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AN IDENTITY IN ROTA–BAXTER ALGEBRAS

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Abstract. We give explicit formulae and study the combinatorics of an identity holding in all Rota–Baxter algebras. We describe the specialization of this identity for a couple of examples of Rota–Baxter algebras.

1. Introduction

The study of Rota–Baxter algebras was initiated by Baxter in his works [2, 3]. The theory was later taken over by Rota [11], who gave an explicit construction of free Rota–Baxter algebras and uncovered the relationship with symmetric functions. Soon after, Cartier studied free Rota–Baxter algebras in [4]. In the last few years the theory of Rota–Baxter algebras has received a great impulse, mainly because of its applications to renormalization, as formalized by Connes and Kreimer in [5, 6, 7]. New techniques and applications of Rota–Baxter algebras have been found by an active group of researches in a number of important works, among which we cite just a few [1, 8, 9, 10].

In this work we consider the seemingly naive problem of writing an element of the form $P^a(x)P^b(y)$ in a Rota–Baxter algebra as a linear combination of terms of the form $P^j(xP^i(y))$ and $P^j(P^i(x)y)$, with i and j varying. The existence of such a linear combination is an immediate consequence of the Rota–Baxter identity satisfied by the operator P. The actual problem is to determine the coefficients involved in such an expression as explicitly as possible. We provide a solution to this problem, and take a look at its meaning in a couple of Rota–Baxter algebras. We approach our problem from a rather pedestrian point of view using a graphical notation to illustrate our ideas.

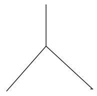
2. Basic ideas

Let us fix a field k of characteristic zero. A Rota–Baxter algebra is a triple (A, λ, P) where A is an associative k-algebra, λ is a constant in k, and $P : A \longrightarrow A$ is a k-linear operator satisfying the identity

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy)$$

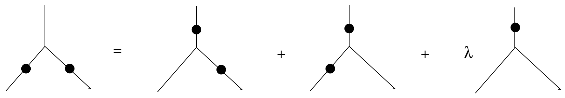
for $x, y \in A$. We find it convenient to use a graphical notation to express our results.

We represent the product on A by

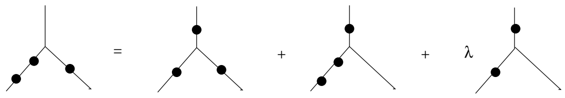


and the Rota–Baxter operator by

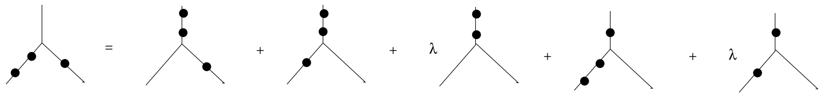
The Rota–Baxter identity satisfied by P is represented graphically by



For example using the graphical form of the Rota–Baxter identity one can see that



A further application of the graphical Rota–Baxter identity yields

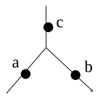


Thus we have shown that the following identity holds in any Rota–Baxter algebra:

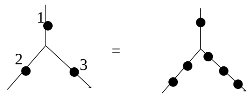
$$P^2(x)P(y) = P^2(xP(y)) + P^2(P(x)y) + \lambda P^2(xy) + P(P^2(x)y) + \lambda P(P(x)y).$$

The symbol $T(a, b, c)$ has two different meanings in this work:

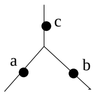
- On the one hand it stands for the operator $P^c(m \circ (P^a \otimes P^b)) : A \otimes A \longrightarrow A$ where m denotes the product on A.
- On the other hand it represents the tree with a dots on the left leg, b dots on the right leg, and c dots on the neck. The tree $T(a, b, c)$ is drawn as follows



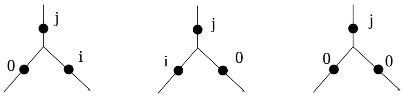
For example the tree $T(1, 2, 3)$ is represented graphically as follows:



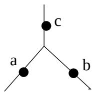
It is clear from the graphical Rota–Baxter identity that each tree



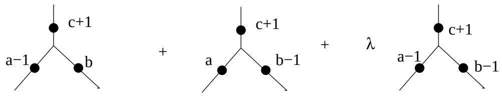
can be written as a linear combination with coefficients in $k[\lambda]$ of trees of the form



Indeed each application of the graphical Rota–Baxter relation replaces the tree



by the sum of trees



From an algorithmic point of view the graphical Rota–Baxter identity can be described as the application of three weighted moves:

(1) A weight 1 move where a dot from the left leg moves up.
(2) A weight 1 move where a dot from the right leg moves up.
(3) A weight λ move where a couple of dots, one from the left leg and another one from the right leg, merge and move up as one dot.

We are ready to formulate our main results.

3. Restricted case $\lambda = 0$

The case of Rota–Baxter algebras with $\lambda = 0$ simplifies considerably. We report on this special case because of its applications and elegant proof.

Theorem 1. Let $a, b > 1$ and $c \geq 0$ be integers. The following identity holds in any Rota–Baxter algebra:

$$T(a,b,c) = \sum_{i=1}^{b} \binom{a-1+b-i}{a-1} T(0,i,a+b+c-i) + \sum_{i=1}^{a} \binom{b-1+a-i}{b-1} T(i,0,a+b+c-i).$$

Proof. We justify only the left summand, the right summand is justified in an analogous way. For $\lambda = 0$ only move 1 and move 2 are allowed. With each move a dot from one of the legs moves up. Suppose that after applying several times the Rota–Baxter identity to $T(a,b,c)$ we arrive at a tree of the form $T(0,i,j)$. Then a total of $a+b-i$ dots from the legs have moved up so $j = a+b+c-i$. Necessarily the last dot moving up comes from the left leg. The other dots moved up in an arbitrary order, so this explain the factor

$$\binom{a-1+b-i}{a-1}.$$

□

Consider the Rota–Baxter algebra $(C(\mathbb{R}), 0, P)$, where $C(\mathbb{R})$ denotes the algebra of continuous functions on $\mathbb{R}$ and P is the Riemann integral operator given by

$$P(f)(y) = \int_0^y f(x)dx.$$

For real numbers $0 \leq x \leq y$ and $a \in \mathbb{N}_+$, let $\Delta^y_{x,a}$ be the convex polytope

$$\Delta^y_{x,a} = \{(x_1, ..., x_a) \in (\mathbb{R}_{\geq 0})^n \ \mid \ x \leq x_1 \leq ... \leq x_a \leq y\}.$$

For $a \geq 1$ we let $v_a(x,y)$ be the volume of $\Delta^y_{x,a}$. By convention we set $v_0(x,y) = 1$. It is easy to check that

$$P^{a+1}(f)(y) = \int_0^y f(x)v_a(x,y)dx.$$

Theorem 1 implies the following result.

Theorem 2.

$$\begin{aligned}
&\int_0^z \int_0^z f(x)g(y)v_a(x,z)v_b(y,z)dxdy \\
&\quad = \sum_{i=1}^{b+1} \binom{a+b+1-i}{a} \int_{0\leq x\leq y\leq z} g(x)f(y)v_{i-1}(x,y)v_{a+b+1-i}(y,z)dxdy \\
&\qquad + \sum_{i=1}^{a+1} \binom{b+a+1-i}{b} \int_{0\leq x\leq y\leq z} f(x)g(y)v_{i-1}(x,y)v_{b+a+1-i}(y,z)dxdy.
\end{aligned}$$

4. Generic case

Now we consider the generic situation, i.e., a Rota–Baxter algebra with $\lambda \neq 0$.

Theorem 3. Let $a, b \geq 1$ and $c \geq 0$ be integers. The following identity holds in any Rota–Baxter algebra:

$$\begin{aligned} T(a,b,c) = &\sum_{(i,j)\in D_1} c_1(a,b;i,j)T(0,i,c+j) \\ &+ \sum_{(i,j)\in D_2} c_2(a,b;i,j)T(0,i,c+j) \\ &+ \sum_{(i,j)\in D_3} c_2(a,b;i,j)T(i,0,c+j) \\ &+ \sum_{(i,j)\in D_4} c_4(a,b;i,j)T(i,0,c+j) \\ &+ \sum_{j\in D_5} c_5(a,b;j)T(0,0,c+j) \end{aligned}$$

where

$$\begin{aligned} D_1 &= \{(i,j) \in \mathbb{N}_+ \times \mathbb{N}_+ \mid 1 \leq i \leq b, \quad a \leq j, \quad b-i+1 \leq j, \quad j \leq a+b-i\}, \\ D_2 &= \{(i,j) \in \mathbb{N}_+ \times \mathbb{N}_+ \mid 1 \leq i \leq b-1, \quad a \leq j, \quad b \leq j, \quad j \leq a+b-1\}, \\ D_3 &= \{(i,j) \in \mathbb{N}_+ \times \mathbb{N}_+ \mid 1 \leq i \leq a, \quad a-i+1 \leq j, \quad b \leq j, \quad j \leq a+b-i\}, \\ D_4 &= \{(i,j) \in \mathbb{N}_+ \times \mathbb{N}_+ \mid 1 \leq i \leq a-1, \quad a-i \leq j, \quad b \leq j, \quad j \leq a+b-i-1\}, \\ D_5 &= \{j \in \mathbb{N} \mid a \leq j, \quad b \leq j, \quad j \leq a+b-1\}, \end{aligned}$$

and

$$\begin{aligned} c_1(a,b;i,j) &= \binom{j-1}{i+j-b-1, \quad j-a, \quad a+b-i-j} \lambda^{a+b-i-j}, \\ c_2(a,b;i,j) &= \binom{j-1}{j-b, j-a, a+b-j-1} \lambda^{a+b-j}, \\ c_3(a,b;i,j) &= \binom{j-1}{j-b, i+j-a-1, a+b-i-j} \lambda^{a+b-i-j}, \\ c_4(a,b;i,j) &= \binom{j-1}{j-b, i+j-a, a+b-i-j-1} \lambda^{a+b-i-j}, \\ c_5(a,b;j) &= \binom{j-1}{j-b, j-a, a+b-j-1} \lambda^{a+b-j}. \end{aligned}$$

Proof. Notice that with each move a dot is added to the neck. If starting from the tree $T(a,b,c)$ we arrive using the allowed moves to the graph $T(0,i,c+j)$, then necessarily we must have applied j moves and the last move must have been either move 1 or move 3. Let us consider the case were the last move is of type 1. The other $j-1$ moves are distributed into k_1 moves of type 1, k_2 moves of type 2, and k_3 moves of type 3, giving rise to the combinatorial number

$$\binom{j-1}{k_1, k_2, k_3}.$$

The numbers k_1, k_2 and k_3 are subject to the constraints

$$k_1 + k_2 + k_3 = j - 1, \quad k_1 + k_3 = a - 1 \text{ and } k_2 + k_3 = b - i - 1.$$

Solving this linear system of equations we find that

$$k_1 = i + j - b - 1, \quad k_2 = j - a \text{ and } k_3 = a + b - i - j.$$

This justifies the expression for $c_1(a, b, c; i, j)$ from the statement of the Theorem. We proceed to justify the expression for $c_2(a, b, c; i, j)$ which arises when the last move taken in the path towards $T(0, i, j)$ is of type 3. The remaining new $j - 1$ dots in the neck move up as consequence of the application of any of the moves, giving rise to the factor $\binom{j-1}{k_1,k_2,k_3}$ where k_1, k_2 and k_3 are subject to the constraints

$$k_1 + k_2 + k_3 = j - 1, \quad k_1 + k_3 = a - 1, \quad k_2 + k_3 = b - i - 1.$$

Solving this equations we find that

$$k_1 = i + j - b, \quad k_2 = j - a, \quad k_3 = a + b - i - j - 1.$$

Thus we have justified the factor

$$\binom{j-1}{j-b, j-a, a+b-i-j-1} \lambda^{a+b-i-j-1}$$

appearing in the formula for $c_2(a, b, c; i, j)$. The formulas for $c_3(a, b, c; i, j)$ and $c_4(a, b, c; i, j)$ are derived in a fairly similar way. Let us consider the formula for $c_5(a, b, c; j)$. In this case the last move is necessarily of type 3 and gives rise to the factor $\binom{j-1}{k_1,k_2,k_3}$, where k_1, k_2 and k_3 satisfy the constraints

$$k_1 + k_2 + k_3 = j - 1, \quad k_1 + k_3 = a - 1, \quad k_2 + k_3 = b - 1.$$

We find that

$$k_1 = j - b, \quad k_2 = j - a, \quad k_3 = a + b - j - 1,$$

which justifies the factor

$$\binom{j-1}{j-b, j-a, a+b-j-1} \lambda^{a+b-j}$$

appearing in the formula for $c_5(a, b, c; j)$.

□

Consider the Rota–Baxter algebra $(\{f : \mathbb{N} \longrightarrow \mathbb{C}\}, -1, P)$, where the operator

$$P : \{f : \mathbb{N} \longrightarrow \mathbb{C}\} \longrightarrow \{f : \mathbb{N} \longrightarrow \mathbb{C}\}$$

is given by the Riemann sum

$$P(f)(m) = \sum_{n=1}^{m} f(n).$$

For $a \geq 1$ one can check that

$$P^a f(m) = \sum_{1 \leq n_1 \leq \ldots \leq n_a \leq m} f(n_1).$$

In particular,

$$P^a 1(m) = |\Omega_a^m|,$$

where

$$\Omega_a^m = \{(n_1, \ldots, n_a) \in \mathbb{N}_+ \ \mid \ 1 \leq n_1 \leq \ldots \leq n_a \leq m\}.$$

It is not hard to show using the Chu–Vandermonde identity that

$$|\Omega_a^m| = \sum_{s=1}^{m} \binom{m}{s}\binom{a}{s} = \binom{a+m}{m} - 1.$$

As a consequence of Theorem 3 we get that the numbers $|\Omega_a^m|$ satisfy the following identity.

Theorem 4. For integers $a, b \geq 1$, we have

$$\begin{aligned} |\Omega_a^m||\Omega_b^m| = &\sum_{(i,j)\in D_1} [c_1(a,b;i,j)(-1)]|\Omega_{i+j}^m| \\ &+ \sum_{(i,j)\in D_2} [c_2(a,b;i,j)(-1)]|\Omega_{i+j}^m| \\ &+ \sum_{(i,j)\in D_3} [c_2(a,b;i,j)(-1)]|\Omega_{i+j}^m| \\ &+ \sum_{(i,j)\in D_4} [c_4(a,b;i,j)(-1)]|\Omega_{i+j}^m| \\ &+ \sum_{j\in D_5} [c_5(a,b;j)(-1)]|\Omega_j^m|. \end{aligned}$$

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