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KRONECKER PRODUCT IDENTITIES FROM D-FINITE SYMMETRIC FUNCTIONS

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ABSTRACT. Using an algorithm for computing the symmetric function Kronecker product of D-finite symmetric functions we find some new Kronecker product identities. The identities give closed form formulas for trace-like values of the Kronecker product.

INTRODUCTION

In the process of showing how the scalar product of symmetric functions can be used for enumeration purposes, Gessel [3], proved that this product, and the Kronecker product, *preserve D-finiteness*. Roughly, this means that if F and G are symmetric functions which both satisfy a particular kind of system of linear differential equations, then so will the scalar and Kronecker products of these functions. In an earlier work [2], we give algorithms to calculate both of these systems of differential equations.

In this short note we use this algorithm in a symbolic way to find explicit expressions for Kronecker products of pairs of several common series of symmetric functions, such as complete ($\mathsf{H} = \sum_n h_n$), elementary ($\mathsf{E} = \sum_n e_n$) and Schur ($\mathsf{S} = \sum_n \sum_{\lambda \vdash n} s_\lambda$). Proposition 12 of [2], is the following identity,

$$\left(\sum_\lambda s_\lambda\right) * \left(\sum_\lambda s_\lambda\right) = \exp\left(\sum_{n\geq 1} \frac{p_{2n-1}}{(2n-1)(1-p_{2n-1})}\right)\left(\prod_{n\geq 1}(1-p_n^2)\right)^{-1/2}.$$

This is the generating series of $\sum_n \left(\sum_{\lambda\vdash n, \mu\vdash n, \lambda<\mu} s_\lambda * s_\mu\right)$. Here, we apply the same technique to give a table of new identities of the same flavour.

1. SYMMETRIC FUNCTIONS

We use notation as in Macdonald [7] for our symmetric functions. A *partition* of a positive integer n is a decreasing sequence of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ whose sum is n. This is denoted $\lambda \vdash n$. A partition is written in either vector

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or power notation, for example $(7,7,4,4,1) = [1\,4^2\,7^2]$ are both partitions of 23. A *symmetric function* is a sum of monomials in some variable set, that is invariant under any permutation of that variable set. We can write any symmetric function as a sum of *monomial symmetric functions*, defined for the variable set $\{x_1, x_2, \ldots\}$ with respect to some partition λ as

$$m_\lambda := \sum_{\sigma \in \mathcal{S}_{\mathbb{N}\setminus\{0\}}} (r_1!\, r_2! \cdots)^{-1}\, x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(k)}^{\lambda_k}.$$

For example, $m_{(3,2,2)} = x_1^3 x_2^2 x_3^2 + x_3^3 x_2^2 x_1^2 + x_4^3 x_1^2 x_3^2 + \ldots$. We also have the *elementary symmetric functions*, $e_n = m_{\langle 1^n \rangle}$, and $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$; the *complete symmetric functions* $h_n = \sum_{\lambda \vdash n} m_\lambda$, and $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k}$; and *power sum symmetric functions* $p_n = m_{(n)} = x_1^n + x_2^n + \ldots$, $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$. We postpone the definition of the Schur symmetric functions to the next section, where we shall be better equipped. Any of the h_λ, p_λ, e_λ, or s_λ can form a $\mathbb{Q}$-basis of the vector space Λ of symmetric functions. We can also view Λ as the ring $\mathbb{Q}[p_1, p_2 \ldots]$ and, finally, we also work in the ring $\hat{\Lambda} = \mathbb{Q}[[p_1, p_2, \ldots]]$.

1.1. **The scalar product of symmetric functions.**

The ring of symmetric series is endowed with a scalar product defined as a symmetric bilinear form such that the bases (h_λ) and (m_λ) are dual to each other:

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}. \tag{1}$$

It turns out that

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu},$$

with $z_\lambda = (1^{r_1} r_1!)(2^{r_2} r_2!) \cdots$ when $\lambda = [1^{r_1} 2^{r_2} \cdots]$.

The Schur basis is an orthonormal symmetric function basis under this scalar product. In fact, Schur functions can be defined as the result of applying the Gram–Schmidt process for orthogonalizing a basis, applied to the monomial basis with the partitions ordered lexicographically[1].

1.2. **Plethysm of symmetric functions.**

To conclude this brief recollection of symmetric functions, we describe one type of composition that turns out to be quite useful here: *plethysm*, written $f[g]$. We can most easily define it using the power sum symmetric functions. It is defined by $p_n[\psi(p_1, p_2, \ldots)] = \psi(p_n, p_{2n}, \ldots)$, along with $(\phi_1 + c\phi_2)[\psi] = \phi_1[\psi] + c\phi_2[\psi]$ and $(\phi_1 \cdot \phi_2)[\psi] = \phi_1[\psi] \cdot \phi_2[\psi]$.

[1] In such an ordering, $1^n < 1^{n-1}2 < \cdots < n$.

1.3. **The Kronecker product of symmetric functions.** In the ring of symmetric functions the usual polynomial multiplication serves as a product, but there is also a second product which arises from the connection between symmetric function and the characters of the symmetric group. This product has several names, including the *Kronecker product*, the tensor product and the internal product. Although we mostly follow the notation of Macdonald [7] for most matters relating to symmetric functions, we shall refer to it here as the Kronecker product, and denote it by $*$. It was first described by Redfield as the *cap product* of symmetric functions and was rediscovered by Littlewood [6]. This product can be defined in representation theory as the pointwise product of characters, which corresponds to tensor products of representations, however here we use the following relation to the power sum symmetric functions, and extend linearly:

$$p_\lambda * p_\mu = \delta_{\lambda\mu} z_\lambda p_\lambda. \tag{2}$$

Calculating the connection coefficients $\gamma_{\lambda,\mu}^{(\rho)}$ for the Kronecker product in the Schur basis

$$s_\lambda * s_\mu = \sum_\rho \gamma_{\lambda,\mu}^{(\rho)} s_\rho$$

is also challenging, and quite interesting. There are some combinatorial interpretations of $\gamma_{\lambda,\mu}^{(\rho)}$ which have obtained results when λ, ρ and μ are of a particular form, such as work of Goupil and Schaeffer [5], Rosas [8], or Chauve and Goupil [4]. The interest originates from the correspondence with irreducible representations,

$$\chi(\mathcal{V}_\lambda \otimes \mathcal{V}_\mu) = \sum_\rho \gamma_{\lambda,\mu}^{(\rho)} \chi(\mathcal{V}_\rho).$$

When λ, μ and ρ are all partitions of n, $\gamma_{\lambda,\mu}^{(\rho)}$ is the multiplicity of a character in the representation. For more details, the reader is pointed towards the text of Sagan [9].

To compute $\gamma_{\lambda,\mu}^{(\rho)}$ using computer algebra systems, one typically expands the symmetric function into the power sum basis and then applies (2) to a pairwise comparison of terms. (For example, in the SF package of Stembridge.) As we mentioned in the introduction, we introduced a generating function approach [2]. The algorithm in [2] that we use is called ITENSOR_DE, and a Maple implementation on the author's web page is available.

A second approach, summarized in [11], uses a reduced notation that allows calculations with series of the form $\sum_n s_{(n,\lambda_2,\dots,\lambda_k)} z^n$, for fixed $\lambda_2, \dots, \lambda_k$. These computations are quite efficient; far more so than expanding the power-sum basis.

2. Applications of D-finite symmetric series

The algorithms we use do not compute the products directly, rather they determine differential equations satisfied by the resulting function. The existence of such differential equations is a consequence of the D-finite closure properties of the scalar product. A function ϕ is said to be *D-finite* in $\mathbb{K}[[x_1, \ldots, x_r]]$ if and only if the partial derivatives $\partial_1^{\alpha_1} \cdots \partial_r^{\alpha_r} \phi$ generate a finite dimensional vector space over $\mathbb{K}(x_1, \ldots, x_r)$. In this case, ϕ is determined by a system of linear differential equations.

In order to treat symmetric functions, however, we must consider functions with an infinite number of variables. The function $\phi(x_1, x_2, \ldots)$ is D-finite in $K[[x_1, x_2, \ldots]]$ if for all r, $\phi(x_1, \ldots, x_r, 0, \cdots)$ is D-finite in $K[[x_1, x_2, \ldots, x_r]]$. This case does not enjoy all of the closure properties of the previous, nonetheless we have closure under $+$, $\times$, ∂_i, extension of coefficients, rational substitution, and exponentials of polynomials. We say that a symmetric function $\phi \in \hat{\Lambda}$ is D-finite if it is D-finite in $\mathbb{Q}[[p_1, p_2, \ldots]]$. For example, under this definition the two following famous symmetric function sums H and E which we introduced earlier satisfy the following relations,

$$\mathsf{H} = \exp\left(\sum_n \frac{p_n}{n}\right) \text{ and } \mathsf{E} = \exp\left(\sum_n (-1)^n \frac{p_n}{n}\right),$$

and thus are both D-finite.

It was Gessel [3] who first showed that the scalar product and the Kronecker product both preserve D-finiteness. The work [2] makes this effective by transforming the system of differential equations satisfied by F and G in to one satisfied by $F * G$, or $\langle F, G\rangle$.

2.1. Kronecker product calculations.

Many interesting problems which use the Kronecker product involve symmetric functions, which once they are expressed in the power sum basis, require an infinite number of p_n. Thus, at first glance they are seemingly unsuitable for direct application of our algorithms which, after all, require finite input! One approach is to apply these algorithms for several truncations of the symmetric functions and generate information upon which reasonable conjectures can be formulated. For each of these, we render the problem applicable by setting most p_n's to 0. That is, we solve a sequence of problems involving an increasing number of p_n, and hope to identify a pattern.

However, far more satisfying are the cases where there is sufficient form and structure which can be exploited to find exact results. We shall be more specific about precisely the "form and structure" we can exploit in a moment. First we remark that one important such class comes from symmetric series arising from plethysms. In this case, we can reduce the Kronecker product of functions each with an infinite number of p_n variables to a finite number of symbolic calculations.

For example, if two symmetric functions F and G can be expressed respectively in the form

$$F(p_1, p_2, \ldots) = \prod_{n\geq 1} f_n(p_n) \qquad \text{and} \qquad G(p_1, p_2, \ldots) = \prod_{n\geq 1} g_n(p_n),$$

then one can easily deduce that

$$F * G = \prod_{n\geq 1} f_n(p_n) * g_n(p_n). \tag{3}$$

Essentially this follows from the fact that the Kronecker product of two power sum symmetric functions of differing order is 0. If, furthermore, the f_n and g_n are such that one can describe them in a finite way using D-finite functions, we can apply this method.

Series which arise as plethysms of the form $\mathsf{H}[u]$ or $\mathsf{E}[u]$, where u is a polynomial in the p_i, are precisely of this form. For example the sum of all Schur functions is of this type:

$$\mathsf{S} = \sum_\lambda s_\lambda = \mathsf{H}[p_1 + \frac{1}{2}p_1^2 - \frac{1}{2}p_2] = \exp\left(\sum_n \frac{p_n^2}{2n} + \frac{p_{2n-1}}{2n-1}\right).$$

Thus,

$$\mathsf{S} = \left(\prod_{n \text{ even}} \exp\left(\frac{p_n^2}{2n}\right)\right)\left(\prod_{n \text{ odd}} \exp\left(\frac{p_n^2}{2n} + \frac{p_n}{n}\right)\right)$$

We assign f_n as follows

$$f_{2n} = \exp\left(\frac{p_{2n}^2}{4n}\right) \qquad \text{and} \qquad f_{2n-1} = \exp\left(\frac{p_{2n-1}^2}{2} + \frac{p_{2n-1}}{2n-1}\right).$$

Thus, to compute $\mathsf{S}*\mathsf{S}$, we compute in turn $g_{2n} = f_{2n} * f_{2n}$, and $g_{2n+1} = f_{2n+1} * f_{2n+1}$. We find g_{2n} by determining the differential equation that it satisfies, using ITENSOR_DE adapted to handle a formal parameter. The adaptation amounts to performing a scalar product with adjunction formula $p^\diamond = n\partial$ for a formal parameter n. This gives

$$(1 - p_n^2)\frac{\partial g_n(p_n)}{\partial p_n} + p_n g_n(p_n) = 0, \qquad \text{for even } n.$$

We then solve for g_n. We do likewise for odd n, and then the identity in the introduction follows.

Carbonara *et al.* [1] are interested in the trace of $\sum_n \left(\sum_{\lambda\vdash n, \mu\vdash n, \lambda<\mu} s_\lambda * s_\mu\right)$ given by $\sum_n \left(\sum_{\lambda\vdash n} s_\lambda * s_\lambda\right)$. It is not immediately clear to me if our method could be adapted directly to this kind of calculation.

2.2. **A family of identities.** We now apply the above approach to create a number of different identities. The following table summarizes results. These formulas for $\mathsf{H}, \mathsf{E}, \mathsf{S}, \mathsf{SE}^{-1}$ and SH^{-1} are all derived in Macdonald [7]:

$$\begin{aligned}
&\mathsf{H} = \sum h_n t^n = \exp\left(\sum_n \tfrac{p_n}{n}\right)\\
&\mathsf{E} = \sum e_n t^n = \exp\left(\sum_n (-1)^{n+1} \tfrac{p_n}{n}\right)\\
&\mathsf{S} = \sum_\lambda s_\lambda t^{|\lambda|} = \exp\left(\sum_n \tfrac{p_n^2 t^{2n}}{2n} + \tfrac{p_{2n-1} t^{2n-1}}{2n-1}\right)\\
&\mathsf{SE}^{-1} = \sum_{\lambda \text{ all parts odd}} s_\lambda t^{|\lambda|}\\
&\mathsf{SH}^{-1} = \sum_{\lambda' \text{all parts even}} s_\lambda t^{|\lambda|}
\end{aligned}$$

Theorem 2.1. *Given the above definitions for* H, E *and* S*. Then, there is the following multiplication table for the Kronecker product,*

$*$	H	E	S	SH^{-1}	SE^{-1}
H	H	E	S	SH^{-1}	SE^{-1}
E		H	S	SE^{-1}	SH^{-1}
S			GM_{odd}	GN	GN
SH^{-1}				GM_{even}	GP
SE^{-1}					GM_{even}

.

The products are expressed in terms of the following:

$$\begin{aligned}
&\mathsf{M}_{odd(even)} = \exp\left(\sum_{n\ odd(even)} \tfrac{p_n}{n(1-p_n)}\right)\\
&\mathsf{N} = \exp\left(\sum_n \tfrac{p_n^2}{2n(1-p_n^2)}\right)\\
&\mathsf{P} = \exp\left(\sum_{n\ even} \tfrac{p_n}{n(1+p_n)}\right)\\
&\mathsf{G} = \prod_{n\geq 1} \left(1-p_n^2\right)^{-1/2}.
\end{aligned}$$

A Maple worksheet with the calculations behind the above table is available at the author's website. We welcome all suggestions for other series of interest. It would be equally easy to treat plethysms of the form $\mathsf{H}[\phi]$ for some symmetric polynomial ϕ. A preliminary review of the work of Scharf, Thibon, and Wybourne, for example [10, 11] suggests that there may be more to do with series of the form $\sum_n s_{(n,\lambda_2,\ldots,\lambda_k)} z^n$, for fixed $\lambda_2, \ldots, \lambda_k$ if they can be shown to be D-finite.

We are able to compute, with this method, expressions satisfied by some powers of $\mathsf{S} = \sum_\lambda s_\lambda$, with respect to the Kronecker product, for example $\mathsf{S} * \mathsf{S} * \mathsf{S}$, but these result in differential equations which we are presently unable to solve into explicit expressions.

Conclusion

The symbolic application of tensor product calculation yields, rather easily, families of Kronecker product identities. It is possible that these identities could be exploited for group theoretic gain, however, this remains to be investigated, as does finding connections between our formulas, and that of the trace co-characters.

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References

[1] J. O. Carbonara, L. Carini, and J. B. Remmel. Trace cocharacters and the Kronecker products of Schur functions. *J. Algebra*, 260(2):631–656, 2003.

[2] Frédéric Chyzak, Marni Mishna, and Bruno Salvy. Effective scalar products of D-finite symmetric series. *J Combin. Theory Ser. A*, 112:1 – 43, 2005.

[3] Ira M. Gessel. Symmetric functions and P-recursiveness. *J. Combin. Theory Ser. A*, 53(2):257–285, 1990.

[4] Alain Goupil and Cédric Chauve. Combinatorial operators for Kronecker powers of representations of S_n. *Séminaire Lotharingien Combin.*, 54, 2006. Article B54j, 13 pages.

[5] Alain Goupil and Gilles Schaeffer. Factoring n-cycles and counting maps of given genus. *European J. Combin.*, 19(7):819–834, 1998.

[6] D. E. Littlewood. The Kronecker product of symmetric group representations. *J. London Math Soc.*, 31:89–93, 1956.

[7] Ian G. Macdonald. *Symmetric functions and Hall polynomials*. The Clarendon Press Oxford University Press, New York, second edition, 1995.

[8] Mercedes H. Rosas. The Kronecker product of Schur functions indexed by two-row shapes or hook shapes. *J. Algebraic Combin.*, 14(2):153–173, 2001.

[9] Bruce E. Sagan. *The symmetric group*, volume 203 of *Graduate Texts in Mathematics*. Springer–Verlag, New York, second edition, 2001.

[10] T. Scharf, and J.-Y. Thibon. A Hopf-algebra approach to inner plethysm. *Adv. Math.* 104:30–58 (1994).

[11] T. Scharf, J.-Y. Thibon and B. G. Wybourne. Reduced notation, inner plethysms and the symmetric group. *J. Phys. A: Math. Gen.*, 26(24):7461–7478, 1993.

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