of the classical definition, taking into account both real and imaginary roots (with multiplicities) of $\mathfrak{g}$. However, it is not immediately clear that (1.1) is well-defined since W and Δ_+ are infinite sets in general. Our first task (Proposition 1) is to establish the well-definedness of $P_\lambda(t)$. The coefficient of a typical term e^μ on the right-hand

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side of (1.1) is a priori only a power series in t, but it will in fact be shown to be a polynomial.

Many of the properties of the classical Hall–Littlewood polynomials remain true for the general $P_\lambda(t)$. For instance $P_\lambda(0) = \chi_\lambda$, the character of the irreducible representation $L(\lambda)$ of $\mathfrak{g}$. One can then define the Kostka–Foulkes polynomials $K_{\lambda\mu}(t)$ for $\mathfrak{g}$ in the usual manner:

$$\chi_\lambda = \sum_{\substack{\mu \in P^+ \\ \mu \leq \lambda}} K_{\lambda\mu}(t) P_\mu(t).$$

Here, the sum on the right is typically infinite, $K_{\lambda\mu}(t) \in \mathbb{Z}[[t]]$ and $K_{\lambda\lambda}(t) = 1$.

One of the fundamental facts concerning classical Kostka–Foulkes polynomials is that they coincide with Lusztig's t-analog of weight multiplicities [13] given by

$$m^\lambda_\mu(t) := \sum_{w \in W} (-1)^{\ell(w)} \mathcal{P}(w(\lambda+\rho) - (\mu+\rho);\, t),$$

where $\mathcal{P}(\,.\,;\, t)$ is the t-analog of Kostant's partition function. In this classical case, the equality $K_{\lambda\mu}(t) = m^\lambda_\mu(t)$ was proved by Kato [8], and later by R. Brylinski [5] (see also [17]). We prove this equality (Theorem 1) in our more general setting of symmetrizable Kac–Moody algebras. Here, $m^\lambda_\mu(t)$ for $\mathfrak{g}$ is obtained in the natural manner by replacing $\mathcal{P}(\,.\,;\, t)$ with the t-analog of the (generalized) Kostant partition function for $\mathfrak{g}$ (Equation (4.4)). Our proof is based on an adaptation of ideas of Macdonald in [14]. For the case of finite dimensional $\mathfrak{g}$, this provides an elementary proof different from those of Kato and Brylinski. For general $\mathfrak{g}$, as nice consequences of this theorem, one obtains that $K_{\lambda\mu}(t) \in \mathbb{Z}[t]$ and $K_{\lambda\mu}(1) = \dim(L(\lambda)_\mu)$.

Next, we specialize to the case that $\mathfrak{g}$ is an untwisted affine Kac–Moody algebra. Given dominant weights λ, μ, we consider the generating function $a^\lambda_\mu(t) := \sum_{k\geq 0} K_{\lambda,\mu-k\delta}(t)\, q^k$. We call these *$t$-string functions* of $\mathfrak{g}$. When $t = 1$, they reduce (up to a power of q) to usual string functions. We exhibit a surprising connection of certain $a^\lambda_\mu(t)$ to Cherednik's famous *constant term identities*. In fact, we show that Cherednik's constant term identities of Macdonald and Macdonald–Mehta type [1, 2] are the precise ingredients necessary to determine the t-string functions associated to the trivial and basic representations of $\mathfrak{g}$. The theory of Macdonald polynomials and the double affine Hecke algebra thus seem to play key background roles in the affine case. In the classical case, the connection is more direct; the classical Hall-Littlewood polynomials are just special cases of Macdonald polynomials.

Finally, we study a relation between Kostka–Foulkes polynomials for affine Lie algebras and their classical counterparts, thereby establishing some partial results concerning *positivity*.

The paper is organized as follows: §§2-3 are concerned with the definitions, and §4 proves the equality $K_{\lambda,\mu}(t) = m^{\lambda}_{\mu}(t)$ in our setting. In §5, t-string functions of levels 0,1 for affine Kac–Moody algebras $\mathfrak{g}$ are computed. §6 and §7 prove the partial positivity results and some assorted facts concerning the $P_\lambda(t)$.

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Postscript. After this paper was completed, we were informed of an earlier unpublished manuscript [4] of Ian Grojnowski, which overlaps with our present work. Together with [3], this manuscript forms part of Grojnowski's larger program of extending the notion of *geometric Satake isomorphism* to double loop groups.

Our approach to the combinatorial results in the present work is different from and more complete than that of [4]. While our proof of the main theorem (Theorem 1) is based on ideas of Macdonald [14], the proof sketched in [4] attempts to directly generalize Brylinski's arguments for the finite dimensional case [5]. This latter approach however presents "well-definedness" difficulties for arbitrary symmetrizable Kac–Moody algebras; these are not fully resolved in [4].

2. Definitions

We will use the notations of Kac's book [7]. Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra with (finite dimensional) Cartan subalgebra $\mathfrak{h}$. Let $\alpha_i, i = 1, \ldots, n$ be the simple roots of $\mathfrak{g}$ and P, Q, P^+, Q^+ be the weight lattice, the root lattice and the sets of dominant weights and non-negative integer linear combinations of simple roots, respectively. We will denote the Weyl group of $\mathfrak{g}$ by W and let $(\cdot, \cdot)$ be a nondegenerate, W-invariant symmetric bilinear form on $\mathfrak{h}^*$. Let Δ (respectively Δ_+, Δ_-) be the set of roots (respectively positive, negative roots) of $\mathfrak{g}$. Given $\lambda, \mu \in \mathfrak{h}^*$, we say $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$. Given $\lambda \in \mathfrak{h}^*$, define $D(\lambda) := \{\gamma \in \mathfrak{h}^* | \gamma \leq \lambda\}$.

Our goal here is to generalize the classical definition of Hall–Littlewood polynomials $P_\lambda(t)$ to all symmetrizable Kac–Moody algebras. However, when passing to arbitrary Kac–Moody algebras, we will have to deal with well-definedness issues concerning infinite sums, products etc.

First, define $\mathcal{E}_t$ to be the set of all series of the form

$$\sum_{\lambda \in \mathfrak{h}^*} c_\lambda(t)\, e^{\lambda}, \tag{2.1}$$

where each $c_\lambda(t) \in \mathbb{C}[[t]]$ and $c_\lambda = 0$ outside the union of a finite number of sets of the form $D(\mu)$, $\mu \in \mathfrak{h}^*$. The e^λ are formal exponentials, which obey the usual rules $e^0 = 1, e^{\lambda+\mu} = e^\lambda e^\mu$. Extending this multiplication $\mathbb{C}[[t]]$-bilinearly to all of $\mathcal{E}_t$ makes it into a commutative, associative $\mathbb{C}[[t]]$-algebra. We also let $\mathcal{E}$ denote the $\mathbb{C}$-subalgebra of $\mathcal{E}_t$ consisting of series where all $c_\lambda \in \mathbb{C}$. Formal characters of $\mathfrak{g}$ modules from category $\mathcal{O}$ are elements of $\mathcal{E}$ [7].

2.1. Let Δ^{re} (respectively $\Delta^{re}_\pm$) denote the set of real roots of $\mathfrak{g}$ (respectively positive/negative real roots), and similarly Δ^{im} (respectively $\Delta^{im}_\pm$) be the corresponding subsets of imaginary roots. For each $\alpha \in \Delta$, let m_α denote the root multiplicity of α. Let $C \subset \mathfrak{h}^*$ be the fundamental Weyl chamber and $X = \bigcup_{w\in W} wC$ be the Tits cone. Let ρ be a Weyl vector of $\mathfrak{g}$ defined by $(\rho, \alpha_i^\vee) = 1$ for all $i = 1, \ldots, n$, where as usual $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i,\alpha_i)}$. Finally, given $\lambda \in P^+$, let $L(\lambda)$ be the integrable $\mathfrak{g}$-module with highest weight λ.

Given $\lambda \in P^+$, let

$$f_\lambda := e^{\lambda+\rho} \prod_{\alpha\in\Delta_+} \left(1 - te^{-\alpha}\right)^{m_\alpha} \in \mathcal{E}_t.$$

We can write $f_\lambda = \sum_{\mu\in P} b_{\lambda\mu}(t)\, e^\mu$; let $\operatorname{supp}(f_\lambda)$ denote the set of all μ for which $b_{\lambda\mu}(t) \neq 0$. Each $\mu \in \operatorname{supp}(f_\lambda)$ can be written as $\mu = \lambda + \rho - |A|$. Here $|A| := \sum_{\alpha\in A} \alpha$ and

(2.2) A is a finite multiset of positive roots such that each $\alpha \in \Delta_+$ occurs at most m_α times in A. For convenience, we will let $\mathcal{A}$ denote the set of all such multisets.

A straightforward calculation gives $b_{\lambda\mu}(t) = \sum(-t)^{\#A}$, where the sum is over all multisets A as described in (2.2) such that $\lambda + \rho - |A| = \mu$.

Our first observation (the general version of the argument of [5]) is the following.

Lemma 1. *Let $\mu \in \operatorname{supp}(f_\lambda)$. Then*

(1) *$\mu \in X$ (the Tits cone)*
(2) *For all $w \in W$, we have $w\mu \leq \lambda + \rho$.*

Proof. We only prove the second assertion, since it clearly implies the first. Let $\mu = \lambda + \rho - |A|$ for some $A \in \mathcal{A}$. Let $L(\rho)$ denote the integrable highest weight representation of $\mathfrak{g}$ with highest weight ρ. It is well-known [7, Ex. 10.1] that the formal character $\operatorname{ch} L(\rho) := \sum_{\mu\in P} \dim(L(\rho)_\mu)e^\mu$ of the module $L(\rho)$ is given by $e^\rho \prod_{\alpha\in\Delta_+}(1 + e^{-\alpha})^{m_\alpha}$. Since $A \in \mathcal{A}$, the element $\gamma := \rho - |A|$ is a weight of $L(\rho)$. Consequently, for all $w \in W, w\gamma \leq \rho$. Now, $\lambda \in P^+$ implies that $w\lambda \leq \lambda$ as well. Since $\mu = \lambda + \gamma$, our assertion is proved. $\square$

2.2. As in the classical finite dimensional situation, we now consider the operator

$$J := \sum_{w\in W} (-1)^{\ell(w)} w,$$

where $\ell(\cdot)$ is the length function on W. Given $f = \sum_{\mu\in P} c_\mu(t)e^\mu \in \mathcal{E}_t$, we let $J(f) := \sum_{w\in W}\sum_{\mu\in P} c_\mu(t)(-1)^{\ell(w)}e^{w\mu}$ whenever the right-hand side is defined. We make a couple of remarks about this definition.

(1) Let $\mu \in X$; then $J(e^\mu) := \sum_{w\in W}(-1)^{\ell(w)}e^{w\mu}$ is clearly a well-defined element of $\mathcal{E}_t$ provided the stabilizer W_μ of μ in W is finite. Further, $1 < \#W_\mu < \infty$ implies that $J(e^\mu) = 0$.
(2) For arbitrary $f \in \mathcal{E}_t$ however, $J(f)$ may no longer be well-defined. As an example, let $f := J(e^\rho) \in \mathcal{E}_t$. Then it is easy to see that $J(f)$ is not defined.

Our main well-definedness result is the following.

Proposition 1. *For each $\lambda \in P^+$, $J(f_\lambda)$ is a well-defined element of $\mathcal{E}_t$.*

Before we prove this proposition, we will use it to define our main objects of interest.

Definition 1. Given $\lambda \in P^+$, let $W_\lambda(t) := \sum_{\sigma\in W_\lambda} t^{\ell(\sigma)} \in \mathbb{C}[[t]]$. Then

$$P_\lambda(t) := \frac{1}{W_\lambda(t)}\frac{J(f_\lambda)}{J(e^\rho)}$$

will be called the *Hall–Littlewood function* associated to λ.

Observe, by the Weyl–Kac character formula that

$$J(e^\rho)^{-1} = e^{-\rho}\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})^{-m_\alpha}.$$

Here we interpret $(1-e^{-\alpha})^{-m_\alpha}$ as the power series

$$(1+e^{-\alpha}+e^{-2\alpha}+\cdots)^{m_\alpha}.$$

Further, since $W_\lambda(0) = 1$, the inverse of $W_\lambda(t)$ is an element of $\mathbb{C}[[t]]$. It is thus clear from Proposition 1 that $P_\lambda(t) \in \mathcal{E}_t$.

The rest of this subsection will be devoted to the proof of Proposition 1. Recall that $f_\lambda = \sum_{\mu\in P} b_{\lambda\mu}(t)\, e^\mu$. As a first step toward showing well-definedness of $J(f_\lambda)$, we prove a necessary condition of being an element of the support of f_λ.

Claim 1. *$\mu \in \operatorname{supp}(f_\lambda)$ implies that $\#W_\mu < \infty$.*

Proof. Write $\mu = \lambda + \rho - |A|$ with A a fixed multiset in $\mathcal{A}$. Then $w \in W_\mu$ implies that $w(\lambda+\rho-|A|) = \lambda+\rho-|A|$, or, equivalently, that

$$[\rho - w\rho + w(|A|)] + (\lambda - w\lambda) = |A|.$$

For an element $\beta = \sum_{i=1}^n a_i\alpha_i \in Q$, the "height" of beta is defined to be $\operatorname{ht}(\beta) := \sum_{i=1}^n a_i$. Since $\lambda \geq w\lambda$, we have $\operatorname{ht}(\rho - w\rho + w(|A|)) \leq \operatorname{ht}(|A|)$. Lemma 2 (below) implies that $|\,\ell(w) - \#A| \leq \operatorname{ht}(|A|)$ and hence that $\ell(w) \leq \operatorname{ht}(|A|) + \#A$. Since A was fixed to begin with, this implies that $\ell(w)$ is bounded above. Thus there can only be finitely many elements in W_μ. $\square$

We now state and prove the lemma that was referred to in the above proof. This simple lemma will come into play again and enable us to complete the proof of Proposition 1.

Lemma 2. *Let $w \in W$ and $A \in \mathcal{A}$. Then*

$$\mathrm{ht}(\rho - w\rho + w(|A|)) \geq |\,\ell(w) - \#A|.$$

Proof. Let $S(w) := \{\beta \in \Delta_+ : w^{-1}\beta \in \Delta_-\}$, and put $A_{re} := \{\alpha \in A : \alpha \in \Delta^{re}\}$ and $A_{im} := \{\alpha \in A : \alpha \in \Delta^{im}\}$. We recall that $\rho - w\rho = \sum_{\beta \in S(w)} \beta$ and $\#S(w) = \ell(w)$. Thus

$$\rho - w\rho + w(|A|) = \sum_{\beta \in S(w)} \beta + \sum_{\substack{\alpha \in A_{re} \\ w\alpha \in \Delta_-}} w\alpha + \sum_{\substack{\alpha \in A_{re} \\ w\alpha \in \Delta_+}} w\alpha + \sum_{\alpha \in A_{im}} w\alpha. \tag{2.3}$$

We observe that (i) $\alpha \in A_{im}$ implies that $w\alpha \in \Delta_+$; (ii) $\alpha \in A_{re}$ such that $w\alpha \in \Delta_-$ implies that $-w\alpha \in S(w)$. Thus the terms in the second sum on the right-hand side of (2.3) cancel with their negatives in the first sum. Let the number of terms in the second sum be denoted $k(w, A)$. Clearly

$$k(w,A) \leq \min\{\ell(w),\, \#A_{re}\} \leq \min\{\ell(w),\, \#A\} = \frac{1}{2}(\ell(w) + \#A - |\,\ell(w) - \#A|).$$

Thus, after this cancellation, the right-hand side of (2.3) is a sum of r positive roots, where $r = \ell(w) + \#A - 2k(w, A) \geq |\,\ell(w) - \#A|$. Since each positive root has height at least 1, our lemma is proved. □

We now complete the proof of Proposition 1. Since $f_\lambda = \sum_{\mu \in P} b_{\lambda\mu}(t)e^\mu$, we have $J(f_\lambda) = \sum_{\mu \in P} b_{\lambda\mu}(t)J(e^\mu)$. Note that each $J(e^\mu)$ is well-defined by Claim 1. To show the well-definedness of the entire sum, observe that

$$J(f_\lambda) = \sum_{w \in W} \sum_{A \in \mathcal{A}} (-1)^{\ell(w)}(-t)^{\#A} e^{w(\lambda + \rho - |A|)}.$$

This is now a well-defined element of $\mathcal{E}_t$ if, for each term e^γ that occurs on the right-hand side, its coefficient is a well-defined power series in t. In other words, for fixed $\gamma \in P$ (by Lemma 1, we can assume $\gamma \leq \lambda + \rho$) and $p \geq 0$, we need to show that the set

$$\{(w, A) \in W \times \mathcal{A} : \#A = p \text{ and } w(\lambda + \rho - |A|) = \gamma\}$$

is finite. Given (w, A) in the above set, we have $\lambda + \rho - \gamma = (\lambda - w\lambda) + (\rho - w\rho + w(|A|))$. By Lemma 2, $|\,\ell(w) - \#A| \leq \mathrm{ht}(\lambda + \rho - \gamma)$. So, $\ell(w) \leq p + \mathrm{ht}(\lambda + \rho - \gamma)$, which means that only finitely many w's are possible. Fix one such w; then $|A| = \lambda + \rho - w^{-1}\gamma$ is also fixed. Since the elements of A are positive roots, the number of possibilities for A corresponding to this w is also finite. This completes the proof of Proposition 1. □

3. Kostka–Foulkes polynomials

3.1. We refer back to the definition of the Hall–Littlewood function $P_\lambda(t)$ in §2.2. Observe that when $t = 0$, $P_\lambda(t)$ reduces to $J(e^{\lambda+\rho})/J(e^\rho) =: \chi_\lambda$, the formal character of the integrable highest weight module $L(\lambda)$. Given $\gamma \in X$ such that $\#W_\gamma < \infty$, consider $J(e^\gamma)/J(e^\rho)$; this equals $\epsilon_\gamma \chi_{[\gamma]-\rho}$, where $[\gamma]$ is the unique dominant weight in the Weyl group orbit of γ, and $\epsilon_\gamma = 0$ if $\#W_\gamma > 1$, and equals $(-1)^{\ell(\sigma)}$ if $\#W_\gamma = 1$ and $\sigma \in W$ is such that $\sigma\gamma = [\gamma]$. This discussion implies that $P_\lambda(t) = W_\lambda(t)^{-1} \sum_{\gamma \in P} b_{\lambda\gamma}(t) \epsilon_\gamma \chi_{[\gamma]-\rho}$. It is clear then that there exists $c_{\lambda\mu}(t) \in \mathbb{C}[[t]]$ such that

$$P_\lambda(t) = \sum_{\mu \in P^+} c_{\lambda\mu}(t) \chi_\mu. \tag{3.1}$$

The right-hand side is an infinite sum in general. We also get

$$\begin{aligned} c_{\lambda\mu}(t) &= W_\lambda(t)^{-1} \sum_{w \in W} (-1)^{\ell(w)}\, b_{\lambda, w^{-1}(\mu+\rho)}(t) \\ &= W_\lambda(t)^{-1} \sum\sum (-1)^{\ell(w)} (-t)^{\#A}, \end{aligned} \tag{3.2}$$

where the double sum in the last equation is over all $w \in W$ and $A \in \mathcal{A}$ such that $w(\lambda + \rho - |A|) = \mu + \rho$.

We have the following easy facts: (i) $c_{\lambda\mu}(t) \neq 0$ implies that $\mu \leq \lambda$. This is an easy consequence of Lemma 1. Also see §7 for a strengthening of this fact. (ii) $c_{\lambda\lambda}(t) = 1$. This calculation can be performed by suitably modifying R. Brylinski's argument in [5, 2.7] for the finite case.

Thus, the change of coordinate matrix from the $P_\lambda(t)$ to the χ_μ is an infinite rank, lower triangular matrix (after suitably ordering the $\lambda \in P^+$) with ones on the diagonal. Inverting the matrix, we can write

$$\chi_\lambda = \sum_{\mu \in P^+} K_{\lambda\mu}(t) P_\mu(t). \tag{3.3}$$

By analogy to the classical case, we will call $K_{\lambda\mu}(t)$ *Kostka–Foulkes polynomials* for the symmetrizable Kac–Moody algebra $\mathfrak{g}$. At the moment, we only know that $K_{\lambda\mu}(t)$ is a power series in t, but Theorem 1 will establish that $K_{\lambda\mu}(t) \in \mathbb{Z}[t]$. Observe that $K_{\lambda\mu}(t) \neq 0$ implies that $\mu \leq \lambda$, and $K_{\lambda\lambda}(t) = 1$.

3.2. We now derive an alternative expression for $P_\lambda(t)$. If $w \in W$, recall that $S(w) = \{\beta \in \Delta_+ : w^{-1}\beta \in \Delta_-\}$. Given $\alpha \in \Delta_+$, either (i) $w\alpha \in \Delta_+$, in which case $\beta := w\alpha \notin S(w)$ or (ii) $w\alpha \in \Delta_-$ in which case $\beta := -w\alpha \in S(w)$. We also have (iii) $w\rho - \rho = -\sum_{\beta \in S(w)} \beta$. So

$$
\begin{aligned}
w(f_\lambda) &= e^{w(\lambda+\rho)} \prod_{\alpha \in \Delta_+} (1 - te^{-w\alpha})^{m_\alpha} \\
&= (-1)^{\ell(w)} e^{w\lambda} e^{\rho} \prod_{\substack{\beta \in \Delta_+ \\ \beta \notin S(w)}} (1 - te^{-\beta})^{m_\beta} \prod_{\beta \in S(w)} (t - e^{-\beta}).
\end{aligned}
$$

Here, we used the additional facts that (iv) $\ell(w) = \#S(w)$, (v) $\beta \in S(w)$ implies that $\beta \in \Delta^{re}$ and so $m_\beta = 1$ and finally (vi) $m_\alpha = m_{w\alpha}$ for all $w \in W$. Together with the Weyl–Kac denominator formula $J(e^\rho) = e^\rho \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{m_\alpha}$ and Definition 1, this gives

$$
P_\lambda(t) = \frac{\sum_{w \in W} \left[e^{w\lambda} \prod_{\substack{\beta \in \Delta_+ \\ \beta \notin S(w)}} (1 - te^{-\beta})^{m_\beta} \prod_{\beta \in S(w)} (t - e^{-\beta}) \right]}{W_\lambda(t) \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{m_\alpha}}. \tag{3.4}
$$

4. The main theorem

4.1. Our main goal in this section will be to establish the relation between the Kostka–Foulkes polynomial $K_{\lambda\mu}(t)$ for $\mathfrak{g}$ and Lusztig's t-analog of weight multiplicities. We use an adaptation of ideas of Macdonald [14]. First define the element $\tilde{\boldsymbol{\Delta}} \in \mathcal{E}_t$ by

$$
\tilde{\boldsymbol{\Delta}} := \frac{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{m_\alpha}}{\prod_{\alpha \in \Delta_+} (1 - te^{-\alpha})^{m_\alpha}} = \prod_{\alpha \in \Delta_+} \left[(1 - e^{-\alpha})(1 + te^{-\alpha} + t^2 e^{-2\alpha} + \cdots) \right]^{m_\alpha}. \tag{4.1}
$$

Using Equation (3.4), we have

$$
\begin{aligned}
\tilde{\boldsymbol{\Delta}} P_\lambda(t) &= \frac{\sum_{w \in W} e^{w\lambda} \prod_{\substack{\beta \in \Delta_+ \\ \beta \notin S(w)}} (1 - te^{-\beta})^{m_\beta} \prod_{\beta \in S(w)} (t - e^{-\beta})}{W_\lambda(t) \prod_{\alpha \in \Delta_+} (1 - te^{-\alpha})^{m_\alpha}} \\
&= \frac{1}{W_\lambda(t)} \sum_{w \in W} e^{w\lambda} \prod_{\alpha \in S(w)} \frac{t - e^{-\alpha}}{1 - te^{-\alpha}}.
\end{aligned} \tag{4.2}
$$

Clearly $\tilde{\boldsymbol{\Delta}} P_\lambda(t)$ is also an element of $\mathcal{E}_t$. Our main observation concerning (4.2) is the following fact.

Proposition 2. *Let $\mu \in P^+$. The coefficient of e^μ in $\tilde{\boldsymbol{\Delta}} P_\lambda(t)$ equals $\delta_{\lambda\mu}$ (i.e., is 1 if $\lambda = \mu$ and 0 otherwise).*

Proof. For convenience, we define (for $w \in W$)

$$p_w := e^{w\lambda} \prod_{\alpha \in S(w)} \frac{t - e^{-\alpha}}{1 - te^{-\alpha}} = e^{w\lambda} \prod_{\alpha \in S(w)} (t - e^{-\alpha})(1 + te^{-\alpha} + t^2 e^{-2\alpha} + \cdots).$$

Thus $\tilde{\boldsymbol{\Delta}} P_\lambda(t) = W_\lambda(t)^{-1} \sum_{w \in W} p_w$. Now, suppose the term e^μ occurs in p_w for some $w \in W$. Then μ can be written as

$$\mu = w\lambda - \sum_{\alpha \in S(w)} n_\alpha \alpha$$

with $n_\alpha \in \mathbb{Z}^{\geq 0}$. Thus $\mu \leq w\lambda$. On the other hand, applying w^{-1} we have

$$w^{-1}\mu = \lambda - \sum_{\alpha \in S(w)} n_\alpha w^{-1}(\alpha) \geq \lambda,$$

since $\alpha \in S(w)$ implies that $w^{-1}(\alpha) \in \Delta_-$. Since $\mu \in P^+$ we also have $\mu \geq w^{-1}\mu$. Putting all these together, we get $\lambda \leq w^{-1}\mu \leq \mu \leq w\lambda$. Finally $\lambda \in P^+$ implies that $\lambda \geq w\lambda$ as well, thereby forcing

$$\lambda = w^{-1}\mu = \mu = w\lambda. \tag{4.3}$$

As a consequence, we obtain: (i) If $\mu \neq \lambda$, then the coefficient of e^μ in p_w is zero for all $w \in W$. (ii) If $\mu = \lambda$, then by Equation (4.3), w must satisfy $\lambda = w\lambda$, i.e., $\lambda \in W_\lambda$. In this case, the coefficient of e^λ in p_w is clearly $t^{\#S(w)} = t^{\ell(w)}$. Thus the coefficient of e^λ in $\tilde{\boldsymbol{\Delta}} P_\lambda(t)$ is $W_\lambda(t)^{-1} \sum_{w \in W_\lambda} t^{\ell(w)} = 1$. □

Remark. Setting $t = 0$ in Proposition 2, we have

$$P_\lambda(t) = \chi_\lambda, \quad \tilde{\boldsymbol{\Delta}} = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{m_\alpha}.$$

In this case, we recover the well-known fact that the coefficient of $e^{\mu+\rho}$ in $J(e^{\lambda+\rho})$ is $\delta_{\lambda\mu}$.

4.2. Let $\mathcal{P}(\,.\,;\, t)$ denote the t-analog of the (generalized) Kostant partition function for $\mathfrak{g}$. This is defined via the relation

$$\frac{1}{\prod_{\alpha \in \Delta_+} (1 - te^{-\alpha})^{m_\alpha}} =: \sum_{\gamma \in Q^+} \mathcal{P}(\gamma;\, t)\, e^{-\gamma}. \tag{4.4}$$

Let $\lambda \in P^+$ and $\mu \in P$. Then Lusztig's t-analog of weight multiplicity which we will denote by $m_\mu^\lambda(t)$, is defined to be

$$m_\mu^\lambda(t) := \sum_{w \in W} (-1)^{\ell(w)} \mathcal{P}(w(\lambda + \rho) - (\mu + \rho);\, t).$$

The right-hand side has only finitely many nonzero terms and hence $m^\lambda_\mu(t) \in \mathbb{Z}[t]$. It is a theorem of Kato [8] that for a finite dimensional simple Lie algebra $\mathfrak{g}$, we have $K_{\lambda\mu}(t) = m^\lambda_\mu(t)$ for all $\lambda, \mu \in P^+$. Our main theorem extends this to all symmetrizable Kac–Moody algebras.

Theorem 1. *Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra and $\lambda, \mu \in P^+$. Then $K_{\lambda\mu}(t) = m^\lambda_\mu(t)$.*

Proof. We will employ the standard notation $[e^\mu]\ f$ to denote the coefficient of e^μ in an expression f. Using Equation (3.3), we have

$$\begin{aligned}[e^\mu]\ \tilde{\boldsymbol{\Delta}}\chi_\lambda &= \sum_{\gamma \in P^+} K_{\lambda\gamma}(t)\ ([e^\mu]\ \tilde{\boldsymbol{\Delta}}\, P_\gamma(t)) \\ &= K_{\lambda\mu}(t),\end{aligned}$$

where for the last equality, we used Proposition 2. Now, the Weyl–Kac character formula gives

$$\begin{aligned}\tilde{\boldsymbol{\Delta}}\chi_\lambda &= \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta_+}(1-te^{-\alpha})^{m_\alpha}} \\ &= \left(\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)-\rho}\right)\left(\sum_{\gamma\in Q^+}\mathcal{P}(\gamma;\, t)\, e^{-\gamma}\right). \qquad (4.5)\end{aligned}$$

Using (4.5), a direct calculation gives that the coefficient of e^μ in $\tilde{\boldsymbol{\Delta}}\chi_\lambda$ is

$$\sum_{w\in W}(-1)^{\ell(w)}\mathcal{P}(w(\lambda+\rho)-(\mu+\rho);\, t),$$

which is precisely $m^\lambda_\mu(t)$. ☐

Remark. For finite dimensional $\mathfrak{g}$, Ranee Brylinski [5] gave a proof of this theorem that was more elementary than Kato's original proof. Ours is yet another proof of this theorem, which works just as well for all symmetrizable Kac–Moody algebras.

Corollary 1. *Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra and $\lambda, \mu \in P^+$. Then*

(1) $K_{\lambda\mu}(t) \in \mathbb{Z}[t]$ *and*
(2) $K_{\lambda\mu}(1) = m^\lambda_\mu(1) = \dim(L(\lambda)_\mu)$.

Proof. The first part follows from the fact that $m^\lambda_\mu(t)$ is an element of $\mathbb{Z}[t]$. Recall that, a priori, we could only say that $K_{\lambda\mu}(t)$ was a power series in t. The second part is a direct consequence of Kostant's weight multiplicity formula. ☐

On account of Theorem 1, we will often refer to the $K_{\lambda,\mu}(t)$ as t-weight multiplicities.

5. Kostka–Foulkes polynomials for affine Kac–Moody algebras

We now specialize to the case that $\mathfrak{g}$ is an untwisted affine Kac–Moody algebra. Let $\mathring{\mathfrak{g}}$ be the underlying finite dimensional simple Lie algebra of rank l, say. If δ is the null root, then $\Delta_+^{im} = \{k\delta : k \geq 1\}$ and each imaginary root has multiplicity l.

Suppose λ, μ are dominant weights of $\mathfrak{g}$ such that $\mu \leq \lambda$. We would like to study $K_{\lambda,\mu}(t)$; observe here that for each $k \geq 0$, $\mu - k\delta$ is also a dominant weight $\leq \lambda$. Let $\mathrm{Max}(\lambda) := \{\mu \in P^+ : \mu \leq \lambda;\ \mu + \delta \not\leq \lambda\}$ (note: this notion is slightly different from that of Kac [7, §12.6] in that we do not require μ to be a weight of the $\mathfrak{g}$-module $L(\lambda)$). For each $\mu \in \mathrm{Max}(\lambda)$, we form the generating function

$$a_\mu^\lambda(t) := \sum_{k\geq 0} K_{\lambda,\mu-k\delta}(t)\, e^{-k\delta} \tag{5.1}$$

of t-weight multiplicities along the δ-string through μ. We will find it convenient to let $q := e^{-\delta}$ in the rest of the paper; thus for example, $a_\mu^\lambda(t) = \sum_{k\geq 0} K_{\lambda,\mu-k\delta}(t)\, q^k$. When $t = 1$, $a_\mu^\lambda(t)$ reduces to the generating function for ordinary weight multiplicities along the δ-string. Thus $a_\mu^\lambda(1)$ is (up to multiplication by a power of q) a *string function* for the module $L(\lambda)$ [7]. By mild abuse of terminology, we will call $a_\mu^\lambda(t)$ a *t-string function*.

We now recall the definition of the *constant term* map $\mathrm{ct}(\cdot)$ from [14]. Given $f = \sum_\lambda f_\lambda e^\lambda \in \mathcal{E}_t$, define $\mathrm{ct}(f) := \sum_{k\in\mathbb{Z}} f_{k\delta}\, e^{k\delta}$. If $g \in \mathcal{E}_t$ is such that $g = \mathrm{ct}(g)$, we recall the observation of [14, (3.1)] that $\mathrm{ct}(fg) = g\,\mathrm{ct}(f)$ for all $f \in \mathcal{E}_t$.

5.1. t-string function of level 0.

The simplest t-string function is that of level 0. It is enough to consider $\lambda = 0$; in this case $\mathrm{Max}(0) = \{0\}$. The t-string function $a_0^0(t)$ is easy to describe. Observe that Theorem 1 implies that $K_{0,-k\delta}(t)$ is the coefficient of $e^{-k\delta}$ in

$$\frac{\sum_{w\in W}(-1)^{\ell(w)} e^{w\rho-\rho}}{\prod_{\alpha\in\Delta_+}(1-te^{-\alpha})^{m_\alpha}} = \frac{\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})^{m_\alpha}}{\prod_{\alpha\in\Delta_+}(1-te^{-\alpha})^{m_\alpha}} = \tilde{\boldsymbol{\Delta}}$$

(notation from (4.1)). In other words, we have

$$a_0^0(t) = \mathrm{ct}(\tilde{\boldsymbol{\Delta}}). \tag{5.2}$$

Next, we shall show that Equation (5.2) is equivalent to an identity due to Macdonald [14, (3.8)]. Since $w\delta = \delta$ for all $w \in W$, Equation (3.4) gives

$$P_{\lambda+c\delta}(t) = e^{c\delta} P_\lambda(t) \ \ \forall \lambda \in P^+, c \in \mathbb{C}. \tag{5.3}$$

As a consequence, $1 = \chi_0 = a_0^0(t)\, P_0(t)$. From Equation (4.2), we have

$$W(t)\, \tilde{\boldsymbol{\Delta}}\, P_0(t) = \sum_{w\in W} \prod_{\alpha\in S(w)} \frac{t-e^{-\alpha}}{1-te^{-\alpha}}.$$

Let $\hat{\boldsymbol{\mu}} := \prod_{\alpha\in\Delta_+^{re}} \frac{1-e^{-\alpha}}{1-te^{-\alpha}}$ and $\tilde{\boldsymbol{\Delta}}^{im} := \prod_{k\geq 1}\left(\frac{1-e^{-k\delta}}{1-te^{-k\delta}}\right)^l$. Observe that (i) $\mathrm{ct}(\tilde{\boldsymbol{\Delta}}^{im}) = \tilde{\boldsymbol{\Delta}}^{im}$ and hence, (ii) $\mathrm{ct}(\tilde{\boldsymbol{\Delta}}) = \mathrm{ct}(\hat{\boldsymbol{\mu}}\,\tilde{\boldsymbol{\Delta}}^{im}) = \tilde{\boldsymbol{\Delta}}^{im}\ \mathrm{ct}(\hat{\boldsymbol{\mu}})$. Putting all this together and using (5.2), we deduce the following result.

Proposition 3.

$$\frac{1}{W(t)} \sum_{w\in W} \prod_{\alpha\in S(w)} \frac{t-e^{-\alpha}}{1-te^{-\alpha}} = \frac{\hat{\boldsymbol{\mu}}}{\mathrm{ct}(\hat{\boldsymbol{\mu}})}. \tag{5.4}$$

This is essentially identity (3.8) of Macdonald [14] (in the case where all t_α are equal). In fact, [14] was the starting point for much of the present article. For many interesting consequences of this identity, see [14].

Finally, we recall the following (special case of an) identity due to Cherednik [1]:

$$\mathrm{ct}(\hat{\boldsymbol{\mu}}) = \prod_{\alpha\in\mathring{\Delta}_+} \frac{(t^{(\mathring{\rho},\alpha^\vee)}\,q;\,q)_\infty^2}{(t^{(\mathring{\rho},\alpha^\vee)+1}\,q;\,q)_\infty (t^{(\mathring{\rho},\alpha^\vee)-1}\,q;\,q)_\infty}, \tag{5.5}$$

where we have used the usual notation $(x;q)_\infty := \prod_{n=0}^\infty (1-xq^n)$. Here $\mathring{\rho}$, $\mathring{\Delta}_+$ are the Weyl vector and set of positive roots (respectively) of $\mathring{\mathfrak{g}}$ and $\alpha^\vee = 2\alpha/(\alpha,\alpha)$. This allows us to write (5.2) and (5.4) as explicit infinite products (cf (3.14) of [14]).

5.2. The level 1 t-string function. Let $L(\Lambda_0)$ be the *basic representation* of $\mathfrak{g}$. Here Λ_0 is the fundamental weight corresponding to the extended (zeroth) node of the Dynkin diagram of $\mathfrak{g}$. From [7, Chap 12], we again have $\mathrm{Max}(\Lambda_0) = \{\Lambda_0\}$. We would like to study the t-string function $a_{\Lambda_0}^{\Lambda_0}(t)$.

Now, suppose $\mathfrak{g}$ is one of the *simply-laced* affine Kac–Moody algebras $A_l^{(1)}, D_l^{(1)}, E_{6/7/8}^{(1)}$. In this case, the multiplicity of the weight $\Lambda_0 - k\delta$ ($k\geq 0$) in the basic representation $L(\Lambda_0)$ equals $p_l(k)$, the number of partitions of k into parts of l colors [7, Prop. 12.13]. Since $K_{\Lambda_0,\Lambda_0-k\delta}(1)$ is equal to this weight multiplicity, this means

$$a_{\Lambda_0}^{\Lambda_0}(1) = \frac{1}{(q;q)_\infty^l}. \tag{5.6}$$

In light of identity (5.6), we can now ask if $a_{\Lambda_0}^{\Lambda_0}(t)$ has a nice closed form expression? We have the following theorem.

Theorem 2. *Let $\mathfrak{g}$ be an untwisted affine Kac–Moody algebra. Then*

$$\frac{a_{\Lambda_0}^{\Lambda_0}(t)}{a_{\Lambda_0}^{\Lambda_0}(1)} = \prod_{i=1}^{l} \frac{(q;\,q)_\infty}{(t^{d_i}q;\,q)_\infty} = \prod_{i=1}^{l}\prod_{n=1}^{\infty} \frac{1-q^n}{1-t^{d_i}q^n} \tag{5.7}$$

where $d_1, d_2, \ldots, d_l$ *are the degrees of the underlying finite dimensional simple Lie algebra* $\mathring{\mathfrak{g}}$.

Proof. Let $\lambda \in P^+$. Using (5.3), we can write $\chi_\lambda = \sum_{\gamma \in \mathrm{Max}(\lambda)} a_\gamma^\lambda(t)\, P_\gamma(t)$. For $\mu \in \mathrm{Max}(\lambda)$, it is easy to see from Proposition 2 that $\mathrm{ct}(e^{-\mu}\, \tilde{\boldsymbol{\Delta}}\, P_\gamma(t)) = \delta_{\gamma,\mu}$, and hence

$$a_\mu^\lambda(t) = \mathrm{ct}(e^{-\mu}\, \tilde{\boldsymbol{\Delta}}\, \chi_\lambda). \tag{5.8}$$

Applying this to our situation gives $a_{\Lambda_0}^{\Lambda_0}(t) = \mathrm{ct}(e^{-\Lambda_0}\, \tilde{\boldsymbol{\Delta}}\, \chi_{\Lambda_0})$. We now let

$$\Theta := \sum_{\alpha \in \mathring{Q}} e^\alpha q^{-(\alpha,\alpha)/2}$$

be the theta function of the root lattice $\mathring{Q}$ of $\mathring{\mathfrak{g}}$. We then have [7, Lemma 12.7]

$$e^{-\Lambda_0}\, \chi_{\Lambda_0} = a_{\Lambda_0}^{\Lambda_0}(1)\, \Theta. \tag{5.9}$$

Recalling $\tilde{\boldsymbol{\Delta}} = \hat{\boldsymbol{\mu}}\, \tilde{\boldsymbol{\Delta}}^{im}$ and using the aforementioned properties of the constant term map gives

$$\frac{a_{\Lambda_0}^{\Lambda_0}(t)}{a_{\Lambda_0}^{\Lambda_0}(1)} = \tilde{\boldsymbol{\Delta}}^{im}\ \mathrm{ct}(\hat{\boldsymbol{\mu}}\, \Theta). \tag{5.10}$$

We recognize the last term from Cherednik's work on the "difference analog of the Macdonald–Mehta conjecture" [2]. Specifically, adjusting for our sign convention and taking all q_α equal in [2, (5.10)], we have the following theorem.

Theorem 3 (Cherednik)**.**

$$\mathrm{ct}(\hat{\boldsymbol{\mu}}\, \Theta) = \prod_{\alpha \in \mathring{\Delta}_+} \frac{(t^{(\mathring{\rho},\alpha^\vee)}\, q;\, q)_\infty}{(t^{(\mathring{\rho},\alpha^\vee)+1}\, q;\, q)_\infty},$$

where $\mathring{\rho}$, $\mathring{\Delta}_+$ *are the Weyl vector and set of positive roots (respectively) of* $\mathring{\mathfrak{g}}$ *and* $\alpha^\vee = 2\alpha/(\alpha,\alpha)$.

For each $j \geq 1$, let p_j denote the number of $\alpha \in \mathring{\Delta}_+$ for which $(\mathring{\rho}, \alpha^\vee) = j$ (i.e., the number of *height* j coroots). We have the following well-known fact [10]:

$$p_j - p_{j+1} = \text{ number of times } j+1 \text{ occurs as a degree of } \mathring{\mathfrak{g}}.$$

Letting $d_1, d_2, \cdots, d_l$ denote the degrees of $\mathring{\mathfrak{g}}$, Theorem 3 then gives

$$\mathrm{ct}(\hat{\boldsymbol{\mu}}\, \Theta) = (tq;\, q)_\infty^l \prod_{i=1}^{l} \frac{1}{(t^{d_i}\, q;\, q)_\infty}.$$

Finally, observing by definition that $\tilde{\boldsymbol{\Delta}}^{im} = (q;\, q)_\infty^l\, (tq;\, q)_\infty^{-l}$ and putting everything into Equation (5.10), Theorem 2 is proved. $\square$

Equation (5.6) immediately gives the following corollary.

Corollary 2. *Let $\mathfrak{g}$ be one of the simply laced untwisted affine Lie algebras $A_l^{(1)}, D_l^{(1)}, E_l^{(1)}$. Then*

$$a_{\Lambda_0}^{\Lambda_0}(t) = \frac{1}{\displaystyle\prod_{i=1}^{l}(t^{d_i}q;\, q)_\infty} = \frac{1}{\displaystyle\prod_{i=1}^{l}\prod_{n=1}^{\infty}(1 - t^{d_i}q^n)}$$

where $d_1, d_2, \ldots, d_l$ are the degrees of the underlying finite dimensional simple Lie algebra $\mathring{\mathfrak{g}}$ $(= A_l, D_l, E_l)$.

Comparing coefficients of $q = e^{-\delta}$ in (5.7) and using $K_{\Lambda_0,\Lambda_0-\delta}(1) = l$, we get another corollary.

Corollary 3. *Let $\mathfrak{g}$ be an untwisted affine Lie algebra. Then,*

$$K_{\Lambda_0,\Lambda_0-\delta}(t) = \sum_{i=1}^{l} t^{d_i},$$

where the d_i's are the degrees of $\mathring{\mathfrak{g}}$.

A strengthened version of this corollary will be proved in §6. We remark that this corollary is analogous to the classical result of Hesselink [6] and (independently) Peterson [16] which states that $\mathring{K}_{\theta,0}(t) = \displaystyle\sum_{i=1}^{l} t^{d_i-1}$ where $\mathring{K}$ denotes a Kostka–Foulkes polynomial for $\mathring{\mathfrak{g}}$ and θ is the highest long root of $\mathring{\mathfrak{g}}$.

6. On positivity of $K_{\lambda\mu}(t)$

For finite dimensional simple Lie algebras, the classical theory shows that the Kostka–Foulkes polynomials are essentially certain Kazhdan-Lusztig polynomials associated to the corresponding extended affine Weyl group. Via their geometric interpretation [9], one obtains the non-negativity of their coefficients.

Let $\mathfrak{g}$ be a fixed untwisted affine Kac–Moody algebra with underlying finite dimensional simple Lie algebra $\mathring{\mathfrak{g}}$. In this section, we will relate the Kostka–Foulkes polynomials $K_{\lambda,\mu}(t)$ of $\mathfrak{g}$ to classical Kostka–Foulkes polynomials $\mathring{K}_{\beta,\gamma}(t)$ associated to $\mathring{\mathfrak{g}}$.

If V is a $\mathring{\mathfrak{g}}$-module and γ is in the weight lattice $P(\mathring{\mathfrak{g}})$ of $\mathring{\mathfrak{g}}$, define $\mathring{K}_{V,\gamma}(t)$ in the natural manner: decompose $V = \bigoplus_\pi \mathring{L}(\pi)^{\oplus n_\pi}$ into a direct sum of irreducible $\mathring{\mathfrak{g}}$

modules $\overset{\circ}{L}(\pi)$ with multiplicities n_π and set $\overset{\circ}{K}_{V,\gamma}(t) := \sum_\pi n_\pi \overset{\circ}{K}_{\pi,\gamma}(t)$. Given a weight $\lambda \in P(\mathfrak{g})$, let $\bar{\lambda}$ denote its restriction to the Cartan subalgebra of $\overset{\circ}{\mathfrak{g}}$; clearly $\bar{\lambda} \in P(\overset{\circ}{\mathfrak{g}})$.

The main result of this section is the following.

Proposition 4. *Let $\mathfrak{g}$ be an untwisted affine Kac–Moody algebra and let $\lambda \in P^+(\mathfrak{g})$ such that $(\lambda, \alpha_0^\vee) \geq 1$. Then $K_{\lambda,\lambda-\delta}(t) = t\,\overset{\circ}{K}_{V,\bar{\lambda}}(t)$ where $V := \overset{\circ}{\mathfrak{g}} \otimes \overset{\circ}{L}(\bar{\lambda})$.*

Before giving the proof, we mention the following immediate corollaries.

Corollary 4. *For $\mathfrak{g}, \lambda$ as in the proposition, we have $K_{\lambda,\lambda-\delta}(t) \in \mathbb{Z}_{\geq 0}[t]$.*

We note that it is not true that $K_{\lambda\mu}(t) \in \mathbb{Z}_{\geq 0}[t]$ for *all pairs* of dominant weights λ, μ. As an example, let $\mathfrak{g} = A_1^{(1)}$ (affine sl_2), $\lambda = 0$, $\mu = -\delta$; then an easy calculation using Theorem 1 gives $K_{\lambda\mu}(t) = t^2 - t$.

Corollary 5. *Let $\mathfrak{g}$ be an untwisted affine Kac–Moody algebra (not necessarily simply laced). Then for each $p \geq 1$, $K_{p\Lambda_0, p\Lambda_0-\delta}(t) = \sum_{i=1}^{l} t^{d_i}$ where the d_i are the degrees of $\overset{\circ}{\mathfrak{g}}$.*

This follows since $\lambda = p\Lambda_0$ implies that $\bar{\lambda} = 0$ and $V = \overset{\circ}{\mathfrak{g}}$. We also note that this is a strengthening of Corollary 3.

We will now prove Proposition 4. The proof is a straightforward (if lengthy) calculation.

Proof of Proposition 4. We assume that the underlying finite dimensional simple Lie algebra $\overset{\circ}{\mathfrak{g}}$ has rank l, highest long root θ and Weyl group $\overset{\circ}{W}$. We have

$$K_{\lambda,\lambda-\delta}(t) = [e^0]\quad \frac{\sum_{w\in W}(-1)^{\ell(w)} e^{w(\lambda+\rho)-(\lambda+\rho)} \cdot e^\delta}{\prod_{\alpha\in\Delta_+}(1-te^{-\alpha})^{m_\alpha}}.$$

If $w \in W$ has nonzero contribution to the sum on the right, then it must satisfy $(\lambda+\rho) - w(\lambda+\rho) \leq \delta$.

We now claim: This implies that $w \in \overset{\circ}{W}$.

To prove the claim, we write $\Lambda := \lambda + \rho$ and let $w = r_{i_1} r_{i_2} \cdots r_{i_k}$ be a reduced word for w. Here, the r_{i_j}'s are simple reflections in W $(0 \leq i_j \leq l)$. Observe that

$$\Lambda - w\Lambda = \sum_{p=1}^{k} (\Lambda, \alpha_{i_p}^\vee)\beta_p, \tag{6.1}$$

where $\beta_p := r_{i_1} r_{i_2} \cdots r_{i_{p-1}}(\alpha_{i_p}) \in \Delta_+$ for all j. Suppose $w \notin \overset{\circ}{W}$, and let j be the least index such that $i_j = 0$. Then $\beta_j = \alpha_0 + \bar{\alpha}$ where $\bar{\alpha}$ is a non-negative integer linear combination of $\alpha_{i_1}, \ldots, \alpha_{i_{j-1}}$. Now $(\Lambda, \alpha_{i_j}^\vee) = (\lambda + \rho, \alpha_0^\vee) \geq 2$. Thus the

j^{th} term on the right-hand side of Equation (6.1) is $\geq 2(\alpha_0+\bar{\alpha})$. This is clearly $\not\leq \delta$, since the coefficient of α_0 in δ is 1. This proves our claim.

Further, $w \in \overset{\circ}{W}$ implies that $w\lambda - \lambda = w\bar{\lambda} - \bar{\lambda}$. This implies that $K_{\lambda,\lambda-\delta}(t)$ is the coefficient of e^0 in the product

$$\frac{\sum_{w\in \overset{\circ}{W}} (-1)^{\ell(w)} e^{w(\bar{\lambda}+\rho)-(\bar{\lambda}+\rho)}}{\prod_{\beta\in\overset{\circ}{\Delta}_+}(1-e^{-\beta})} \prod_{\beta\in\overset{\circ}{\Delta}_+} \frac{1-e^{-\beta}}{1-te^{-\beta}} \; \frac{e^{\delta}}{\prod_{\alpha\in\Delta_+\backslash\overset{\circ}{\Delta}_+}(1-te^{-\alpha})^{m_\alpha}}.$$

By the Weyl character formula, the first term equals $e^{-\bar{\lambda}}\overset{\circ}{\chi}_{\bar{\lambda}}$. Put

(6.2)

$$\xi := \prod_{\beta\in\overset{\circ}{\Delta}_+} \frac{1-e^{-\beta}}{1-te^{-\beta}} = \prod_{\beta\in\overset{\circ}{\Delta}_+} (1+(t-1)e^{-\beta}+(t^2-t)e^{-2\beta}+\cdots) \text{ and}$$

(6.3)

$$\eta := \frac{e^{\delta}}{\prod_{\alpha\in\Delta_+\backslash\overset{\circ}{\Delta}_+}(1-te^{-\alpha})^{m_\alpha}} = e^{\delta} \prod_{\alpha\in\Delta_+\backslash\overset{\circ}{\Delta}_+} (1+m_\alpha t\, e^{-\alpha} + \binom{m_\alpha+1}{2} t^2\, e^{-2\alpha}+\cdots).$$

Thus $K_{\lambda,\lambda-\delta}(t) = [e^0]\ (e^{-\bar{\lambda}}\overset{\circ}{\chi}_{\bar{\lambda}})\ \xi\eta$. Note that $(e^{-\bar{\lambda}}\,\overset{\circ}{\chi}_{\bar{\lambda}})\,\xi$ is a power series only involving $e^{-\gamma}, \gamma \in \left(\overset{\circ}{Q}\right)^+$. We analyze η more closely. Given $\alpha \in Q$, write $\alpha = \sum_{i=0}^{l} n_i(\alpha)\alpha_i$ with $n_i(\alpha) \in \mathbb{Z}$. Then $\alpha \in \Delta_+\backslash\overset{\circ}{\Delta}_+$ implies that $n_0(\alpha) \geq 1$. Consider the expression on the far right of Equation (6.3). For each $\alpha \in \Delta_+\backslash\overset{\circ}{\Delta}_+$, suppose we throw away the terms $e^{-p\alpha}$ $(p \geq 1)$ for which $p\alpha \not\leq \delta$, it is clear that this does not affect the value of $K_{\lambda,\lambda-\delta}(t)$. If $n_0(\alpha) \geq 2$, we can throw away $e^{-p\alpha}$ for all $p \geq 1$ while if $n_0(\alpha) = 1$, we can throw away $e^{-p\alpha}$ for all $p \geq 2$. This means that if we define

$$\tilde{\eta} := e^{\delta} \prod_{\substack{\alpha\in\Delta_+\\ n_0(\alpha)=1}} (1+m_\alpha t\, e^{-\alpha}),$$

then the coefficients of e^0 in $(e^{-\bar{\lambda}}\overset{\circ}{\chi}_{\bar{\lambda}})\xi\eta$ and $(e^{-\bar{\lambda}}\overset{\circ}{\chi}_{\bar{\lambda}})\xi\tilde{\eta}$ are the same. Observe now that $\{\alpha \in \Delta_+ : n_0(\alpha) = 1\} = \{\delta\} \cup \{-\beta+\delta : \beta \in \overset{\circ}{\Delta}_+\}$. Since $m_\delta = l$ and $m_{-\beta+\delta} = 1$, this implies that

$$\tilde{\eta} := e^{\delta}(1+lte^{-\delta}) \prod_{\beta\in\overset{\circ}{\Delta}_+} (1+te^{\beta-\delta}).$$

Expanding out, we get

$$\tilde{\eta} := lt + t \sum_{\beta \in \overset{\circ}{\Delta}_+} e^{\beta} + \sum_{\substack{\gamma \in Q \\ n_0(\gamma) \neq 0}} p_\gamma(t) e^{\gamma}$$

for some $p_\gamma(t) \in \mathbb{Z}[t]$, i.e., the last sum runs over $\gamma \in Q \backslash \overset{\circ}{Q}$. As remarked before, $(e^{-\bar{\lambda}} \overset{\circ}{\chi}_{\bar{\lambda}})\xi$ is a power series involving $e^{-\gamma}$ with $n_0(\gamma) = 0$. Thus, finally,

$$K_{\Lambda_0, \Lambda_0 - \delta}(t) = lt + t \sum_{\beta \in \overset{\circ}{\Delta}_+} [e^{-\beta}]\, (e^{-\bar{\lambda}} \overset{\circ}{\chi}_{\bar{\lambda}})\, \xi\,. \tag{6.4}$$

We now turn to $\overset{\circ}{\mathfrak{g}}$. We have $V = \overset{\circ}{L}(\theta) \otimes \overset{\circ}{L}(\bar{\lambda})$. This decomposes into

$$\overset{\circ}{L}(\theta) \otimes \overset{\circ}{L}(\bar{\lambda}) = \bigoplus_{\pi \in \left(\overset{\circ}{P}\right)^+} c_\pi \overset{\circ}{L}(\pi) \quad \text{with } c_\pi \in \mathbb{Z}^{\geq 0}.$$

Thus $\overset{\circ}{\chi}_\theta \overset{\circ}{\chi}_{\bar{\lambda}} = \sum_\pi c_\pi \overset{\circ}{\chi}_\pi$. Consider the sum

$$\begin{aligned}
\sum_{\pi \in \overset{\circ}{P}^+} c_\pi \overset{\circ}{K}_{\pi, \bar{\lambda}}(t) &= [e^0] \quad \frac{\sum_\pi c_\pi \sum_{w \in \overset{\circ}{W}} (-1)^{\ell(w)} e^{w(\pi+\rho)-(\bar{\lambda}+\rho)}}{\prod_{\alpha \in \overset{\circ}{\Delta}_+} (1 - te^{-\alpha})} \\
&= [e^0] \quad e^{-\bar{\lambda}}\, \xi \sum_\pi c_\pi \overset{\circ}{\chi}_\pi \\
&= [e^0] \quad e^{-\bar{\lambda}}\, \xi\, \overset{\circ}{\chi}_\theta\, \overset{\circ}{\chi}_{\bar{\lambda}} \\
&= [e^0] \quad (e^{-\bar{\lambda}} \overset{\circ}{\chi}_{\bar{\lambda}}\, \xi)(l + \sum_{\alpha \in \overset{\circ}{\Delta}_+} (e^{\alpha} + e^{-\alpha})) \\
&= l + \sum_{\alpha \in \overset{\circ}{\Delta}_+} [e^{-\alpha}]\, (e^{-\bar{\lambda}}\, \overset{\circ}{\chi}_{\bar{\lambda}}\, \xi).
\end{aligned} \tag{6.5}$$

Comparing Equations (6.4) and (6.5), Proposition 4 is proved. □

Remark. The above proof works with no essential change if we replace the node 0 by a node p which satisfies the property that, if $\delta = \sum_{j=0}^{l} a_j \alpha_j$, then $a_p = 1$. Since all nodes of the Dynkin diagram of $A_n^{(1)}$ satisfy this property, we have the following corollary.

Corollary 6. *Let $\mathfrak{g} = A_l^{(1)}$, and λ be a dominant weight such that not all $(\lambda, \alpha_i^\vee)$ are zero (equivalently $\lambda \notin \mathbb{C}\delta$). Then $K_{\lambda, \lambda-\delta}(t) \in \mathbb{Z}_{\geq 0}[t]$.*

7. Miscellaneous facts

In this subsection, we collect together an assortment of facts concerning the Hall–Littlewood functions $P_\lambda(t)$. We work in the full generality of $\mathfrak{g}$ being a symmetrizable Kac–Moody algebra.

7.1. Recall from (3.1) that $P_\lambda(t) = \sum_{\mu \in P^+} c_{\lambda\mu}(t)\, \chi_\mu$. The $c_{\lambda\mu}(t)$ are the entries of the inverse matrix of $[K_{\lambda\mu}(t)]_{\lambda,\mu}$. Since this latter matrix is lower triangular with ones on the diagonal, each $c_{\lambda\mu}(t)$ is a polynomial in the $K_{\pi\gamma}(t)$ with integer coefficients. Theorem 1 then implies that $c_{\lambda\mu}(t) \in \mathbb{Z}[t]$ too (note that (3.2) only guaranteed $c_{\lambda\mu}(t) \in \mathbb{Z}[[t]]$). This means we can specialize t to any complex number a and $P_\lambda(a)$ would end up being a well-defined element of $\mathcal{E}$.

7.2. When $\mathfrak{g}$ is finite dimensional, it follows from an argument due to Stembridge [17] that $c_{\lambda\mu}(t)$ has the following expression.

$$c_{\lambda\mu}(t) := \sum_{\substack{A \\ w(\lambda+\rho-|A|)=\mu+\rho}} \sum_{w \in W} (-1)^{\ell(w)} (-t)^{\#A}, \tag{7.1}$$

where the outer sum only ranges over those $A \subset \Delta_+$ for which $(\lambda, \alpha) > 0$ for all $\alpha \in A$ (cf. Equation (3.2)). This expression explicitly demonstrates that $c_{\lambda\mu}(t)$ is a polynomial in t. Stembridge's elegant argument can be adapted without change to our present situation; in our case, it shows that for any symmetrizable Kac–Moody algebra $\mathfrak{g}$ and $\lambda, \mu \in P^+$ such that $\#W_\lambda < \infty$, $c_{\lambda\mu}(t)$ is given by the same (explicitly polynomial) expression of Equation (7.1), where we now let the outer sum range over all multisets $A \in \mathcal{A}$ which satisfy $(\lambda, \alpha) > 0$ for all $\alpha \in A$.

7.3. Next, we prove a necessary condition on (λ, μ) in order that $c_{\lambda\mu}(t) \neq 0$.

Proposition 5. *Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra, $\lambda, \mu \in P^+$. Then $c_{\lambda\mu}(t) \neq 0$ implies that there exists $B \in \mathcal{A}$ such that $\mu = \lambda - |B|$.*

Proof. From Equation (3.2), we have $c_{\lambda\mu}(t) \neq 0$ implies that there exists $w \in W, A \in \mathcal{A}$ such that $\mu + \rho = w(\lambda) + w(\rho - |A|)$. Since the formal character of $L(\rho)$ is $e^\rho \prod_{\alpha \in \Delta_+} (1 + e^{-\alpha})^{m_\alpha}$, the weights of $L(\rho)$ are $\{\rho - |B| : B \in \mathcal{A}\}$. Let $\gamma \in P^+$ be the unique dominant weight in the Weyl group orbit of $\rho - |A|$. Then $\gamma \leq \rho$ and there exists $\tau \in W$ such that $w(\rho - |A|) = \tau\gamma$. So, we have $\mu + \rho = w\lambda + \tau\gamma$ with $\lambda, \gamma, \mu + \rho \in P^+$. By the Parthasarathy–Ranga Rao–Varadarajan (PRV) conjecture (see [11] for a recent proof), this implies that $L(\mu + \rho)$ occurs as a summand in the decomposition of $L(\lambda) \otimes L(\gamma)$. It is well-known (see, for example, [12]) that this implies $\mu + \rho = \lambda + \eta$ for a weight η of $L(\gamma)$.

We now claim that η is also a weight of $L(\rho)$.

To prove the claim, let η^+ be the dominant weight in the Weyl orbit of η. Then $\eta^+ \leq \gamma \leq \rho$. Since $(\rho, \alpha_i) > 0$ for all i, Proposition 11.2 of [7] implies that η^+ is in fact a weight of $L(\rho)$. Hence, so is η.

This of course means that there exists $B \in \mathcal{A}$ such that $\eta = \rho - |B|$. Thus $\mu + \rho = \lambda + \rho - |B|$, proving our proposition. $\square$

We now deduce a corollary that was proved by R. Brylinksi in [5].

Corollary 7. *Let $\mathfrak{g}$ be finite dimensional. Then $c_{\lambda\mu}(t) \neq 0$ implies that $\mu \geq \lambda - 2\rho$.*

Proof. By our proposition above, $\mu = \lambda - |B|$ where B is a subset of Δ_+. This means $|B| \leq \sum_{\alpha\in\Delta_+} \alpha = 2\rho$. $\square$

7.4. Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra and consider the special case that $\lambda \in P^{++}$ (a regular dominant weight), i.e., $(\lambda, \alpha_i) > 0$ for all $i = 1, \dots, n$. Then $W_\lambda(t) = 1$. Consider the specialization of $P_\lambda(t)$ at $t = -1$ (for classical type A, these are the Schur Q-functions). Now,

$$P_\lambda(-1) = \frac{\sum_{w\in W} (-1)^{\ell(w)} w(e^{\lambda+\rho} \prod_{\alpha\in\Delta_+} (1 + e^{-\alpha})^{m_\alpha})}{e^\rho \prod_{\alpha\in\Delta_+} (1 - e^{-\alpha})^{m_\alpha}}.$$

As mentioned before, $\chi_\rho = e^\rho \prod_{\alpha\in\Delta_+} (1+e^{-\alpha})^{m_\alpha}$ is $\operatorname{ch} L(\rho)$ and thus W-invariant. Thus

$$P_\lambda(-1) = \frac{J(e^\lambda)}{J(e^\rho)} \chi_\rho.$$

By the Weyl–Kac character formula, this is just $\chi_{\lambda-\rho}\,\chi_\rho$ (observe that $\lambda \in P^{++}$ ensures $\lambda - \rho \in P^+$). Since $c_{\lambda\mu}(-1)$ is the coefficient of χ_μ in $P_\lambda(-1)$, we obtain the following generalization of a well-known result in the classical case.

Proposition 6. *Let $\lambda \in P^{++}$ and $\mu \in P^+$. Then $c_{\lambda\mu}(-1)$ is the multiplicity of $L(\mu)$ in the tensor product $L(\lambda - \rho) \otimes L(\rho)$. In particular, $c_{\lambda\mu}(-1) \in \mathbb{Z}^{\geq 0}$.*

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