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EXCEDANCE NUMBER FOR INVOLUTIONS IN COMPLEX REFLECTION GROUPS

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ABSTRACT. We define the excedance number on the complex reflection groups and compute its multidistribution with the number of fixed points on the set of involutions in these groups. We use some recurrences and generating functions manipulations to obtain our results.

1. INTRODUCTION

Let V be a complex vector space of dimension n. A *pseudo-reflection* on V is a linear transformation on V of finite order which fixes a hyperplane in V pointwise. A *complex reflection group* on V is a finite subgroup W of $\mathrm{GL}(V)$ generated by pseudo-reflections.

Irreducible finite complex reflection groups have been classified by Shephard-Todd [3]. In particular, there is a single infinite family of groups and exactly 34 other "exceptional" complex reflection groups. The infinite family $G_{r,p,n}$, where r, p, n are positive integers with $p \mid r$, consists of the groups of $n \times n$ matrices such that:

(1) The entries are either 0 or r^{th} roots of unity;
(2) There is exactly one nonzero entry in each row and each column;
(3) The $(r/p)^{\text{th}}$ power of the product of the nonzero entries is 1.

The classical Weyl groups appear as special cases: $G_{1,1,n} = S_n$ the symmetric group, $G_{2,1,n} = B_n$ the hyperoctahedral group, and $G_{2,2,n} = D_n$, the group of even-signed permutations.

In S_n one can define the following well-known parameters: given $\sigma \in S_n$, $i \in [n]$ is *an excedance of* σ if and only if $\sigma(i) > i$. The number of excedances is denoted by $\mathrm{exc}(\sigma)$. Another natural parameter on S_n is the number of fixed points, denoted by $\mathrm{fix}(\sigma)$.

We say that a permutation $\pi \in G_{r,p,n}$ is *an involution* if $\pi^2 = 1$. Let $I_{r,p,n}$ be the set of involutions in the complex reflection group $G_{r,p,n}$.

In this paper we are interested in computing the number of involutions having specific numbers of fixed points and excedances. We do this by producing recurrences, and computing them explicitly by the corresponding generating functions.

Here are our main results (the relevant definitions can be found in Section 3).

Theorem 1.1. (*See Corollaries 5.4, 5.5 and 5.6*).

(1) *The number of involutions* $\pi \in G_{r,p,n}$ *where* r *is odd and* $p \mid r$ *with* $\mathrm{exc}^{\mathrm{Clr}}(\pi) = m$ *is*

$$\sum_{j=\frac{n}{2}}^{n} (n-j)! \binom{n}{n-j,\ n-j,\ j-k,\ k-n+j} \left(\frac{r}{2}\right)^{n-j}.$$

(2) *The number of involutions* $\pi \in G_{r,p,n}$ *where* r *is even and* $p \mid \frac{r}{2}$ *with* $\mathrm{exc}^{\mathrm{Clr}}(\pi) = m$ *is*

$$k! \binom{n}{k,\ k,\ n-2k} \left(\frac{r}{2}\right)^{k},$$

where $k = \frac{m}{r}$.

Theorem 1.2. (*See Corollary 5.10*).

The number of involutions $\pi \in G_{r,p,n}$ (r *is even,* $p \nmid \frac{r}{2}$) *with* $\mathrm{exc}^{\mathrm{Clr}}(\pi) = m$ *is*

$$\frac{(\frac{m}{r})!}{2^{\frac{m}{r}}} \binom{n}{\frac{m}{r},\ \frac{m}{r},\ n-2\frac{m}{r}} (r+1)^{\frac{m}{r}}.$$

This paper is organized as follows. In Section 2, we recall some properties of $G_{r,p,n}$. In Section 3 we define some parameters on $G_{r,n}$ and hence also on $G_{r,p,n}$. In Section 4 we classify the involutions of $G_{r,n}$ and $G_{r,p,n}$, and finally in Section 5 we compute the corresponding recurrences and explicit formulas.

2. Preliminaries

2.1. Complex reflection groups.

Definition 2.1. Let r and n be positive integers. *The group of colored permutations of* n *digits with* r *colors* is the wreath product

$$G_{r,n} = \mathbb{Z}_r \wr S_n = \mathbb{Z}_r^n \rtimes S_n,$$

consisting of all the pairs (z, τ) where z is an n-tuple of integers between 0 and $r-1$ and $\tau \in S_n$. The multiplication is defined by the following rule: for $z = (z_1, \ldots, z_n)$ and $z' = (z'_1, \ldots, z'_n)$

$$(z, \tau) \cdot (z', \tau') = ((z_1 + z'_{\tau^{-1}(1)}, \ldots, z_n + z'_{\tau^{-1}(n)}), \tau \circ \tau')$$

(here $+$ is taken modulo r).

We use some conventions along this paper. For an element $\pi = (z,\tau) \in G_{r,n}$ with $z = (z_1,\dots,z_n)$ we write $z_i(\pi) = z_i$. For $\pi = (z,\tau)$, we denote $|\pi| = (0,\tau), (0 \in \mathbb{Z}_r^n)$.

A much more natural way to present $G_{r,n}$ is the following: consider the alphabet $\Sigma = \{1^{[0]},\dots,n^{[0]},1^{[1]},\dots,n^{[1]},\dots,1^{[r-1]},\dots,n^{[r-1]}\}$ as the set $[n] = \{1,\dots,n\}$ colored by the colors $0,\dots,r-1$. Then, an element of $G_{r,n}$ is a *colored permutation*, i.e., a bijection $\pi : \Sigma \to \Sigma$ such that if $\pi(i) = k^{[t]}$ then $\pi(i^{[j]}) = k^{[t+j]}$ where $0 \le j \le r-1$ and the addition is taken modulo r. Occasionally, we write j bars over a digit i instead of $i^{[j]}$. For example, an element $(z,\tau) = ((1,0,3,2),(2,1,4,3)) \in G_{3,4}$ will be written as $(\bar{2}1\bar{\bar{\bar{4}}}\bar{\bar{3}})$.

For each $p \mid r$ we define the *complex reflection group*

$$G_{r,p,n} := \{g \in G_{r,n} \mid \operatorname{csum}(g) \equiv 0 \bmod p\}, \tag{2.1}$$

where

$$\operatorname{csum}(\sigma) = \sum_{i=1}^{n} z_i(\sigma).$$

3. Statistics on $G_{r,n}$ and its subgroups

In this section we define some parameters on $G_{r,n}$. The complex reflection group $G_{r,p,n}$ inherits all of them. Given any ordered alphabet Σ', we recall the definition of the *excedance set* of a permutation π on Σ',

$$\operatorname{Exc}(\pi) = \{i \in \Sigma' \mid \pi(i) > i\},$$

and the *excedance number* is defined to be $\operatorname{exc}(\pi) = |\operatorname{Exc}(\pi)|$.

Definition 3.1. We define the color order on the set

$$\Sigma = \{1,\dots,n,\bar{1},\dots,\bar{n},\dots,1^{[r-1]},\dots,n^{[r-1]}\}$$

by

$$1^{[r-1]} < \cdots < n^{[r-1]} < 1^{[r-2]} < 2^{[r-2]} < \cdots < n^{[r-2]} < \cdots < 1^{[0]} < \cdots < n^{[0]}.$$

We note that there are some other possible ways of defining orders on Σ, some of them lead to other versions of the excedance number, see for example [1].

Example 3.2. *Given the color order*

$$\bar{\bar{1}} < \bar{\bar{2}} < \bar{\bar{3}} < \bar{1} < \bar{2} < \bar{3} < 1 < 2 < 3,$$

we write $\sigma = (3\bar{1}\bar{\bar{2}}) \in G_{3,3}$ *in an extended form:*

$$\begin{pmatrix} \bar{\bar{1}} & \bar{\bar{2}} & \bar{\bar{3}} & \bar{1} & \bar{2} & \bar{3} & 1 & 2 & 3 \\ \bar{\bar{3}} & 1 & \bar{2} & \bar{3} & \bar{\bar{1}} & 2 & 3 & \bar{1} & \bar{\bar{2}}. \end{pmatrix}$$

We calculate: $\mathrm{Exc}(\sigma) = \{\bar{\bar{1}}, \bar{\bar{2}}, \bar{\bar{3}}, \bar{1}, \bar{3}, 1\}$ *and* $\mathrm{exc}(\sigma) = 6$*.*

Let $\sigma \in G_{r,n}$. Recall the definition of the parameter csum from the end of Section 2 as

$$\mathrm{csum}(\sigma) = \sum_{i=1}^{n} z_i(\sigma),$$

and define

$$\mathrm{Exc}_A(\sigma) = \{i \in [n-1] \mid \sigma(i) > i\},$$

where the comparison is with respect to the color order.

Define also

$$\mathrm{exc}_A(\sigma) = |\mathrm{Exc}_A(\sigma)|.$$

Example 3.3. *Take* $\sigma = (\bar{1}\bar{\bar{3}}4\bar{2}) \in G_{3,4}$*. Then* $\mathrm{csum}(\sigma) = 4$*,* $\mathrm{Exc}_{\mathrm{A}}(\sigma) = \{3\}$ *and hence* $\mathrm{exc}_A(\sigma) = 1$*.*

Let $\sigma \in G_{r,n}$. Recall that for $\sigma = (z, \tau) \in G_{r,n}$, $|\sigma|$ is the permutation of $[n]$ satisfying $|\sigma|(i) = \tau(i)$. For example, if $\sigma = (\bar{2}\bar{\bar{3}}1\bar{4})$ then $|\sigma| = (2314)$.

Now we can define the colored excedance number for $G_{r,n}$.

Definition 3.4. Define

$$\mathrm{exc}^{\mathrm{Clr}}(\sigma) = r \cdot \mathrm{exc}_A(\sigma) + \mathrm{csum}(\sigma).$$

One can view $\mathrm{exc}^{\mathrm{Clr}}(\sigma)$ in a different way (see [1]).

Lemma 3.5. *Let* $\sigma \in G_{r,n}$*. Consider the set* Σ *ordered by the color order. Then*

$$\mathrm{exc}(\sigma) = \mathrm{exc}^{\mathrm{Clr}}(\sigma).$$

We say that $i \in [n]$ is an *absolute fixed point* of $\sigma \in G_{r,n}$ if $|\sigma|(i) = i$.

4. Involutions in $G_{r,p,n}$

As was already mentioned, we say that σ is an *involution* if $\sigma^2 = 1$.

In this section we classify the involutions of $G_{r,p,n}$. Note that each involution of $G_{r,p,n}$ can be decomposed into a product of 'atomic' involutions of two types: absolute fixed points and 2-cycles.

We start with the absolute fixed points. In the case $p = 1$, i.e., $\mathbb{Z}_r \wr S_n = G_{r,n}$, we distinguish between two subcases according to the parity of r. In the case of even r, an absolute fixed point can be one of the following two kinds: $\pi(i) = i$ or $\pi(i) = i^{\left[\frac{r}{2}\right]}$. If r is odd, an absolute fixed point can be only of the first kind.

If $p > 1$ and r is odd, we have the same absolute fixed points as in the case $p = 1$. On the other hand, if r is even, then we have to distinguish again between two subcases. If $p \mid \frac{r}{2}$, then the absolute fixed points in $I_{r,p,n}$ are exactly as those of $I_{r,1,n}$. If $p \nmid \frac{r}{2}$, then an element of $G_{r,p,n}$ with an odd number of fixed points of the form $\pi(i) = i^{\left[\frac{r}{2}\right]}$ is not an element of $I_{r,p,n}$ and thus the only absolute fixed points are of the form $\pi(i) = i$ or pairs of absolute fixed points of the form $\pi(i) = i^{\left[\frac{r}{2}\right]}; \pi(j) = j^{\left[\frac{r}{2}\right]}$.

In all cases, the 2-cycles have the form $\pi(i) = j^{[k]}$; $\pi(j) = i^{[r-k]}$ where $0 \leq k \leq r - 1$.

We conclude this section with an example.

Example 4.1. *Let $r = 18, p = 6, n = 7$ and let*

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7^{[2]} & 3 & 2 & 4^{[9]} & 5^{[9]} & 6 & 1^{[16]} \end{pmatrix} \in I_{18,6,7}.$$

Then π can be decomposed into the absolute fixed points $\begin{pmatrix} 6 \\ 6 \end{pmatrix}$, the pair of absolute fixed points $\begin{pmatrix} 4 & 5 \\ 4^{[9]} & 5^{[9]} \end{pmatrix}$ and the following two 2-cycles: $\begin{pmatrix} 1 & 7 \\ 7^{[2]} & 1^{[16]} \end{pmatrix}$ and $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$.

5. Recurrences and explicit formulas

In this section, we compute recurrences and explicit formulas for

$$f_{r,p,n}(u, v, w) = \sum_{\pi \in I_{r,p,n}} u^{\mathrm{fix}(\pi)} v^{\mathrm{exc}_A(\pi)} w^{\mathrm{csum}(\pi)}$$

for all r and p where $p \mid r$.

5.1. **Recurrences for $G_{r,n} = G_{r,1,n}$.** Let π be any colored involution in $I_{r,n} = I_{r,1,n}$. Then we have either $\pi(n) = n^{[j]}$ or $\pi(n) = k^{[j]}$ with $k < n$.

If $\pi(n) = n^{[j]}$, then we distinguish between two subcases according to the parity of r, as we have seen in Section 4. If r is even we have $j = 0$ or $j = \frac{r}{2}$. If r is odd then $j = 0$.

For $\pi \in I_{r,1,n}$ such that $\pi(n) = n^{[j]}$, define $\pi' \in I_{r,1,n-1}$ by ignoring the last digit of π. For $\pi \in I_{r,1,n}$ with $\pi(n) = k^{[j]}$ and $\pi(k) = n^{[r-j]}$, define $\pi'' \in I_{r,1,n-2}$ in the following way: write π in its complete notation, i.e., as a matrix of two rows, as in Example 4.1. The first row of

π'' is $(1, 2, \ldots, n-2)$ while the second row is obtained from the second row of π by ignoring the digits n and k and the other digits are placed in an order preserving way with respect to the second row of π. Here is an explicit formula for the map $\pi \mapsto \pi''$:

$$\pi''(i) = \begin{cases} \pi(i) & \text{if } 1 \le i < k \text{ and } \pi(i) < k, \\ \pi(i) - 1 & \text{if } 1 \le i < k \text{ and } \pi(i) > k, \\ \pi(i-1) & \text{if } k \le i < n \text{ and } \pi(i) < k, \\ \pi(i-1) - 1 & \text{if } k \le i < n \text{ and } \pi(i) > k. \end{cases}$$

Note that the map $\pi \mapsto \pi'$ is a bijection from the set

$$\{\pi \in I_{r,1,n} \mid \pi(n) = n^{[j]}\} \quad (j \text{ fixed})$$

to $I_{r,1,n-1}$, while $\pi \mapsto \pi''$ is a bijection from the set

$$\{\pi \in I_{r,1,n} \mid \pi(n) = k^{[j]}\} \quad (j \text{ fixed})$$

to $I_{r,1,n-2}$.

For any r, if $\pi(n) = n^{[j]}$ then

$$\begin{aligned} \mathrm{fix}(\pi) &= \mathrm{fix}(\pi') + 1, \\ \mathrm{exc}_A(\pi) &= \mathrm{exc}_A(\pi'), \\ \mathrm{csum}(\pi) &= \mathrm{csum}(\pi') + j. \end{aligned}$$

If $\pi(n) = t^{[j]}$, then the parameters satisfy

$$\begin{aligned} \mathrm{fix}(\pi) &= \mathrm{fix}(\pi''), \\ \mathrm{exc}_A(\pi) &= \mathrm{exc}_A(\pi'') + \delta_{j,0}, \\ \mathrm{csum}(\pi) &= \mathrm{csum}(\pi'') + r(1 - \delta_{j,0}), \end{aligned}$$

where $\delta_{i,j}$ is the Kronecker delta defined by

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The above consideration gives the following recurrences, where we define $\mu_r = 1 + w^{\frac{r}{2}}$ for even r, and $\mu_r = 1$ otherwise:

$$\begin{aligned} f_{r,1,n}(u, v, w) = {} & u\mu_r f_{r,1,n-1}(u, v, w) \\ & + (n-1)(v + (r-1)w^r) f_{r,1,n-2}(u, v, w), \quad n \ge 1. \end{aligned}$$

5.2. **Explicit formulas for $G_{r,n}=G_{r,1,n}$.** We turn now to the explicit formula. Define

$$\begin{aligned}F_{r,p}(x;u,v,w)&=\sum_{n\geq 0} f_{r,p,n}(u,v,w)\frac{x^n}{n!}\\&=\sum_{n\geq 0}\sum_{\pi\in I_{r,p,n}}\left(u^{\mathrm{fix}(\pi)}v^{\mathrm{exc}_A(\pi)}w^{\mathrm{csum}(\pi)}\right)\frac{x^n}{n!}.\end{aligned}$$

Rewriting the recurrence in terms of generating functions, we obtain that

$$\begin{aligned}x\frac{\partial}{\partial x}F_{r,1}(x;u,v,w)&=\sum_{n\geq 1}\frac{f_{r,1,n}(u,v,w)}{(n-1)!}x^n\\&=ux\mu_r\sum_{n\geq 1}\frac{x^{n-1}}{(n-1)!}f_{r,1,n-1}(u,v,w)+\\&\quad+x^2(v+(r-1)w^r)\sum_{n\geq 2}\frac{x^{n-2}}{(n-2)!}f_{r,1,n-2}(u,v,w)\\&=ux\mu_r F_{r,1}(x;u,v,w)+\\&\quad+x^2(v+(r-1)w^r)F_{r,1}(x;u,v,w).\end{aligned}$$

Thus, the generating function $F_{r,1}(x;u,v,w)$ satisfies

$$\frac{\frac{\partial}{\partial x}F_{r,1}(x;u,v,w)}{F_{r,1}(x;u,v,w)}=u\mu_r+x(v+(r-1)w^r).$$

Integrating with respect to x on both sides of the above differential equation, using the fact that $F_{r,1}(0;u,v,w)=1$, we obtain the following proposition.

Proposition 5.1. Let $r\geq 1$. The generating function $F_{r,1}(x;u,v,w)$ is given by

$$e^{ux\mu_r+\frac{1}{2}x^2(v+(r-1)w^r)}.$$

We are looking for an explicit expression for the polynomial $f_{r,1,n}(u,v,w)$. From the definitions we have that $\frac{f_{r,1,n}(u,v,w)}{n!}$ is the coefficient of x^n in $F_{r,1}(x;u,v,w)$, namely $[x^n]F_{r,1}(x;u,v,w)$. Computing the coefficient of x^n in the Maclaurin series of $F_{r,1}(x;u,v,w)$ one gets

$$\begin{aligned} f_{r,1,n}(u,v,w) &= n![x^n]\sum_{j\geq 0}\frac{1}{j!}\left(ux\mu_r+\frac{1}{2}x^2(v+(r-1)w^r)\right)^j \\ &= n![x^n]\sum_{j\geq 0}\sum_{i=0}^{j}\binom{j}{i}\frac{x^{i+j}u^{j-i}(v+(r-1)w^r)^i}{j!2^i}\mu_r^{j-i} \\ &= n!\sum_{j=n/2}^{n}\binom{j}{n-j}\frac{u^{2j-n}(v+(r-1)w^r)^{n-j}}{j!2^{n-j}}\mu_r^{2j-n}. \end{aligned}$$

Hence, we have the following corollary.

Corollary 5.2. *The polynomial $f_{r,1,n}(u,v,w)$ is given by*
(5.1)

$$\sum_{j=n/2}^{n}(n-j)!\binom{n}{n-j,n-j,2j-n}\frac{u^{2j-n}(v+(r-1)w^r)^{n-j}}{2^{n-j}}\mu_r^{2j-n}.$$

If we substitute $w=1$ and compute the coefficient of u^mv^ℓ in Formula (5.1), we get the following result.

Corollary 5.3. *Let $r\geq 1$. The number of colored involutions in $G_{r,n}$ with exactly m absolute fixed points and $\mathrm{exc}_A(\pi)=\ell$ is given by*

$$(\frac{n-m}{2})!(r-1)^{\frac{n-m}{2}-\ell}\binom{n}{\frac{n-m}{2},m,\frac{n-m}{2}-\ell,\ell}2^{\frac{m(3-2k)-n}{2}},$$

where $k\in\{0,1\}$ and $k\equiv r \pmod 2$.

It is easy to see that if $r=1$, then $2\mathrm{exc}_A(\pi)+\mathrm{fix}(\pi)=n$ for each involution π of S_n. From the above corollary we have then that the number of involutions in S_n with exactly ℓ excedances is given by $\frac{l!}{2^l}\binom{n}{l,l,n-2l}$.

We turn now to the computation of the number of involutions with a fixed number of excedances. We do this by substituting $u=1$ and $v=w^r$ in Formula (5.1).

Corollary 5.4. *The number of involutions $\pi\in G_{r,n}$ with $\mathrm{exc}^{\mathrm{Clr}}(\pi)=m$ is*

$$\begin{cases} k!\binom{n}{k,\ k,\ n-2k}\left(\frac{r}{2}\right)^k & \text{if } r\equiv 1 \pmod 2, \\ \displaystyle\sum_{j=\frac{n}{2}}^{n}(n-j)!\binom{n}{n-j,\ n-j,\ j-k,\ k-n+j}\left(\frac{r}{2}\right)^{n-j} & \\ & \text{if } r\equiv 0 \pmod 2, \end{cases}$$

where $k=\frac{m}{r}$.

Note that k is an integer, since $\mathrm{exc}^{\mathrm{Clr}}(\pi)$ is an integral multiplicity of r, for $\pi \in I_{r,n}$.

5.3. **Recurrences and explicit formulas for $G_{r,p,n}$ where r is odd, $p > 1$.** As we have seen in Section 4, the involutions in this case coincide with the involutions of $G_{r,1,n}$ where r is odd and thus we have the following corollary.

Corollary 5.5. *The recurrence for $f_{r,p,n}(u,v,w)$ for odd r is*

$$\begin{aligned} f_{r,p,n}(u,v,w) &= uf_{r,p,n-1}(u,v,w) \\ &\qquad + (n-1)(v+(r-1)w^r)f_{r,p,n-2}(u,v,w), \quad n \geq 1, \end{aligned}$$

and thus its explicit formula is

$$\begin{aligned} &f_{r,p,n}(u,v,w) \\ &\quad = \sum_{j=n/2}^{n} (n-j)! \binom{n}{n-j,\ n-j,\ 2j-n} \frac{u^{2j-n}(v+(r-1)w^r)^{n-j}}{2^{n-j}}. \end{aligned}$$

5.4. **Recurrences and explicit formulas for $G_{r,p,n}$ where r is even and $p > 1$, $p \mid \frac{r}{2}$.** Also in this case, we have that the involutions coincide with the involutions of $G_{r,1,n}$ where r is even, and thus we have the following corollary.

Corollary 5.6. *The recurrence for $f_{r,p,n}(u,v,w)$ for even r and $p > 1$, $p \mid \frac{r}{2}$ is*

$$\begin{aligned} f_{r,p,n}(u,v,w) &= u(1+w^{\frac{r}{2}})f_{r,p,n-1}(u,v,w) \\ &\qquad + (n-1)(v+(r-1)w^r)f_{r,p,n-2}(u,v,w), \quad n \geq 1, \end{aligned}$$

and thus its explicit formula is

$$\begin{aligned} f_{r,p,n}(u,v,w) = \sum_{j=n/2}^{n} (n-j)! &\binom{n}{n-j,\ n-j,\ 2j-n} \\ &\cdot \frac{u^{2j-n}(v+(r-1)w^r)^{n-j}}{2^{n-j}}(1+w^{\frac{r}{2}})^{2j-n}. \end{aligned}$$

5.5. **Recurrences and explicit formulas for $G_{r,p,n}$ where r is even and $p > 1, p \nmid \frac{r}{2}$.** Let π be any colored involution in $I_{r,p,n}$. Then, according to Section 4, we have either $\pi(n) = n^{[j]}$ (where $j = 0$ or $j = \frac{r}{2}$) or $\pi(n) = k^{[j]}$ with $k < n$.

We start with the recurrence. Let π be any colored involution in $I_{r,p,n}$. Then we have several cases:

(1) $\pi(n) = n$. In this case define $\pi' \in I_{r,p,n-1}$ by ignoring the last digit of π. The map $\pi \mapsto \pi'$ is a bijection from the set $\{\pi \in I_{r,p,n} \mid \pi(n) = n\}$ to $I_{r,p,n-1}$.
We have
$$\begin{aligned}\text{fix}(\pi) &= \text{fix}(\pi') + 1,\\ \text{exc}_A(\pi) &= \text{exc}_A(\pi'),\\ \text{csum}(\pi) &= \text{csum}(\pi').\end{aligned}$$

(2) $\pi(n) = n^{[\frac{r}{2}]}$ and there exists some $k < n$ such that $\pi(k) = k^{[\frac{r}{2}]}$. Define $\pi'' \in I_{r,p,n-2}$ as in Section 5.1.
Note that $\pi \mapsto \pi''$ is a bijection from the set
$$\{\pi \in I_{r,p,n} \mid \pi(n) = n^{[\frac{r}{2}]}\}$$
to $I_{r,p,n-2}$.
We have
$$\begin{aligned}\text{fix}(\pi) &= \text{fix}(\pi'') + 2,\\ \text{exc}_A(\pi) &= \text{exc}_A(\pi''),\\ \text{csum}(\pi) &= \text{csum}(\pi'') + r.\end{aligned}$$

(3) $\pi(n) = k^{[j]}$ with $k < n$ and we have $\pi(k) = n^{[r-j]}$. In this case, we use $\pi'' \in I_{r,p,n-2}$ as above. Note that in this case $\pi \mapsto \pi''$ is a bijection from the set
$$\{\pi \in I_{r,p,n} \mid \pi(n) = k^{[j]}\}$$
to $I_{r,p,n-2}$. We get in this case
$$\begin{aligned}\text{fix}(\pi) &= \text{fix}(\pi''),\\ \text{exc}_A(\pi) &= \text{exc}_A(\pi'') + \delta_{j,0},\\ \text{csum}(\pi) &= \text{csum}(\pi'') + r(1 - \delta_{j,0}).\end{aligned}$$

The above consideration gives the following recurrence:
$$\begin{aligned}f_{r,p,n}(u, v, w) &= uf_{r,p,n-1}(u, v, w)\\ &\qquad + (n - 1)(u^2w^r + (r - 1)w^r + v)f_{r,p,n-2}(u, v, w),\end{aligned}$$
where $n \geq 1$.

By similar arguments to the ones we have used in Section 5.2, we get the following generating function and explicit formula.

Proposition 5.7. Let $r \geq 1$. The generating function $F_{r,p}(x; u, v, w)$ is given by
$$e^{ux+\frac{1}{2}x^2((u^2+(r-1))w^r+v)}.$$

Corollary 5.8. *The polynomial $f_{r,p,n}(u,v,w)$ is given by*
(5.2)
$$\sum_{j=n/2}^{n}(n-j)!\binom{n}{n-j,\ n-j,\ 2j-n}\frac{u^{2j-n}(v+(u^2+(r-1))w^r)^{n-j}}{2^{n-j}}.$$

If we substitute $w=1$ and compute the coefficient of $u^m v^\ell$ in Formula (5.2), we get the following result.

Corollary 5.9. *Let $r \geq 1$. The number of colored involutions in $G_{r,p,n}$ (r is even, $p \nmid \frac{r}{2}$) with exactly m absolute fixed points and* $\mathrm{exc}_A(\pi)=\ell$ *is given by*
$$\sum_{j=\frac{n}{2}}^{n}\frac{(n-j)!}{2^{n-j}}\binom{n}{n-j,\ 2j-n,\ l,\ \frac{n-m}{2}-l,\ \frac{m+n}{2}-j}(r-1)^{\frac{n-m}{2}-l}.$$

For computing the number of involutions with a fixed number of excedances, we substitute $u=1$ and $v=w^r$ in Formula (5.2).

Corollary 5.10. *The number of involutions $\pi \in G_{r,p,n}$ (r is even, $p \nmid \frac{r}{2}$) with* $\mathrm{exc}^{\mathrm{Clr}}(\pi)=m$ *is*
$$\frac{(\frac{m}{r})!}{2^{\frac{m}{r}}}\binom{n}{\frac{m}{r},\ \frac{m}{r},\ n-2\frac{m}{r}}(r+1)^{\frac{m}{r}}.$$

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