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MARK SEQUENCES IN DIGRAPHS

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Abstract. A k-digraph is an orientation of a multi-graph that is without loops and contains at most k edges between any pair of distinct vertices. We obtain necessary and sufficient conditions for a sequence of non-negative integers in non-decreasing order to be a sequence of numbers, called marks (k-scores), attached to vertices of a k-digraph. We characterize irreducible and uniquely realizable mark sequences in k-digraphs.

1. Introduction

Let D be a k-digraph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and let $d^+(v_i)$ and $d^-(v_i)$ denote the outdegree and indegree, respectively, of a vertex v_i. Define p_{v_i} (or p_i) $= k(n-1) + d^+(v_i) - d^-(v_i)$, as the *mark* of v_i, so that $0 \le p_{v_i} \le 2k(n-1)$. The sequence $P = [p_i]_1^n$ in non-decreasing order is called the *mark sequence* of D.

A k-digraph can be interpreted as the result of a competition in which the participants play each other at most k times, with an arc from u to v if and only if u defeats v. A player receives two points for each win, and one point for each tie (draw), that is the case in which the two players do not play one another or the competition between the players yields no result. With this marking system, player v obtains a total of p_v points.

A sequence P of non-negative integers in non-decreasing order is said to be *realizable* if there exists a k-digraph with mark sequence P.

Any undefined terms are found in [3,5], and one should also take into account the non-standard definitions and notations introduced in this paper.

In a k-digraph, if there are x_1 arcs directed from vertex u to vertex v, and x_2 arcs directed from vertex v to vertex u, with $0 \le x_1, x_2 \le k$ and $0 \le x_1 + x_2 \le k$, we denote this by $u(x_1 - x_2)v$.

We have one of the following six possibilities between any two vertices u and v in a 2-digraph:

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(i) exactly two arcs directed from u to v, and no arc directed from v to u; this is denoted by $u(2-0)v$;
(ii) exactly two arcs directed from v to u, and no arc directed from u to v; this is denoted by $u(0-2)v$;
(iii) exactly one arc from u to v, and exactly one arc from v to u; this is denoted by $u(1-1)v$;
(iv) exactly one arc from u to v, and no arc from v to u; this is denoted by $u(1-0)v$;
(v) exactly one arc from v to u, and no arc from u to v; this is denoted by $u(0-1)v$;
(vi) no arc from u to v, and no arc from v to u; this is denoted by $u(0-0)v$.

We note that a 1-digraph is an oriented graph, and a complete 1-digraph is a tournament. A k-digraph D is said to be *complete* if there are exactly k arcs between any pair of vertices of D.

A *k-triple* in a k-digraph is an induced k-subdigraph with three vertices, and is of the form $u(x_1-x_2)v(y_1-y_2)w(z_1-z_2)u$, where, for $i=1,2$, we have $0 \le x_i, y_i, z_i \le k$ and $0 \le \sum_{i=1}^{2} x_i, \sum_{i=1}^{2} y_i, \sum_{i=1}^{2} z_i \le k$. Also, in a k-digraph a 1-triple is an induced 1-subdigraph with three vertices. A 1-triple is said to be *transitive* if it is of the form $u(1-0)v(1-0)w(0-1)u$, or $u(1-0)v(0-1)w(0-0)u$, or $u(1-0)v(0-0)w(0-1)u$, or $u(1-0)v(0-0)w(0-0)u$, or $u(0-0)v(0-0)w(0-0)u$, otherwise it is said to be *intransitive*. A k-triple is said to be *transitive* if it contains only transitive 1-triples, and a k-digraph is said to be *transitive* if every of its k-triples is transitive.

A *tournament* is an irreflexive, complete, asymmetric digraph. The *score* s_v of a vertex v in a tournament is the number of arcs directed away from that vertex, and the *score sequence* $S(T)$ of a tournament T is formed by listing the vertex scores in non-decreasing order. The following criterion is given by Landau [4].

Theorem 1.1 ([4]). *A sequence $[s_i]_1^n$ of non-negative integers in non-decreasing order is the score sequence of a tournament if and only if*

$$\sum_{i=1}^{k} s_i \ge \binom{k}{2}, \qquad \text{for } 1 \le k \le n,$$

with equality for $k=n$.

With the marking system, the mark p_v of a vertex v in a tournament is given by $p_v = 2s_v + n - 1$, and Landau's conditions become

$$\sum_{i=1}^{k} p_i \ge k(n+k-2), \qquad \text{for } 1 \le k \le n,$$

with equality for $k=n$.

An *oriented graph* is a digraph with no symmetric pairs of directed arcs and without self-loops. Avery [2] defined $a_v = n-1+d^+(v)-d^-(v)$, $0 \le a_v \le 2n-2$, as the score of a vertex v in an oriented graph D, and $A=[a_1,a_2,\ldots,a_n]$ in non-decreasing order is the score sequence of D. The following result is due to Avery, and a constructive proof can be found in [8].

Theorem 1.2 ([2]). *A sequence $A = [a_i]_1^n$ of non-negative integers in non-decreasing order is the score sequence of an oriented graph if and only if*

$$\sum_{i=1}^{k} a_i \geq k(k-1), \qquad \text{for } 1 \leq k \leq n,$$

with equality for $k = n$.

Once again, with the marking system, the mark p_v of a vertex v in an oriented graph is given by $p_v = a_v + n - 1$, and Avery's conditions become

$$\sum_{i=1}^{k} p_i \geq k(n+k-2), \qquad \text{for } 1 \leq k \leq n,$$

with equality for $k = n$.

2. Mark sequences in digraphs

A k-digraph D is said to be *complete* if there are exactly k arcs between every pair of vertices of D. If in a k-digraph D there are exactly k arcs, which are parallel, between every pair of vertices of D, then D is called a *k tournament.* A double tournament can be treated as a tournament whose arcs have been duplicated.

The following result can be easily established, and is analogous to Theorem 2.2 of Avery [2].

Lemma 2.1. *If D and D' are two k-digraphs with the same mark sequence, then D can be transformed to D'*

(i) *by successively transforming 1-triples in one of the following ways:*
either (a) *by changing the intransitive 1-triple $u(1-0)v(1-0)w(1-0)u$ to a transitive 1-triple $u(0-0)v(0-0)w(0-0)u$, which has the same mark sequence, or vice versa,*
or (b) *by changing an intransitive 1-triple $u(1-0)v(1-0)w(0-0)u$ to a transitive 1-triple $u(0-0)v(0-0)w(0-1)u$, which has the same mark sequence, or vice versa.*

or (ii) *by changing a double $u(1-1)v$ to a double $u(0-0)v$ which has the same mark sequence, or vice versa.*

We note here that, in a transitive tournament T, all its 1-triples are of the form $u(1-0)v(1-0)w(0-1)u$, for all vertices u, v and w in T. Similarly, in a transitive oriented graph, all the 1-triples are of the form $u(1-0)v(1-0)w(0-1)u$, $u(1-0)v(0-1)w(0-0)u$, $u(1-0)v(0-0)w(0-1)u$, $u(1-0)v(0-0)w(0-0)u$, $u(0-0)v(0-0)w(0-0)u$. Clearly, in the transitive double tournament D, we have $u(2-0)v(2-0)w(0-2)u$ for all vertices u, v and w in D.

Now, we have the following observation.

Theorem 2.1. *Among all k-digraphs with a given mark sequence those with the fewest arcs are transitive.*

Proof. Let P be a mark sequence, and let D be a realization of P that is not transitive. Then D contains an intransitive 1-triple. If it is of the form $u(1-0)v(1-0)w(1-0)u$, it can be transformed by operation (i)(a) of Lemma 2.1 to a transitive 1-triple $u(0-0)v(0-0)w(0-0)u$ with the same mark sequence and three arcs fewer. If D contains an intransitive 1-triple of the form $u(1-0)v(1-0)w(0-0)u$, it can be transformed by operation (i)(b) of Lemma 2.1 to a transitive 1-triple of the form $u(0-0)v(0-0)w(0-1)u$ with the same mark sequence and one arc fewer. If D contains both types of intransitive 1-triples, then again they can be transformed to transitive 1-triples, and certainly with lesser arcs. In case D contains a double $u(1-1)v$, it can be transformed to $u(0-0)v$ by operation (ii) of Lemma 2.1 with the same mark sequence and two arcs fewer. □

The following result is the existence criteria for realizability of mark sequences in k-digraphs.

Theorem 2.2. *A sequence $[p_i]_1^n$ of non-negative integers in non-decreasing order is the mark sequence of a k-digraph if and only if*

$$\sum_{i=1}^{t} p_i \geq kt(t-1), \qquad \text{for } 1 \leq t \leq n,$$

with equality for $t = n$.

Proof. (i) Sufficiency. Let $q_i = p_i - k(n-1)$. Then $\sum_{i=1}^{n} q_i = 0$, and we may assume that $q_1 \leq q_2 \leq \cdots \leq q_r < 0 \leq q_{r+1} \leq \cdots \leq q_n$.

Construct a network with vertex set $\{s, v_1, v_2, \ldots, v_n, t\}$ of cardinality $n+2$ as follows.

1. There are arcs (s, v_i), $1 \leq i \leq r$ from the source s to vertex v_i. The arc (s, v_i) has capacity $-q_i$, $1 \leq i \leq r$.
2. Arcs (v_i, t) from v_i to the sink t, $r+1 \leq i \leq n$. The arc (v_i, t) has capacity $-q_i$.
3. For each pair v_i, v_j of distinct vertices $(i \neq j)$, we have one arc from v_i to v_j and one arc from v_j to v_i, each with capacity k.

It is easy to check that a k-digraph with mark sequence $[p_i]_i^n$ can be obtained from an integral flow of value $-\sum_{i=1}^{r} q_i = \sum_{i=r+1}^{n} q_i$ by reducing the flow on cycles of length 2 until one of the two edges has flow value zero.

In view of the max-flow-min-cut-Theorem, it suffices to check that each cut has capacity at least $\sum_{i=r+1}^{n} q_i$.

We thus assume that $\{s\} \cup C$ is a cut, $C \subseteq \{v_1, v_2, \ldots, v_n\}$, $|C| = t$, and that $|C \cap \{v_1, v_2, \ldots, v_r\}| = a$ and $|C \cap \{v_{r+1}, v_{r+2}, \ldots, v_n\}| = b = t - a$.

For its capacity, we have the following estimate.

$$\begin{aligned}\operatorname{cap}(\{s\} \cup C) &= \sum_{i: i \leq r, v_i \notin C} -q_i + \sum_{i: i > r, v_i \in C} q_i + t(n-t) \cdot k \\ &\geq -\sum_{i=a+1}^{r} q_i + \sum_{i=r+1}^{r+b} q_i + t(n-t) \cdot k.\end{aligned}$$

This expression is bounded from below by $-\sum_{i=1}^{r} q_i = \sum_{i=r+1}^{n} q_i$ if and only if

$$\sum_{i=1}^{a} q_i + \sum_{i=r+1}^{r+b} q_i + t(n-t) \cdot k \geq 0,$$

if and only if

$$\sum_{i=1}^{a} p_i + \sum_{i=r+1}^{r+b} p_i + t(n-t)\cdot k \geq t\cdot k(n-1)$$

(since $p_i = k(n-1) + q_i$), if and only if

$$\sum_{i=1}^{a} p_i + \sum_{i=r+1}^{r+b} p_i \geq kt(t-1).$$

This latter inequality is certainly implied by the inequality

$$\sum_{i=1}^{t} p_i \geq kt(t-1),$$

since the p_i are non-decreasing.

(ii) Necessity. Follows from the construction in (i) if we use the cuts $\{s\} \cup \{v_1, v_2, \dots, v_t\}$, $1 \leq t \leq n$. □

The following result is the existence criteria for realizability of mark sequences in 2-digraphs. The proof follows from Theorem 2.2. Here we give a different proof.

Theorem 2.3. *A sequence $[p_i]_1^n$ of non-negative integers in non-decreasing order is the mark sequence of a 2-digraph if and only if*

$$\sum_{i=1}^{k} p_i \geq 2k(k-1), \qquad \textit{for } 1 \leq k \leq n, \tag{2.3.1}$$

with equality for $k = n$.

Proof. Necessity. Let D be a 2-digraph with mark sequence $[p_i]_1^n$. Let W be the 2-subdigraph induced by any set of k vertices $w_1, w_2, \dots, w_k$ of D. Let α denote the number of arcs of D that start in W and end outside W, and let β denote the number of arcs of D that start outside of W and end in W. Note that each vertex w in W, and for every vertex v of D not in W, there are at most two arcs from v to w, so that $\beta \leq 2k(n-k)$. Therefore, we have $\beta \leq 2nk - 2k^2$. Then,

$$\begin{aligned}
\sum_{i=1}^{k} p_{w_i} &= \sum_{i=1}^{k}(2n-2+d_D^+(w_i) - d_D^-(w_i)) \\
&= 2nk - 2k + \sum_{i=1}^{k} d_D^+(w_i) - \sum_{i=1}^{k} d_D^-(w_i) \\
&= 2nk - 2k + \left[\sum_{i=1}^{k} d_W^+(w_i) + \alpha\right] - \left[\sum_{i=1}^{k} d_W^-(w_i) + \beta\right] \\
&= 2nk - 2k + (\text{number of arcs of } W) + \alpha - (\text{number of arcs of } W) - \beta \\
&= 2nk - 2k + \alpha - \beta \qquad (2.3.2) \\
&\geq 2nk - 2k - \beta \\
&\geq 2nk - 2k - 2nk + 2k^2 = 2k(k-1).
\end{aligned}$$

Applying this result to the k vertices with marks $p_1, p_2, \ldots, p_k$ yields the desired inequality. If $k = n$, then $\alpha = \beta = 0$, and the required equality follows from Equation (2.3.2).

Sufficiency. This is proved by contradiction. Assume all sequences of non-negative integers in non-decreasing order of length fewer than n satisfying conditions (2.3.1) be the mark sequences. Let n be the smallest and with this choice of n, p_1 be the smallest possible such that $P = [p_i]_1^n$ is not a mark sequence. Two cases arise,

(a) equality in (2.3.1) holds for some $k < n$, and
(b) each inequality in (2.3.1) is strict for all $k < n$.

Case (a). Assume k $(k < n)$ is the smallest such that

$$\sum_{i=1}^{k} p_i = 2k(k-1).$$

Clearly, the sequence $[p_1, p_2, \ldots, p_k]$ satisfies conditions (2.3.1), and is a sequence with length less than n. So, by assumption, $[p_i]_1^k$ is a mark sequence of some 2-digraph, say D_1. Further,

$$\begin{aligned}\sum_{i=1}^{m}(p_{k+i} - 4k) &= \sum_{i=1}^{m+k} p_i - \sum_{i=1}^{k} a_i - 4mk \\ &\geq 2(m+k)(m+k-1) - 2k(k-1) - 4mk \\ &= 2m(m-1),\end{aligned}$$

for each m, $1 \leq m \leq n-k$, with equality when $m = k$. As $m < n$, thus by the minimality of n, the sequence $[p_{k+1} - 4k, p_{k+2} - 4k, \ldots, p_n - 4k]$ is the mark sequence of some 2-digraph D_2. The 2-digraph D of order n consisting of disjoint copies of D_1 and D_2, such that $u(2-0)v$ for each vertex $u \in D_2$ and for each vertex $v \in D_1$, has mark sequence $P = [p_i]_1^n$, which is a contradiction.

Case (b). Assume that each inequality in condition (2.3.1) is strict for all $k < n$. Obviously, $p_1 > 0$. Consider the sequence $P' = [p'_i]_1^n$, defined by

$$p'_i = \begin{cases} p_i - 1, & \text{if } i = 1, \\ p_i + 1, & \text{if } i = n, \\ p_i, & \text{otherwise.} \end{cases}$$

Then,

$$\sum_{i=1}^{k} p'_i = \left(\sum_{i=1}^{k} p_i\right) - 1 > 2k(k-1) - 1 \geq 2k(k-1),$$

for all k, $1 \leq k < n$, and

$$\sum_{i=1}^{n} p'_i = \left(\sum_{i=1}^{n} p_i\right) - 1 + 1 = 2n(n-1).$$

This shows that the sequence $P' = [p'_i]_1^n$ satisfies condition (2.3.1), and therefore is a mark sequence of some 2-digraph D. Let u and v denote the vertices with marks $p'_i = p_i - 1$ and $p'_n = p_n - 1$ respectively.

If in D, $u(0-2)v$, or $u(1-1)v$, or $u(1-0)v$, or $u(0-1)v$, or $u(0-0)v$, then transforming them respectively to $u(0-1)v$, or $u(1-0)v$, or $u(2-0)v$, or $u(1-1)v$, or $u(1-0)v$, we obtain a 2-digraph with mark sequence P, a contradiction.

In D, let $u(2-0)v$. We have $p'_v \geq p'_u + 2$. If there exists at least one vertex $w \in D - \{u, v\}$ such that the 2-triples formed by the vertices u, v and w contain an intransitive 1-triple of the form $u(1-0)v(1-0)w(1-0)u$, or $u(1-0)v(1-0)w(0-0)u$, or $u(1-0)v(0-0)w(1-0)u$, transforming them respectively to $u(1-0)v(0-0)w(0-0)u$, or $u(1-0)v(0-0)w(0-1)u$, or $u(1-0)v(0-1)w(0-0)u$, we obtain a 2-digraph with mark sequence P, which is a contradiction.

Assume for each vertex $w \in D - \{u, v\}$, the 2-triples formed by the vertices u, v and w contain only transitive 1-triples of the form

(i) $u(1-0)v(1-0)w(0-1)u$,
(ii) $u(1-0)v(0-1)w(1-0)u$,
(iii) $u(1-0)v(0-1)w(0-1)u$,
(iv) $u(1-0)v(0-0)w(0-1)u$,
(v) $u(1-0)v(0-1)w(0-0)u$,
(vi) $u(1-0)v(0-0)w(0-0)u$.

Then, clearly $p'_v < p'_u + 2$, since $d^+_u > d^+_v$ and $d^-_u < d^-_v$, and we get a contradiction.

If (i) appears for every vertex $w \in D - \{u, v\}$, so that the 2-triples formed by u, v and w is of the form $u(2-0)v(2-0)w(0-1)u$, then

$$p'_v = 2n - 2 + d^+_v - d^-_v = 2n - 2 + 2(n-2) - 2 = 4n - 8,$$

and

$$p'_u = 2n - 2 + d^+_u - d^-_u = 2n - 2 + n - 2 + 2 = 3n - 2.$$

Therefore, $p'_v = p'_u + n - 6$.

For $n < 8$, clearly $p'_v \leq p'_u + 1$, a contradiction.

For $n \geq 8$, we do have $p'_v \geq p'_u + 2$, but then $u(2-0)v(2-0)w(0-1)u$ can be transformed to $u(2-0)v(1-0)w(0-2)u$, and we get a 2-digraph with mark sequence P, a contradiction.

If (ii) appears for every vertex $w \in D - \{u, v\}$ such that the 2-triple formed by u, v and w is of the form $u(2-0)v(0-1)w(2-0)u$, then

$$p'_v = 2n - 2 + d^+_v - d^-_v = 2n - 2 - (n-2) - 2 = n - 2,$$

and

$$p'_u = 2n - 2 + d^+_u - d^-_u = 2n - 2 - 2(n-2) = 4.$$

Therefore, $p'_v - p'_u = n - 6$, so that $p'_v = p'_u + n - 6$.

For $n < 8$, clearly $p'_v \leq p'_u + 1$, a contradiction.

For $n \geq 8$, we have $p'_v \geq p'_u + 2$. Then, transforming $u(2-0)v(0-1)w(2-0)u$ to $u(2-0)v(0-2)w(1-0)u$, we obtain a 2-digraph with mark sequence P, again a contradiction. □

Some stronger inequalities on marks in 2-digraphs can be found in [7]. The next result is the analogue of Havel–Hakimi theorem on degree sequences of simple graphs.

Theorem 2.4. *Let $P = [p_i]_1^n$ be a sequence of non-negative integers in non-decreasing order, where for each i, $0 \leq p_i \leq 2k(n-1)$. Let P' be obtained from P by deleting the greatest entry p_n ($= 2k(n-1) - r$, say) and* (a) *if $r \leq n-1$, reducing the r greatest* *remaining entries by one each, or* (b) *if $r > n-1$, reducing the $r-(n-1)$ greatest remaining entries by two each, and the $2n-2-r$ remaining entries by one. Then, P is a mark sequence of some k-digraph if and only if P' (arranged in non-decreasing order) is a mark sequence of some k-digraph.*

Proof. Let P' be a mark sequence of some k-digraph D'. If P' is obtained from P as in (a), then a k-digraph D with mark sequence P is obtained by adding a vertex v in D' such that $v((k-1)-0)v_i$ for those vertices v_i in D' with mark $v_i = p_i - 1$, and $v(k-0)v_i$ for those vertices v_i in D' with mark $v_i = p_i$. If P' is obtained from P as in (b), then again a k-digraph D with mark sequence P is obtained by adding a vertex v in D' such that $v((k-1)-1)v_i$ for those vertices v_i in D' with mark $v_i = p_i - 2$ and $v((k-1)-0)v_i$ for those vertices v_i in D' with mark $v_i = p_i - 1$.

Conversely, let P be the mark sequence of some k-digraph D. We assume D is transitive, if not D becomes transitive by using Lemma 2.1. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of D, and let $p_n = 2k(n-1) - r$. If $r \leq n-1$, construct D such that $v_n((k-1)-0)v_i$ for all i, $n-r \leq i \leq n-1$, and $v_n(k-0)v_j$ for all j, $1 \leq j \leq n-r-1$. Clearly, $D - v_n$ realizes P' (arranged in non-decreasing order). If $r > n-1$, construct D such that $v_n((k-1)-1)v_i$ for all i, $2n-r-1 \leq i \leq n-1$, and $v_n((k-1)-0)v_j$ for all j, $1 \leq j \leq 2n-r-2$. Then again, $D - v_n$ realizes P' (arranged in non-decreasing order). □

Theorem 2.4 provides an algorithm for determining whether a given non-decreasing sequence P of non-negative integers is a mark sequence, and for constructing a corresponding k-digraph. At each stage, we form P' according to Theorem 2.4 such that P' is in non-decreasing order. If $p_n = 2k(n-1) - r$, deleting p_n, and performing (a) or (b) of Theorem 2.4 according as $r \leq n-1$, or $r > n-1$, we get $P' = [p'_1, p'_2, \ldots, p'_{n-1}]$. If the mark of vertex v_i was decreased by one in this process, then the construction yielded $v_n((k-1)-0)v_i$, and if it was decreased by two, then the construction yielded $v_n((k-1)-1)v_i$. For a vertex v_j whose mark remained unchanged, the construction yielded $v_n(k-0)v_j$. If this procedure is applied recursively, then it tests whether or not P is a mark sequence, and if P is a mark sequence, then a k-digraph with mark sequence P is constructed.

Theorem 2.5. *Let $P = [p_i]_1^n$ be a sequence of non-negative integers in non-decreasing order, where for each i, $0 \leq p_i \leq 2k(n-1)$. Let P' be obtained from P by deleting the greatest entry p_n $(= 2k(n-1) - r$, say) and* (a) *if r is even, say $r = 2t$, reducing t of the next greatest entries by two, or* (b) *if r is odd, say $r = 2t+1$, reducing t greatest remaining entries by two, and reducing the greatest among the remaining entries by one. Then P is a mark sequence if and only if P' (arranged in non-decreasing order) is a mark sequence.*

The proof follows by using the arguments as in Theorem 2.4.

Theorem 2.5 also provides an algorithm of checking whether or not a given non-decreasing sequence P of non-negative integers is a mark sequence and for constructing a corresponding k-digraph. At each stage, we form P' according to Theorem 2.5 such that P' is in non-decreasing order. If $p_n = 2k(n-1) - r$, deleting p_n, and performing (a), or (b), of Theorem 2.5 according as r is even or odd, we get $P' = [p'_1, p'_2, \ldots, p'_{n-1}]$. If the mark of the vertex v_i was decreased by two in the process, then the construction yielded $v_n((k-1)-1)v_i$, and if it was decreased by one, then the construction yielded $v_n((k-1)-0)v_i$. For a vertex v_j whose mark remained unchanged, the construction yielded $v_n(k-0)v_j$. If this procedure is applied recursively, then it tests whether or not P is a mark sequence, and if P is a mark sequence, then a k-digraph with mark sequence P is constructed.

3. Irreducible mark sequences

A k-digraph is *reducible* if it is possible to partition its vertices into two nonempty sets V_1 and V_2 in such a way that there are exactly two arcs directed from every vertex of V_2 to each vertex of V_1, and there is no arc from any vertex of V_1 to any vertex of V_2. If D_1 and D_2 are k-digraphs having vertex sets V_1 and V_2 respectively, then the k-digraph D consisting of all the arcs of D_1, and all the arcs of D_2, and exactly k arcs directed from every vertex of D_2 to each vertex of D_1 is denoted by $D = [D_1, D_2]$. If this is not possible, the k-digraph is said to be *irreducible*. Let $D_1, D_2, \ldots, D_h$ be irreducible k-digraphs with disjoint vertex sets. Then $D = [D_1, D_2, \ldots, D_h]$ is the k-digraph having all arcs of D_i, $1 \le i \le h$, and exactly k arcs from every vertex of D_j to each vertex of D_i, $1 \le i < j \le h$. We call $D_1, D_2, \ldots, D_h$ the *irreducible components* of D, and such a decomposition is called the *irreducible decomposition* of D. A mark sequence P is said to be *irreducible* if all the k-digraphs D with mark sequence P are irreducible.

The following result characterizes irreducible k-digraphs.

Theorem 3.1. *If D is a connected k-digraph with mark sequence $P = [p_i]_1^n$, then D is irreducible if and only if, for $t = 1, 2, \ldots, n-1$,*

$$\sum_{i=1}^{t} p_i > kt(t-1) \tag{3.1.1}$$

and

$$\sum_{i=1}^{n} p_i = kn(n-1). \tag{3.1.2}$$

Proof. Let D be a connected, irreducible k-digraph having mark sequence $P = [p_i]_1^n$. Condition (3.1.2) holds, since Theorem 2.2 has already established it for any k-digraph. Condition (3.1.2) also implies that for any integer $t < n$, the k-subdigraph D' induced by any set of t vertices has a sum of marks in D' equal to $kt(t-1)$. Since D is irreducible, therefore either there is an arc from at least one of these t vertices to at least one of the other $n-t$ vertices, or there is exactly one arc from at least one of the other $n-t$ vertices to at least one vertex in D'. Therefore, for $1 \le t < n-1$,

$$\sum_{i=1}^{t} p_i \ge kt(t-1) + 1 > kt(t-1).$$

For the converse, suppose that conditions (3.1.1) and (3.1.2) hold. It follows from Theorem 2.2 that there exists a k-digraph with mark sequence $P = [p_i]_1^n$. Assume such a k-digraph is reducible, and let $D = [D_1, D_2, \ldots, D_h]$ be the irreducible component decomposition of D. Since there are exactly k arcs from every vertex of D_j to each vertex of D_i, $1 \le i < j \le h$, D is evidently connected. If m is the number of vertices in D_1, then $m < n$, and $\sum_{i=1}^{m} p_i = km(m-1)$, which is a contradiction to the given hypothesis. Hence, D is irreducible. □

We note that a disconnected k-digraph is always irreducible, since if D_1 and D_2 are the components of D, then there are no arcs between vertices of D_1 and vertices of D_2.

The following result can be easily established.

Theorem 3.2. *If D is a k-digraph with mark sequence $P = [p_i]_1^n$, and $\sum_{i=1}^{r} p_i = kr(r-1)$, $\sum_{i=1}^{t} p_i = kt(t-1)$, and $\sum_{i=1}^{q} p_i > kq(q-1)$, for $r+1 \leq q \leq t-1$, $0 \leq r < t \leq n$, then the k-subdigraph induced by the vertices v_{r+1}, v_{r+2}, $\ldots$, v_t is an irreducible component of D with mark sequence $[p_i - kr]_{r+1}^t$.*

The mark sequence P is irreducible if D is irreducible, and the irreducible components of P are the mark sequences of the irreducible components of D. That is, if $D = [D_1, D_2, \ldots, D_h]$ is the irreducible component decomposition of a k-digraph D with mark sequence P, then the irreducible components P_i of P are the mark sequences of the k-subdigraphs induced by the vertices of D_i, $1 \leq i \leq h$. Theorem 3.2 shows that the irreducible components of P are determined by the successive values of k for which

$$\sum_{i=1}^{t} p_i = kt(t-1), \qquad 1 \leq t \leq n. \tag{3.2.1}$$

This is illustrated by the following examples of 2-digraphs.

(i) Let $P = [1, 3, 9, 12, 15, 20]$. Equation (3.2.1) is satisfied for $k = 2, 5, 6$. Therefore, the irreducible components of P are $[0]$, $[1, 4, 7]$, $[0]$ in ascending order.

(ii) Let $P = [0, 5, 8, 11, 17, 19]$. Here Equation (3.2.1) is satisfied for $k = 1, 4, 6$. Therefore, the irreducible components of P are $[0]$, $[1, 4, 7]$ and $[1, 3]$ in ascending order.

A mark sequence is uniquely realizable if it belongs to exactly one k-digraph. The characterization of uniquely realizable score sequences in tournaments is given by Avery [1], and that of oriented graphs by S.Pirzada [6]. Now, as an observation, we have the following result.

Theorem 3.3. *The mark sequence P of a k-digraph D is uniquely realizable if and only if every irreducible component of P is uniquely realizable.*

The next result determines which irreducible mark sequences in 2-digraphs are uniquely realizable.

Theorem 3.4. *The only irreducible mark sequences in 2-digraphs that are uniquely realizable are $[0]$ and $[1, 3]$.*

Proof. Let P be an irreducible mark sequence, and let D with vertex set V be a 2-digraph having mark sequence P. Then D is irreducible. Therefore, D cannot be partitioned into 2-subdigraphs D_1, D_2, $\ldots$, D_k such that there are exactly two arcs from every vertex of D_α to each vertex of D_β, $1 \leq \beta < \alpha \leq k$. First assume D has $n \geq 3$ vertices. Let $W = \{w_1, w_2, \ldots, w_r\}$ and $U = \{u_1, u_2, \ldots, u_s\}$ respectively be any two disjoint subsets of V such that $r + s = n$. Since D is irreducible, (1) there do not exist exactly two arcs from every w_i $(1 \leq i \leq r)$ to each u_j $(1 \leq j \leq s)$, and (2) there do not exist exactly two arcs from every u_j $(1 \le j \le s)$ to each w_i $(1 \le i \le s)$. First of all we consider Case (1), and then Case (2) follows by using the same argument as in (1).

CASE (1). There exists at least one vertex, say w_1, in W, and at least one vertex, say u_1, in U such that either (a) $w_1(1-1)u$, or (b) $w_1(0-2)u_1$, or (c) $w_1(1-0)u_1$, or (d) $w_1(0-1)u_1$, or (e) $w_1(0-0)u_1$.

Assume $w_i(2-0)u_j$ for each i $(1 \le i \le r)$ and j $(1 \le j \le s)$, except for $i = j = 1$.

If in D, either (a) $w_1(1-1)u_1$, or (e) $w_1(0-0)u_1$, then transforming them respectively to $w_1(0-0)u_1$, or $w_1(1-1)u_1$, gives a 2-digraph D' with the same mark sequence. In both cases, D and D' have different number of arcs, and thus are non-isomorphic.

(b) Let $w_1(0-2)u_1$. Since there are only six possibilities between w_1 and w_i, therefore, for any other vertex w_i in W we have one of the following cases:

(i) $w_1(2-0)w_i(2-0)u_1(2-0)w_1$, (ii) $w_1(1-1)w_i(2-0)u_1(2-0)w_1$, (iii) $w_1(1-0)w_i(2-0)u_1(2-0)w_1$, (iv) $w_1(0-1)w_i(2-0)u_1(2-0)w_1$, (v) $w_1(0-0)w_i(2-0)u_1(2-0)w_1$, (vi) $w_1(0-2)w_i(2-0)u_1(2-0)w_1$.

Transforming (i)–(v) respectively to $w_1(1-0)w_i(1-0)u_1(1-0)w_1$, $w_1(0-1)w_i(1-0)u_1(1-0)w_1$, $w_1(0-0)w_i(1-0)u_1(1-0)w_1$, $w_1(0-2)w_i(1-0)u_1(1-0)w_1$, $w_1(0-1)w_i(1-0)u_1(1-0)w_1$, gives a 2-digraph with the same mark sequence. In all these five cases, D and D' have different number of arcs, and thus are non-isomorphic.

If (vi) occurs in D, and also $w_q(2-0)w_i$ for $1 \le i < q \le r$, then the 2-digraph D is reducible with irreducible components $D_1, D_2, \ldots, D_r$ respectively having vertex sets $V_1 = \{u_1, u_2, \ldots, u_s, w_1\}$, $V_2 = \{w_2\}$, $V_3 = \{w_3\}$, $\ldots$, $V_k = \{w_r\}$.

Also for any vertex u_j in U, since there are only six possibilities between u_1 and u_j, we have one of the following cases:

(vii) $w_1(0-2)u_1(0-2)u_j(0-2)w_1$, (viii) $w_1(0-2)u_1(1-1)u_j(0-2)w_1$, (ix) $w_1(0-2)u_1(1-0)u_j(0-2)w_1$, (x) $w_1(0-2)u_1(0-1)u_j(0-2)w_1$, (xi) $w_1(0-2)u_1(0-0)u_j(0-2)w_1$, (xii) $w_1(0-2)u_1(2-0)u_j(0-2)w_1$.

If any one of (vii)–(xi) appears in D, then making respectively the transformations $w_1(0-1)u_1(0-1)u_j(0-1)w_1$, $w_1(0-1)u_1(1-0)u_j(0-1)w_1$, $w_1(0-1)u_1(2-0)u_j(0-1)w_1$, $w_1(0-1)u_1(1-1)u_j(0-1)w_1$, $w_1(0-1)u_1(1-0)u_j(0-1)w_1$, we get a 2-digraph with the same mark sequence, but the numbers of arcs in D and D' are different, and thus D and D' are non-isomorphic.

If (xii) and any of (i)–(v) appear simultaneously, then there exists a 2-digraph D' with the same mark sequence, but D and D' have different numbers of arcs. Thus, D and D' are non-isomorphic.

If (vi) and (xii) appear simultaneously, and also $w_q(2-0)w_i$ for all $1 \le i < q \le r$, then D is reducible with the irreducible components $D_1, D_2, \ldots, D_r$ having vertex sets $V_1 = \{u_1, u_2, \ldots, u_s, w_1\}$, $V_2 = \{w_2\}$, $V_3 = \{w_3\}$, $\ldots$, $V_r = \{w_r\}$ respectively.

(c) Let $w_1(1-0)u_1$. For any vertex w_i in W, since there are only six possibilities between w_1 and w_i, we have one of the following cases:

(i) $w_1(2-0)w_i(2-0)u_1(0-1)w_1$, (ii) $w_1(1-1)w_i(2-0)u_1(0-1)w_1$, (iii) $w_1(1-0)w_i(2-0)u_1(0-1)w_1$, (iv) $w_1(0-1)w_i(2-0)u_1(0-1)w_1$, (v) $w_1(0-0)w_i(2-0)u_1(0-1)w_1$, (vi) $w_1(0-2)w_i(2-0)u_1(0-1)w_1$.

For (i)–(v) making respectively the transformations $w_1(1-0)w_i(1-0)u_1(0-2)w_1$, $w_1(0-1)w_i(1-0)u_1(0-2)w_1$, $w_1(0-1)w_i(1-0)u_1(0-2)w_1$, $w_1(1-1)w_i(1-0)u_1(2-0)w_1$,

$w_1(0-1)w_i(1-0)u_1(2-1)w_1$, we obtain a 2-digraph D' with the same mark sequence, but the numbers of arcs in D and D' are not equal. Thus, D and D' are non-isomorphic.

Now, for any other vertex u_j in U, there are only six possibilities between u_1 and u_j, and we have one of the following cases:

(vii) $w_1(1-0)u_1(0-2)u_j(0-2)w_1$, (viii) $w_1(1-0)u_1(1-1)u_j(0-2)w_1$, (ix) $w_1(1-0)u_1(1-0)u_j(0-2)w_1$, (x) $w_1(1-0)u_1(0-1)u_j(0-2)w_1$, (xi) $w_1(1-0)u_1(0-0)u_j(0-2)w_1$, (xii) $w_1(1-0)u_1(2-0)u_j(0-2)w_1$.

If any one of (vii)–(xi) appears, then making respectively the transformations $w_1(2-0)u_1(0-1)u_j(0-1)w_1$, $w_1(2-0)u_1(1-0)u_j(0-1)w_1$, $w_1(2-0)u_1(2-0)u_j(0-1)w_1$, $w_1(2-0)u_1(1-1)u_j(0-1)w_1$, $w_1(2-0)u_1(1-0)u_j(0-1)w_1$, we get a 2-digraph D' with the same mark sequence, but D and D' have different numbers of arcs. Thus, D and D' are non-isomorphic.

If (xii) and one of (i)-(v) appears simultaneously, we once again arrive to the conclusion that there exists a 2-digraph D' with the mark sequence P, but D and D' are non-isomorphic.

Thus, we are left with the case when (vi) and (xii) appear simultaneously, and also $w_q(2-0)w_i$ for all $1 \leq i < q \leq r$. But, then D is reducible having the irreducible components D_1, D_2, $\ldots$, D_r with vertex sets $V_1 = \{u_1, u_2, \ldots, u_s, w_1\}$, $V_2 = \{w_2\}$, $\ldots$, $V_r = \{w_r\}$ respectively.

(d) Let $w_1(0-1)u_1$. Since there are only six possibilities between w_1 and w_i, therefore for any other vertex w_i in W, we have one of the following cases:

(i) $w_1(2-1)w_i(2-0)u_1(1-0)w_1$, (ii) $w_1(1-1)w_i(2-0)u_1(1-0)w_1$, (iii) $w_1(1-0)w_i(2-0)u_1(1-0)w_1$, (iv) $w_1(0-1)w_i(2-0)u_1(1-0)w_1$, (v) $w_1(0-0)w_i(2-0)u_1(1-0)w_1$, (vi) $w_1(0-2)w_i(2-0)u_1(1-0)w_1$.

If any one of (i)–(v) appears, then making respectively the transformations $w_1(1-0)w_i(1-0)u_1(0-0)w_1$, $w_1(0-1)w_i(1-0)u_1(0-0)w_1$, $w_1(0-0)w_i(1-0)u_1(0-0)w_1$, $w_1(0-2)w_i(1-0)u_1(0-0)w_1$, $w_1(0-1)w_i(1-0)u_1(0-0)w_1$, gives a 2-digraph D' with the same mark sequence, but the numbers of arcs in D and D' are different so that D and D' are non-isomorphic.

If (vi) appears in D, and also if $w_q(2-0)w_i$ for all $1 \leq i < q \leq r$, then D becomes reducible.

Now, for any other vertex u_j in U, there are only six possibilities between u_1 and u_j, and we have one of the following cases:

(vii) $w_1(0-1)u_1(0-2)u_j(0-2)w_1$, (viii) $w_1(0-1)u_1(1-1)u_j(0-2)w_1$, (ix) $w_1(0-1)u_1(1-0)u_j(0-2)w_1$, (x) $w_1(0-1)u_1(0-1)u_j(0-2)w_1$, (ix) $w_1(0-1)u_1(0-0)u_j(0-2)w_1$, (xii) $w_1(0-1)u_1(2-0)u_j(0-2)w_1$.

If any one of (vii)–(xi) appears in D, then making respectively the transformations $w_1(0-0)u_1(0-1)u_j(0-1)w_1$, $w_1(0-0)u_1(1-0)u_j(0-1)w_1$, $w_1(0-0)u_1(2-0)u_j(0-1)w_1$, $w_1(0-0)u_1(0-0)u_j(0-1)w_1$, $w_1(0-0)u_1(1-0)u_j(0-1)w_1$, gives a 2-digraph D' with the same mark sequence, but the numbers of arcs in D and D' are different so that D is not isomorphic to D'.

If (xii) and any one of (i)–(v) appear simultaneously, then once again there exists a 2-digraph D' with the same mark sequence, but D and D' have different numbers of arcs so that D and D' are non-isomorphic.

If (vi) and (xii) appear simultaneously, and also $w_q(2-0)w_i$ for all $1 \leq i < q \leq r$, then D is reducible.

Now, let D have exactly two vertices say u and v. The only irreducible mark sequences realizing D are $[2,2]$, and $[1,3]$. Obviously the sequence $[2,2]$ has two non-isomorphic realizations namely $u(0{-}0)v$ and $u(1{-}1)v$, and $[1,3]$ has the unique realization $u(0{-}1)v$. Thus $P = [1,3]$ is uniquely realizable.

If D has only one vertex, then $P = [0]$, which evidently is uniquely realizable. $\square$

Combining Theorem 3.3 and Theorem 3.4, we have the following result for 2-digraphs.

Theorem 3.5. *The mark sequence P of a 2-digraph is uniquely realizable if and only if every irreducible component of P is of the form $[0]$ or $[1,3]$.*

We observe that in the mark sequence $P = [4i-4]_1^n$ every irreducible component is $[0]$, and thus P is uniquely realizable. We note that the mark sequences of tournaments are not uniquely realizable. To see this, consider the mark sequence $P = [2,4,6]$ realizing the tournament T. The other 2-digraph D realized by P has vertex set $\{v_1, v_2, v_3\}$ with $v_1(0-0)v_2(0-0)v_3(2-0)v_1$.

However, we observe that a mark sequence of a tournament T is uniquely realizable if and only if the mark sequence of the double tournament of T is uniquely realizable.

Now, we have the following generalization of Theorem 3.5, and the proof follows by using arguments as in Theorem 3.5.

Theorem 3.6. *The mark sequence P of a k-digraph is uniquely realizable if and only if every irreducible component of P is of the form $[0]$ or $[1, 2k-1]$.*

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